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# ベルジュ双対のための非二部的 Dulmage-Mendelsohn 分解 (Nonbipartite Dulmage-Mendelsohn Decomposition for Berge Duality)

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**Abstract**

The *Dulmage-Mendelsohn decomposition* is a classical *canonical decomposition* in matching theory applicable for bipartite graphs and is famous not only for its application in the field of matrix computation, but also for providing a prototypal structure in matroidal optimization theory. The Dulmage-Mendelsohn decomposition is stated and proved using the two color classes of a bipartite graph, and therefore generalizing this decomposition for nonbipartite graphs has been a difficult task. In our study, we obtain a new canonical decomposition that is a generalization of the Dulmage-Mendelsohn decomposition for arbitrary graphs using a recently introduced tool in matching theory, the *basilica decomposition*. Our result enables us to understand all known canonical decompositions in a unified way. Furthermore, we apply our result to derive a new theorem regarding *barriers*. The duality theorem for the maximum matching problem is the celebrated *Berge formula*, in which dual optimizers are known as barriers. Several results regarding maximal barriers have been derived by known canonical decompositions; however, no characterization has been known for general graphs. In our study, we provide a characterization of the family of *maximal barriers* in general graphs, in which the known results are developed and unified.

## 1 Introduction

We establish the Dulmage-Mendelsohn decomposition for general graphs. The *Dulmage-Mendelsohn decomposition* [2–4], or the *DM decomposition* in short, is a classical canonical decomposition in matching theory [15] applicable for bipartite graphs. This decomposition is famous for its application for combinatorial matrix theory, especially for providing an efficient solution for a system of linear equations [1,4] and is also important in matroidal optimization theory. Furthermore, its connection with matrices and matroids gave rise to a branch of combinatorial matrix theory known as mixed matrix theory [16].

*Canonical decompositions* of a graph are fundamental tools in matching theory [15]. A canonical decomposition partitions a given graph in a way uniquely determined for the graph and describes the structure of maximum matchings using this partition. The classical canonical decompositions are the *Gallai-Edmonds* [5, 6] and *Kotzig-Lovász* decompositions [11–13] in addition to the DM decomposition. The DM and Kotzig-Lovász decompositions are applicable for bipartite graphs and *factor-connected graphs*, respectively. The Gallai-Edmonds decomposition partitions an arbitrary graph into three parts: that is, the so-called $D(G)$, $A(G)$, and $C(G)$ parts. Comparably recently, a new canonical decomposition was proposed: the *basilica decomposition* [7–9]. This decomposition is applicable for arbitrary graphs and contains a generalization of the Kotzig-Lovász decomposition and a refinement the Gallai-Edmonds decomposition. (The $C(G)$ part can be decomposed nontrivially.)

In our study, we establish an analogue of the DM decomposition for general graphs using the basilica decomposition. Our results accordingly provide a paradigm that enables us to handle any graph and understand the known canonical decompositions in a unified way. In the original theory of DM decomposition, the concept of the *DM components* of a bipartite graph is first defined, and then it is proved that these components form a poset with respect to a certain binary relation. This theory depends heavily on the two color classes of a bipartite graph and cannot be easily generalized for nonbipartite graphs. In our generalization, we first define a generalization of the DM components using the basilica decomposition. To capture the structure formed by these components in nonbipartite graphs, we introduce a slightly more complex concept: *posets with a transitive forbidden relation.* We then prove that the generalized DM components form a poset with a transitive forbidden relation for certain binary relations.

Furthermore, we apply our generalized DM decomposition to derive a characterization of the family of *maximal barriers* in general graphs. The *Berge formula* is a combinatorial min-max theorem in which maximum matchings are the optimizers of one hand, and the optimizers of the other hand are known as *barriers* [15]. That is, barriers are the dual optimizers of the maximum matchings problem. Barriers are heavily employed as a tool for studying matchings. However, not as much is known about barriers themselves [15]. Aside from several observations that are derived rather easily from the Berge formula, several substantial results about (inclusion-wise) maximal barriers have been provided by canonical decompositions.

