

On Weighted Fock Spaces and Distributions

Nobuhiro ASAI (浅井暢宏)*
Department of Mathematics,
Aichi University of Education,
Kariya, 448-8542, Japan.

Abstract

In this note, we shall consider a Gel'fand triple associated with weighted Fock spaces and revisit the characterization theorems for the S -transform and the operator symbol in terms of analytic and growth conditions. In addition, some results on higher order Bell numbers as a non-trivial example of weight sequences are summarized.

1 Preliminaries

1.1 Weighted Fock Spaces

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|_0$. Let A be a self-adjoint operator in H with dense domain $\text{Dom}(A) \subset H$ satisfying $\inf \text{Spec}(A) \geq 1$. For each $p \geq 0$, a dense subspace of H , $\mathcal{D}_p := \{\xi \in H; |\xi|_p := |A^p \xi|_0 < \infty\}$, is a Hilbert space. It is easy to see $\mathcal{D}_q \subset \mathcal{D}_p \subset H = \mathcal{D}_0$ for $0 \leq p \leq q$. Then, consider $\mathcal{D} := \text{projlim}_{p \rightarrow \infty} \mathcal{D}_p$ and let \mathcal{D}^* denote the dual space of \mathcal{D} . For each $p \geq 0$, let \mathcal{D}_{-p} be the completion of H with respect to the norm $|\xi|_{-p} := |A^{-p} \xi|_0$. Then we get $H = \mathcal{D}_0 \subset \mathcal{D}_{-p} \subset \mathcal{D}_{-q}$ for $0 \leq p \leq q$, and $\mathcal{D}^* \cong \text{indlim}_{p \rightarrow \infty} \mathcal{D}_{-p}$. As a result, with the identification $H \cong H^*$ by the Riesz representation theorem, we obtain a triple, $\mathcal{D} \subset H \subset \mathcal{D}^*$, where the bilinear form on $\mathcal{D}^* \times \mathcal{D}$ is also denoted by $\langle \cdot, \cdot \rangle$.

Let $\mathcal{F}_1(H)$ be a standard Boson Fock space over H and $\alpha = \{\alpha(n)\}_{n=0}^\infty$ be a weight sequence of positive real numbers satisfying the condition,

$$(A1) \quad \alpha(0) = 1, \quad \inf_{n \geq 0} \alpha(n) > 0.$$

Now we introduce a weighted Boson Fock space as follows. Let $\mathcal{F}_\alpha(\mathcal{D}_p)$ be a weighted Boson Fock space over \mathcal{D}_p given by

$$\mathcal{F}_\alpha(\mathcal{D}_p) := \left\{ \phi := (f_n)_{n=0}^\infty; f_n \in \mathcal{D}_p^{\hat{\otimes} n}, \|\phi\|_{p,\alpha}^2 := \sum_{n=0}^\infty n! \alpha(n) |f_n|_p^2 < \infty \right\}$$

where $\cdot^{\hat{\otimes} n}$ for the n -fold symmetric tensor product of \cdot and $|f_n|_p := |(A^p)^{\otimes n} f_n|_0$. The condition $\alpha(0) = 1$ in (A1) is simply to ensure that the norm on $\mathcal{D}_p^{\hat{\otimes} 0}$ coincides with the absolute value on \mathbb{C} . The condition $\inf_{n \geq 0} \alpha(n) > 0$ in (A1) is required to have $\mathcal{F}_\alpha(H) \subset \mathcal{F}_1(H)$. By identifying $\mathcal{F}_1(H)$ with its dual space, we have a chain of weighted Fock spaces,

$$\cdots \subset \mathcal{F}_\alpha(\mathcal{D}_p) \subset \cdots \subset \mathcal{F}_\alpha(H) \subset \mathcal{F}_1(H) \subset \mathcal{F}_{1/\alpha}(H) \subset \cdots \subset \mathcal{F}_{1/\alpha}(\mathcal{D}_{-p}) \subset \cdots, \quad p \geq 0,$$