Our result for maximal barriers proves that our generalization of the DM decomposition has a reasonable consistency with the relationship between each known canonical decomposition and maximal barriers. Each known canonical decomposition can be used to state the structure of maximal barriers. The original DM decomposition provides a characterization of the family of maximal barriers in a bipartite graph in terms of ideals in the poset; minimum vertex covers in bipartite graphs are equivalent to maximal barriers. The Gallai-Edmonds decomposition derives a characterization of the intersection of all maximal barriers (that is, the $A(G)$ part) [15]; this characterization is known as the *Gallai-Edmonds description.* The Kotzig-Lovász decomposition is used for characterizing the family of maximal barriers in factor-connected graphs [15]; this result is known as *Lovász's canonical partition theorem* [14, 15]. The basilica decomposition provides the structure of a given maximal barrier in general graphs, which contains a common generalization of the Gallai-Edmonds description and Lovász's canonical partition theorem. Hence, a generalization of the DM decomposition would be reasonable if it can characterize the family of maximal barriers, and our generalization attains this in a way analogical to the classical DM decomposition, that is, in terms of ideals in the poset with a transitive forbidden relation.

Our results imply a new possibility in matroidal optimization theory. Submodular function theory is a systematic field of study that captures many well-solved problems in terms of submodular functions and generalizations. In this theory, the bipartite maximum matching problem is an important exemplary problem. According to the Hall-Ore theorem [17], which is the duality theorem for the bipartite maximum matching problem, this problem can be understood as a special case of the submodular function minimization. The DM decomposition therefore has a special meaning in this theory as it describes the structure of the family of minimizers of a submodular function. The nonbipartite maximum matching problem is also an important well-solved problem, and is even referred to as the archetype of well-solved problems [15, 17]. In fact, the idea of polyhedral combinatorics and some of its central concepts, such as the total dual integrality, have been discovered from the nonbipartite maximum matching problem. However, the nonbipartite maximum matching problem and its duality shown by the Berge formula are not included in submodular function theory today and nor in any of its generalizations. Our nonbipartite DM decomposition may provide a clue to a new aspect of submodular function theory that can be brought in by capturing these concepts.

## 2 Notation

For basic notation for sets, graphs, and algorithms, we mostly follow Schrijver [17]. In this section, unless otherwise stated, let $G$ be a graph. The vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. We often treat a graph as the set of its vertices.

In the remainder of this section, let $X \subseteq V(G)$. The subgraph of $G$ induced by $X$ is denoted by $G[X]$. The graph $G[V(G) \setminus X]$ is denoted by $G - X$. The contraction of $G$ by $X$ is denoted by $G/X$. Let $F \subseteq E(G)$. The graph obtained by deleting $F$ from $G$ without removing vertices is denoted by $G - F$. Let $H$ be a subgraph of $G$. The graph obtained by adding $F$ to $H$ is denoted by $H + F$. Regarding these operations, we identify vertices, edges, subgraphs of the newly created graph with the naturally corresponding items of old graphs.

A *neighbor* of $X$ is a vertex from $V(G) \setminus X$ that is adjacent to some vertex from $X$. The neighbor set of $X$ is denoted by $N_G(X)$. Let $Y \subseteq V(G)$. The set of edges joining $X$ and $Y$ is denoted by $E_G[X, Y]$. The set $E_G[X, V(G) \setminus X]$ is denoted by $\delta_G(X)$.

A set $M \subseteq E(G)$ is a *matching* if $|\delta_G(v) \cap M| \leq 1$ holds for each $v \in V(G)$. For a matching $M$, we say that $M$ *covers* a vertex $v$ if $|\delta_G(v) \cap M| = 1$; otherwise, we say that $M$ *exposes* $v$. A matching is *maximum* if it consists of the maximum number of edges. A graph can possess an exponentially large number of matchings. A matching is *perfect* if it covers every vertex. A graph is *factorizable* if it has a perfect matchings. A graph is *factor-critical* if, for each vertex $v$, $G - v$ is factorizable. A graph with only one vertex is defined to be factor-critical. The number of edges in a maximum matching is denoted by $\nu(G)$. The number of vertices exposed by a maximum matching is denoted by $\mathrm{def}(G)$; that is, $\mathrm{def}(G) := |V(G)| - 2\nu(G)$.

## 3 Basics on Matchings

We now explain the Berge Formula and the definition of barriers. An *odd component* (resp. *even component*) of a graph is a connected component with an odd (resp. even) number of vertices. The number of odd components of $G - X$ is denoted by $q_G(X)$. The set of vertices from odd components (resp. even components) of $G - X$ is denoted by $D_X$ (resp. $C_X$).

**Theorem 3.1** (Berge Formula [15])**.** For a graph $G$, $\mathrm{def}(G)$ is equal to the maximum value of $q_G(X) - |X|$, where $X$ is taken over all subsets of $V(G)$.

The set of vertices that attains the maximum value in this relation is called a *barrier*. That is, a set of vertices $X$ is a *barrier* if $\mathrm{def}(G) = q_G(X) - |X|$.

The set of vertices that can be exposed by maximum matchings is denoted by $D(G)$. The neighbor set of $D(G)$ is denoted by $A(G)$, and the set $V(G) \setminus D(G) \setminus A(G)$ is denoted by $C(G)$. The following statement about $D(G)$, $A(G)$, and $C(G)$ is the celebrated *Gallai-Edmonds structure theorem* [5, 6, 15].

**Theorem 3.2** (Gallai-Edmonds Structure Theorem)**.** For any graph $G$,

(i) $A(G)$ is a barrier for which $D_{A(G)} = D(G)$ and $C_{A(G)} = C(G)$;

(ii) each odd component of $G - A(G)$ is factor-critical; and,

(iii) any edge in $E_G[A(G), D(G)]$ is allowed.

An edge is *allowed* if it is contained in some maximum matching. Two vertices are *factor-connected* if they are connected by a path whose edges are allowed. A subgraph is *factor-connected* if any two vertices are factor-connected. A maximal factor-connected subgraph is called a *factor-connected component* or *factor-component*. A graph consists of its factor-components and edges joining them that are not allowed. The set of factor-components of $G$ is denoted by $\mathcal{G}(G)$.

A factor-component $C$ is *inconsistent* if $V(C) \cap D(G) \neq \emptyset$. Otherwise, $C$ is said to be *consistent*. We denote the sets of consistent and inconsistent factor-components of $G$ by $\mathcal{G}^+(G)$ and $\mathcal{G}^-(G)$, respectively.

## 4 Basilica Decomposition

We now introduce the basilica decomposition of graphs [8, 9]. The theory of basilica decomposition is made up of the three central concepts:

(i) a canonical partial order between factor-components (Theorem 4.2),

(ii) the general Kotzig-Lovász decomposition (Theorem 4.4), and

(iii) an interrelationship between the two (Theorem 4.5).

In this section, we explain these three concepts and give the definition of the basilica decomposition. Every statement in the following is from Kita [8, 9]. In the following, let $G$ be a graph unless otherwise stated.

**Definition 4.1.** A set $X \subseteq V(G)$ is said to be *separating* if there exist $H_1, \ldots, H_k \in \mathcal{G}(G)$, where $k \geq 1$, such that $X = V(H_1) \cup \cdots \cup V(H_k)$. For $G_1, G_2 \in \mathcal{G}(G)$, we say $G_1 \lhd G_2$ if there exists a separating set $X \subseteq V(G)$ with $V(G_1) \cup V(G_2) \subseteq X$ such that $G[X]/G_1$ is a factor-critical graph.

**Theorem 4.2.** For a graph $G$, the binary relation $\lhd$ is a partial order over $\mathcal{G}(G)$.

**Definition 4.3.** For $u, v \in V(G) \setminus D(G)$, we say $u \sim_G v$ if $u$ and $v$ are identical or if $u$ and $v$ are factor-connected and satisfy $\mathrm{def}(G - u - v) > \mathrm{def}(G)$.

**Theorem 4.4.** For a graph $G$, the binary relation $\sim_G$ is an equivalence relation.

We denote as $\mathcal{P}(G)$ the family of equivalence classes determined by $\sim_G$. This family is known as the *general Kotzig-Lovász decomposition* or just the *Kotzig-Lovász decomposition* of $G$. From the definition of $\sim_G$, for each $H \in \mathcal{G}(G)$, the family $\{S \in \mathcal{P}(G) : S \subseteq V(H)\}$ forms a partition of $V(H) \setminus D(G)$. We denote this family by $\mathcal{P}_G(H)$.