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where the norm on $\mathcal{F}_{1/\alpha}(\mathcal{D}_{-p})$ is given by $\|\cdot\|_{-p,1/\alpha} := \|(A^{-p})^{\otimes n} \cdot\|_{0,1/\alpha}$. Consider the space $\mathcal{F}_\alpha(\mathcal{D})$ of test functions defined by

$$\mathcal{F}_\alpha(\mathcal{D}) = \text{proj} \lim_{p \rightarrow \infty} \mathcal{F}_\alpha(\mathcal{D}_p).$$

The dual space $\mathcal{F}_\alpha(\mathcal{D})^*$ of $\mathcal{F}_\alpha(\mathcal{D})$,

$$\mathcal{F}_\alpha(\mathcal{D})^* \cong \text{ind} \lim_{p \rightarrow \infty} \mathcal{F}_{1/\alpha}(\mathcal{D}_{-p}),$$

is called the space of generalized functions. Then we get a triple,

$$\mathcal{F}_\alpha(\mathcal{D}) \subset \mathcal{F}_1(H) \subset \mathcal{F}_\alpha(\mathcal{D})^*.$$

We adopt the notation $\langle\langle \cdot, \cdot \rangle\rangle$ to denote the bilinear form on $\mathcal{F}_\alpha(\mathcal{D})^* \times \mathcal{F}_\alpha(\mathcal{D})$,

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \quad \Phi = (F_n) \in \mathcal{F}_\alpha(\mathcal{D})^*, \quad \phi = (f_n) \in \mathcal{F}_\alpha(\mathcal{D}).$$

Due to the Cauchy-Schwartz inequality, we have $|\langle\langle \Phi, \phi \rangle\rangle| \leq \|\Phi\|_{-p,1/\alpha} \|\phi\|_{p,\alpha}$.

1.2 Growth Bound of the S -transform

Moreover, let us assume that

$$(A2) \quad \lim_{n \rightarrow \infty} \left(\frac{\alpha(n)}{n!} \right)^{\frac{1}{n}} = 0.$$

This condition implies that

$$G_\alpha(z) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} z^n$$

is an entire function. Then the exponential vector (coherent state) $e(\xi)$ given by

$$e(\xi) := \left(\frac{\xi^{\otimes n}}{n!} \right)_{n=0}^{\infty}, \quad \xi \in \mathcal{D}$$

belongs to $\mathcal{F}_\alpha(\mathcal{D})$ due to $\|e(\xi)\|_{p,\alpha}^2 = G_\alpha(|\xi|_p^2) < \infty$.

Definition 1.1. Assume (A1) and (A2). The S -transform $S\Phi$ of $\Phi = (F_n)_{n=0}^{\infty} \in \mathcal{F}_\alpha(\mathcal{D})^*$ is defined to be the function on \mathcal{D} by

$$(S\Phi)(\xi) := \langle\langle \Phi, e(\xi) \rangle\rangle = \sum_{n=0}^{\infty} \langle F_n, \xi^{\otimes n} \rangle, \quad \xi \in \mathcal{D}.$$

The S -transform can be viewed as the generalization to distributions of the Segal-Bargmann transform.

Lemma 1.2. Assume that conditions (A1)(A2) hold. The S -transform $F = S\Phi$ of a generalized function $\Phi \in \mathcal{F}_\alpha(\mathcal{D})^*$ satisfies the growth condition

$$|(S\Phi)(\xi)| \leq \|\Phi\|_{-p,1/\alpha} G_\alpha(|\xi|_p^2)^{1/2}, \quad \xi \in \mathcal{D}$$

for some $p \geq 0$.

Note that the condition (A1) guarantees that $G_{1/\alpha}$ given by

$$G_{1/\alpha}(z) = \sum_{n=0}^{\infty} \frac{1}{n! \alpha(n)} z^n$$

is an entire function.