Let $H \in \mathcal{G}(G)$. The sets of strict and nonstrict upper bounds of $H$ are denoted by $\mathcal{U}_G(H)$ and $\mathcal{U}^*_G(H)$, respectively. The sets of vertices $\bigcup\{V(I) : I \in \mathcal{U}_G(H)\}$ and $\bigcup\{V(I) : I \in \mathcal{U}^*_G(H)\}$ are denoted by $U_G(H)$ and $U^*_G(H)$, respectively.

**Theorem 4.5.** Let $G$ be a graph, and let $H \in \mathcal{G}(G)$. Then, for each connected component $K$ of $G[U_G(H)]$, there exists $S \in \mathcal{P}_G(H)$ such that $N_G(K) \cap V(H) \subseteq S$.

Under Theorem 4.5, for $S \in \mathcal{P}_G(H)$, we denote by $\mathcal{U}_G(S)$ the set of factor-components that are contained in a connected component $K$ of $G[U_G(H)]$ with $N_G(K) \cap V(H) \subseteq S$. The set $\bigcup\{V(I) : I \in U_G(H)\}$ is denoted by $U_G(S)$. We denote $U_G(H) \setminus S \setminus U_G(S)$ by ${}^{\top}U_G(S)$.

Theorem 4.5 integrates the two structures given by Theorems 4.2 and 4.4 into a structure of graphs that is reminiscent of an architectural building. We call this integrated structure the *basilica decomposition* of a graph.

## 5 TFR Poset

We now introduce the new concept of *posets with a transitive forbidden relation*, which serves as a language to describe the nonbipartite DM decomposition.

**Definition 5.1.** Let $X$ be a set, and let $\preceq$ be a partial order over $X$. Let $\smile$ be a binary relation over $X$ such that,

(i) for each $x, y, z \in X$, if $x \preceq y$ and $y \smile z$ hold, then $x \smile z$ holds (transitivity);

(ii) for each $x \in X$, $x \smile x$ does not hold (nonreflexivity); and,

(iii) for each $x, y \in X$, if $x \smile y$ holds, then $y \smile x$ also holds (symmetry).

We call this poset endowed with this additional binary relation a *poset with a transitive forbidden relation* or *TFR poset* in short, and denote this by $(X, \preceq, \smile)$. We call a pair of two elements $x$ and $y$ with $x \smile y$ *forbidden*.

Let $(X, \preceq, \smile)$ be a TFR poset. For two elements $x, y \in X$ with $x \smile y$, we say that $x \overset{\star}{\smile} y$ if, there is no $z \in X \setminus \{x, y\}$ with $x \preceq z$ and $z \smile y$. We call such a forbidden pair of $x$ and $y$ *immediate*. A TFR poset can be visualized in a similar way to an ordinary posets. We represent $\preceq$ just in the same way as the Hasse diagrams and depict $\smile$ by indicating every immediate forbidden pairs.

**Definition 5.2.** Let $P$ be a TFR poset $(X, \preceq, \smile)$. A lower or upper ideal $Y$ of $P$ is *legitimate* if no elements $x, y \in Y$ satisfy $x \smile y$. Otherwise, we say that $Y$ is *illegitimate*. Let $Y$ be a consistent lower or upper ideal, and let $Z$ be the subset of $X \setminus Y$ such that, for each $x \in Z$, there exists $y \in Y$ with $x \smile y$. We say that $Y$ is *spanning* if $Y \cup Z = X$.

# 6 DM Decomposition Theory for General Graphs

We now provide our new theory of the DM decomposition for general graphs. In this section, unless otherwise stated, let $G$ be a graph.

**Definition 6.1.** A *Dulmage-Mendelsohn component*, or a *DM component* in short, is a subgraph of the form $G[S \cup {}^{\top}U_G(S)]$, where $S \in \mathcal{P}(G)$, endowed with $S$ as an attribute known as the *base*. For a DM component $C$, the base of $C$ is denoted by $\pi(C)$. Conversely, for $S \in \mathcal{P}(G)$, $K(S)$ denotes the DM components whose base is $S$. We denote by $\mathcal{D}(G)$ the set of DM components of $G$.