Lemma 1.3. *Assume that condition (A1) holds. Then the S-transform $F = S\varphi$ of a test function $\varphi \in \mathcal{F}_\alpha(\mathcal{D})$ satisfies the growth condition*

$$|(S\varphi)(\xi)| \leq \|\varphi\|_{p,1/\alpha} G_{1/\alpha}(|\xi|_{-p}^2)^{1/2}, \quad \xi \in \mathcal{D},$$

for any $p \geq 0$.

Up to here, the nuclearity of $\mathcal{F}_\alpha(\mathcal{D})$ is not assumed.

2 Analytic Characterizations

The characterization of generalized functions in terms of analytic and growth conditions is called the analytic characterization, which was first discussed by Potthoff-Streit [24] for Hida distributions (Kuo *et al.* [20] for test functions). From the point of infinite dimensional analytic functions, equivalent results were obtained by Lee [21].

It is well-known that the nuclearity of $\mathcal{F}_\alpha(\mathcal{D})$ is a sufficient condition for the analytic characterization. It is recently proved [1] that the nuclearity of $\mathcal{F}_\alpha(\mathcal{D})$ is a necessary condition for it. In proof, the infinite dimensional Bargmann-Segal space [14][16], the space of square integrable analytic functions on infinite dimensional complex Gaussian space, plays important roles.

From now on, we suppose that the self-adjoint operator A satisfies the condition,

(H1) $\inf \text{Spec}(A) > 1$ and A^{-r} is of Hilbert-Schmidt type for some $r > 0$.

Then \mathcal{D} becomes a nuclear space and so is $\mathcal{F}_\alpha(\mathcal{D})$. In such a case, we denote \mathcal{D} and $\mathcal{F}_\alpha(\mathcal{D})$ by \mathcal{E} and $\mathcal{F}_\alpha(\mathcal{E})$, respectively, and so a Gel'fand triple

$$\mathcal{F}_\alpha(\mathcal{E}) \subset \mathcal{F}_1(H) \subset \mathcal{F}_\alpha(\mathcal{E})^*$$

is referred to as a CKS-space where a condition $\inf_{n \geq 0} \alpha(n) > 0$ in (A1) is assumed in [11]. However, a weaker condition,

(A1)* $\alpha(0) = 1$, $\inf_{n \geq 0} \alpha(n) \sigma^n > 0$ for some $\sigma \geq 1$,

is strong enough to assure that the nuclear space $\mathcal{F}_\alpha(\mathcal{E})$ is a subspace of $\mathcal{F}_1(H)$. This weaker condition was first introduced in [4]. Therefore, the condition (A1) on Theorem 2.1 and Theorem 2.2 can be replaced by (A1)*.

Theorem 2.1 ([11]). *Assume that conditions (A1)*(A2) hold. The S-transform $F = S\Phi$ of a generalized function $\Phi \in \mathcal{F}_\alpha(\mathcal{E})^*$ satisfies the conditions:*

- (a) For any $\xi, \eta \in \mathcal{D}$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.
- (b) There exist constants $K > 0, a > 0, p \geq 0$ such that

$$|F(\xi)| \leq KG_\alpha(a|\xi|_p^2)^{\frac{1}{2}}, \quad \xi \in \mathcal{E}.$$

Conversely, assume that

$$(B1) \limsup_{n \rightarrow \infty} \left(\frac{n!}{\alpha(n)} \inf_{r>0} \frac{G_\alpha(r)}{r^n} \right)^{\frac{1}{n}} < \infty$$

holds and let a \mathbb{C} -valued function F on \mathcal{E} satisfies the above two conditions (a)(b). Then, there exists a unique $\Phi \in \mathcal{F}_\alpha(\mathcal{E})^*$ such that $F = S\Phi$. Moreover, for any $q > p$ with $ae^2 \|A^{-(q-p)}\|_{HS}^2 < 1$, we have the norm estimate

$$\|\Phi\|_{-q, 1/\alpha} \leq K(1 - ae^2 \|A^{-(q-p)}\|_{HS}^2)^{-\frac{1}{2}}.$$

For the space $\varphi \in \mathcal{F}_\alpha(\mathcal{E})$ of test functions, which was not studied in [11], we have

Theorem 2.2 ([3]). *Assume that condition (A1)* holds. Then the S -transform $F = S\varphi$ of a test function $\varphi \in \mathcal{F}_\alpha(\mathcal{E})$ satisfies the conditions:*

- (a) For any $\xi, \eta \in \mathcal{D}$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.
- (b) For any $p \geq 0, a > 0$, there exists a constant $K > 0$ such that

$$|F(\xi)| \leq KG_{1/\alpha}(a|\xi|_{-p}^2)^{\frac{1}{2}}, \quad \xi \in \mathcal{E}.$$

Conversely, assume that

$$(\tilde{B}1) \limsup_{n \rightarrow \infty} \left(n! \alpha(n) \inf_{r>0} \frac{G_{1/\alpha}(r)}{r^n} \right)^{\frac{1}{n}} < \infty$$

holds and let a \mathbb{C} -valued function F on \mathcal{E} satisfies the above two conditions (a)(b). Then there exists a unique $\varphi \in \mathcal{F}_\alpha(\mathcal{E})$ such that $F = S\varphi$. Moreover, for any given $a, p > 0$, choose $q \in [0, p)$ such that $ae^2 \|A^{-(p-q)}\|_{HS}^2 < 1$, then we have the norm estimate

$$\|\varphi\|_{q, \alpha} \leq K(1 - ae^2 \|A^{-(p-q)}\|_{HS}^2)^{-\frac{1}{2}}.$$

Remark 2.3. (1) It will be seen that (A3) and (A4) given in Section 4 are necessary and sufficient conditions for (B1) and ($\tilde{B}1$), respectively.

(2) It was our starting point [5][6] to clarify minimal conditions on $\{\alpha(n)\}_{n=0}^\infty$ to carry out theories of generalized functions and operators associated with a CKS space,

$$\mathcal{F}_\alpha(\mathcal{E}) \subset \mathcal{F}_1(H) \subset \mathcal{F}_\alpha(\mathcal{E})^*$$

such that Theorems 2.1 and 2.2 hold.

3 Examples and Log-concavity Criterion

Example 3.1. It is easy to see that the classical examples,

- (1) $\alpha(n) = 1$ for the Hida-Kubo-Takenaka space [19],

$$\mathcal{F}_1(\mathcal{E}) \subset \mathcal{F}_1(H) \subset \mathcal{F}_1(\mathcal{E})^*,$$

and $\beta(n) = (n!)^\beta$ ($0 \leq \beta < 1$) for the Kondratiev-Streit space [17],

$$\mathcal{F}_\beta(\mathcal{E}) \subset \mathcal{F}_1(H) \subset \mathcal{F}_\beta(\mathcal{E})^*,$$

satisfy (A1)(A2)(B1)($\tilde{B}1$), which can be checked by direct computations.

- (3) Let $\exp_k(x)$ denotes the k -times iterated exponential function for an integer $k \geq 2$, that is,

$$\exp_k(x) = \exp(\exp \cdots (\exp(x))).$$

The k -th order Bell numbers $b_k(n)$ are defined by

$$\frac{\exp_k(x)}{\exp_k(0)} = \sum_{n=0}^{\infty} \frac{b_k(n)}{n!} x^n, \quad k \geq 2,$$

where the numbers $b_2(n), n \geq 0$ are known as the (standard) Bell numbers. Then

$$\mathcal{F}_{b_k}(\mathcal{E}) \subset \mathcal{F}_\beta(\mathcal{E}) \subset \mathcal{F}_1(\mathcal{E}) \subset \mathcal{F}_1(H) \subset \mathcal{F}_1(\mathcal{E})^* \subset \mathcal{F}_\beta(\mathcal{E})^* \subset \mathcal{F}_{b_k}(\mathcal{E})^*.$$

It is not difficult to check (A1)(A2) for $\{b_k(n)\}_{n=0}^{\infty}$. Cochran *et.al* [11] proved by direct computations that the condition (B1) is satisfied, but they did not study $(\tilde{B}1)$. It seems impossible to check by direct computations whether or not $(\tilde{B}1)$ holds for the case of the k -th order bell numbers. Hence it is natural to seek an easy criterion for $(\tilde{B}1)$.