Hence, distinct DM components can be equivalent as a subgraph of $G$. Each member from $\mathcal{P}(G)$ serves as an identifier of a DM component.

**Definition 6.2.** A DM component $C$ is said to be *inconsistent* if $\pi(C) \in \mathcal{P}_G(H)$ for some $H \in \mathcal{G}^-(G)$; otherwise, $C$ is said to be *consistent*. The sets of consistent and inconsistent DM components are denoted by $\mathcal{D}^+(G)$ and $\mathcal{D}^-(G)$, respectively.

**Definition 6.3.** We define binary relations $\preceq^\circ$ and $\preceq$ over $\mathcal{D}(G)$ as follows: for $D_1, D_2 \in \mathcal{D}(G)$, we let $D_1 \preceq^\circ D_2$ if $D_1 = D_2$ or if $N_G({}^{\top}U_G(S_1)) \cap S_2 \neq \emptyset$; we let $D_1 \preceq D_2$ if there exist $C_1, \ldots, C_k \in \mathcal{D}(G)$, where $k \geq 1$, such that $\pi(C_1) = \pi(D_1)$, $\pi(C_k) = \pi(D_2)$, and $C_i \preceq^\circ C_{i+1}$ for each $i \in \{1, \ldots, k\} \setminus \{k\}$.

**Definition 6.4.** We define binary relations $\smile^\circ$ and $\smile$ over $\mathcal{D}(G)$ as follows: for $D_1, D_2 \in \mathcal{D}(G)$, we let $D_1 \smile^\circ D_2$ if $\pi(D_2) \subseteq V(D_1) \setminus \pi(D_1)$ holds; we let $D_1 \smile D_2$ if there exists $D' \in \mathcal{D}(G)$ with $D_1 \preceq D'$ and $D' \smile^\circ D_2$.

**Theorem 6.5.** For a graph $G$, the triple $(\mathcal{D}(G), \preceq, \smile)$ is a TFR poset.

For a graph $G$, the TFR poset $(\mathcal{D}(G), \preceq, \smile)$ is uniquely determined. We denote this TFR poset by $\mathcal{O}(G)$. We call this canonical structure that $\mathcal{O}(G)$ describes the *nonbipartite Dulmage-Mendelsohn (DM) decomposition* of $G$. This is a generalization of the classical DM decomposition for bipartite graphs.

*Remark* 6.6. As mentioned previously, a DM component is identified by its base. Therefore, the nonbipartite DM decomposition is essentially the relations between the members of $\mathcal{P}(G)$.

Immediate forbidded pairs in $\mathcal{O}(G)$ can be characterized as follows:

**Theorem 6.7.** Let $G$ be a graph. Let $S, T \in \mathcal{P}(G)$. Then, $K(S)$ and $K(T)$ are immediate forbidden pairs if and only if $S$ and $T$ are contained in the same factor-component.

Given a graph $G$, its basilica decomposition can be computed in $O(|V(G)| \cdot |E(G)|)$ time [8, 9]. Therefore, the next thereom can be stated.

**Theorem 6.8.** Given a graph $G$, the TFR poset $\mathcal{O}(G)$ can be computed in $O(|V(G)| \cdot |E(G)|)$ time.

## 7 Characterization of Barriers

We now derive the characterization of the family of maximal barriers in general graphs using the nonbipartite DM decomposition. In this section, unless otherwise stated, let $G$ be a graph. It is a known fact that a graph has an exponentially many number of maximal barriers, however the family of maximal barriers can be fully characterized in terms of ideals of $\mathcal{O}(G)$.

**Theorem 7.1.** Let $G$ be a graph. A set of vertices $X \subseteq V(G)$ is a maximal barrier if and only if there exists a spanning legitimate normalized upper ideal $\mathcal{I}$ of the TFR poset $\mathcal{O}(G)$ such that $X = \bigcup\{\pi(C) : C \in \mathcal{I}\}$.

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