Definition 3.2. A sequence $\{\delta(n)\}_{n=0}^{\infty}$ is log-concave if $\delta(n)\delta(n+2) \leq \delta(n+1)^2$ and $\{\delta(n)\}_{n=0}^{\infty}$ is log-convex if $\delta(n+1)^2 \leq \delta(n)\delta(n+2)$.

In fact, the following criterion was mentioned in [11].

Proposition 3.3. *If $\{\alpha(n)/n!\}_{n=0}^{\infty}$ is log-concave, then (B1) holds.*

Due to this proposition, it is easy to see the following.

Corollary 3.4. *If $\{1/n!\alpha(n)\}_{n=0}^{\infty}$ is log-concave, then $(\tilde{B}1)$ holds.*

However, it was not proved in [11] if the sequences $\{b_k(n)/n!\}_{n=0}^{\infty}$ and $\{1/n!b_k(n)\}_{n=0}^{\infty}$ are log-concave or not. We filled up these gaps [2].

Theorem 3.5. (1) $\{b_k(n)/n!\}_{n=0}^{\infty}$ is log-concave.

(2) $\{b_k(n)\}_{n=0}^{\infty}$ is log-convex and hence $\{1/n!b_k(n)\}_{n=0}^{\infty}$ is log-concave.

(3) $\{b_k(n)\}_{n=0}^{\infty}$ satisfies (A1)(A2)(B1)($\tilde{B}1$).

Remark 3.6. One can find a different way of proof by Engel [12] concerning the log-convexity of $\{b_2(n)\}_{n=0}^{\infty}$. Canfield [9] showed that the log-concavity of $\{b_2(n)/n!\}_{n=0}^{\infty}$ holds asymptotically.

In [18], the following “log-additivity” conditions were introduced in order to prove the continuity of various operators acting on $\mathcal{F}_\alpha(\mathcal{E})$ and $\mathcal{F}_\alpha(\mathcal{E})^*$:

(C1) There exists a constant c_1 such that for any $n \leq m$,

$$\alpha(n) \leq c_1^m \alpha(m).$$

(C2) There exists a constant c_2 such that for any n, m ,

$$\alpha(n+m) \leq c_2^{n+m} \alpha(n)\alpha(m).$$

(C3) There exists a constant c_3 such that for any n, m ,

$$\alpha(n)\alpha(m) \leq c_3^{n+m} \alpha(n+m).$$

Theorem 3.7. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\alpha(0) = 1$.

(1) If $\{\alpha(n)\}_{n=0}^{\infty}$ is log-convex, then

$$\alpha(n)\alpha(m) \leq \alpha(n+m), \quad n, m \geq 0.$$

(2) If $\{\alpha(n)/n!\}_{n=0}^{\infty}$ is log-concave, then

$$\alpha(n+m) \leq 2^{n+m}\alpha(n)\alpha(m), \quad n, m \geq 0.$$

Due to Theorem 3.5 and Theorem 3.7, one has the following inequalities.

Corollary 3.8. $\{b_k(n)\}_{n=0}^{\infty}$ satisfies (C1)(C2)(C3) with $c_1 = 1, c_2 = 2, c_3 = 1$, that is,

$$b_k(n)b_k(m) \leq b_k(n+m) \leq 2^{n+m}b_k(n)b_k(m), \quad n, m \geq 0.$$

Remark 3.9. In [18], it was proved that (C3) implies (C1) and the k -th order Bell numbers $\{b_k(n)\}_{n=0}^{\infty}$ satisfies (C1)(C2)(C3) in an asymptotical consideration. Moreover, we proved in [2] that $c_1 = 1, c_3 = 1$ for any $k \geq 2$ and $c_2 = 2$ for $k = 2$ are best constants. It is not known if $c_2 = 2$ for $k \geq 3$ is the best constant.

4 Growth Functions

In this section, we shall recall key notions and results from [5][6]. Let $C_{+, \log}$ denote the collection of all positive continuous functions u on $[0, \infty)$ satisfying

$$\lim_{r \rightarrow \infty} \frac{\log u(r)}{\log r} = \infty.$$

The Legendre transform ℓ_u of $u \in C_{+, \log}$ defined as the function,

$$\ell_u(t) := \inf_{r > 0} \frac{u(r)}{r^t}, \quad t \in [0, \infty).$$

Let $C_{+, 1/2}$ denotes the collection of all positive continuous functions u on $[0, \infty)$ satisfying

$$\lim_{r \rightarrow \infty} \frac{\log u(r)}{\sqrt{r}} = \infty.$$

The dual Legendre transform u^* of $u \in C_{+, 1/2}$ is defined to be the function

$$u^*(r) = \sup_{s \geq 0} \frac{e^{2\sqrt{rs}}}{u(s)}, \quad r \in [0, \infty).$$

It can be proved that $u^* \in C_{+, 1/2}$.

Remark 4.1. One can see that $\exp[\sqrt{r}] \in C_{+, \log}$, but $\notin C_{+, 1/2}$. In addition, $\exp[2\sqrt{r \log \sqrt{r}}] \in C_{+, 1/2}$.

Definition 4.2. We say that two sequences $\{a(n)\}$ and $\{b(n)\}$ are equivalent denoted by $\{a(n)\} \sim \{b(n)\}$ if there exist constants $K_1, K_2, c_1, c_2 > 0$ such that for all n ,

$$K_1 c_1^n a(n) \leq b(n) \leq K_2 c_2^n a(n).$$

Now we state the weaker conditions for the sequence $\{\alpha(n)\}$:

(A3) $\{\alpha(n)\}$ is equivalent to a positive sequence $\{\lambda(n)\}$ such that $\{\lambda(n)/n!\}$ is log-concave.

(A4) $\{\alpha(n)\}$ is equivalent to a positive sequence $\{\lambda(n)\}$ such that $\{1/n!\lambda(n)\}$ is log-concave.

Then it is easy to see the following Lemma.

Lemma 4.3. (1) (B1) is equivalent to (A3).

(2) ($\tilde{B}1$) is equivalent to (A4).

For our discussion, the following conditions on u play important roles:

(U1) $\inf_{r \geq 0} u(r) = 1$.

(U2) $\lim_{r \rightarrow \infty} \frac{\log u(r)}{r} < \infty$.

(U3) $u(r^2)$ is a log-convex function on $[0, \infty)$.

For a given $u \in C_{+, \log}$, define a sequence $\{\alpha_u(n)\}_{n=0}^{\infty}$ given by

$$\alpha_u(n) := \frac{1}{\ell_u(n)n!},$$

which plays a role of a sequence $\{\alpha(n)\}_{n=0}^{\infty}$.

Lemma 4.4. (1) If $u \in C_{+, \log}$ satisfies (U1)(U2), then $\{\alpha_u(n)\}_{n=0}^{\infty}$ satisfies (A1)*.

(2) If $u \in C_{+, 1/2}$ satisfies (U3), then $\{\alpha_u(n)\}_{n=0}^{\infty}$ satisfies (A2).

(3) If $u \in C_{+, \log}$ satisfies (U3), $\{\alpha_u(n)\}_{n=0}^{\infty}$ satisfies (A3).

(4) If $u \in C_{+, \log}$, then $\{\alpha_u(n)\}_{n=0}^{\infty}$ satisfies (A4).

Theorem 4.5. Suppose that $u \in C_{+, 1/2}$ satisfies (U1)(U2)(U3). Then,

(1) a sequence $\{\alpha_u(n)\}_{n=0}^{\infty}$ satisfies conditions (A1)*(A2)(A3)(A4).

(2) (A3) (\iff (B1)) implies (C2).

(3) (A4) (\iff ($\tilde{B}1$)) implies (C3).

(4) (C3) implies (C1).

Definition 4.6. Two positive functions f and g on $[0, \infty)$ are called equivalent, denoted by $f \sim g$, if there exists constants $c_1, c_2, a_1, a_2 > 0$ such that

$$c_1 f(a_1 r) \leq g(r) \leq c_2 f(a_2 r), \quad r \in [0, \infty).$$

Example 4.7. (1) For $0 \leq \beta < 1$, one can see that

$$u_\beta(r) = \exp[(1 + \beta)r^{\frac{1}{1+\beta}}] \in C_{+, 1/2} \iff u_\beta^*(r) = \exp[(1 - \beta)r^{\frac{1}{1-\beta}}] \in C_{+, 1/2}.$$

In fact, the series G_α and $G_{1/\alpha}$ with $\alpha(n) = (n!)^\beta$ cannot have the closed forms unless $\beta = 0$, but we have the following estimates:

$$\begin{cases} \exp[(1 - \beta)r^{\frac{1}{1-\beta}}] \leq G_\alpha(r) \leq 2^\beta \exp[(1 - \beta)2^{\frac{\beta}{1-\beta}} r^{\frac{1}{1-\beta}}], \\ 2^{-\beta} \exp[(1 + \beta)2^{-\frac{\beta}{1+\beta}} r^{\frac{1}{1+\beta}}] \leq G_{1/\alpha}(r) \leq \exp[(1 + \beta)r^{\frac{1}{1+\beta}}]. \end{cases} \quad (4.1)$$

That is, $u_\beta(r) \sim \sum_{n=0}^{\infty} \frac{1}{(n!)^{1+\beta}} r^n$ and $u_\beta^*(r) \sim \sum_{n=0}^{\infty} \frac{1}{(n!)^{1-\beta}} r^n$.

(2) Let $\log_j(\cdot)$ denote the j -th iterated logarithmic function inductively defined by

$$\log_1(r) := \log(\max\{r, e\}), \quad \log_j(r) := \log_1(\log_{j-1}(r)), \quad j \geq 2.$$

Then we have

$$u_k^*(r) := \exp_k(r)/\exp_k(0) \in C_{+,1/2} \iff u_k(r) \sim w_k(r) = \exp[2\sqrt{r \log_{k-1} \sqrt{r}}] \in C_{+,1/2}$$

and $w_k(r) \sim \sum_{n=0}^{\infty} \frac{1}{n!b_k(n)} r^n$.

If one merges everything together with replacements of growth conditions in Theorem 2.1 and Theorem 2.2 respectively by

- $|F(\xi)| \leq K u^*(a|\xi|_p)^{\frac{1}{2}}$ for $\mathcal{F}_\alpha(\mathcal{E})^*$,
- $|F(\xi)| \leq K u(a|\xi|_{-p})^{\frac{1}{2}}$ for $\mathcal{F}_\alpha(\mathcal{E})$,

where $\alpha = \{\alpha_u(n)\}_{n=0}^{\infty}$, then we obtain

Theorem 4.8. *Suppose that $u \in C_{+1/2}$ satisfies (U1)(U2)(U3). The S -transform $F = S\Phi$ of a generalized function $\Phi \in \mathcal{F}_\alpha(\mathcal{E})^*$ satisfies the conditions:*

- (a) *For any $\xi, \eta \in \mathcal{E}$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*
- (b) *There exist constants $K > 0, a > 0, p \geq 0$ such that*

$$|F(\xi)| \leq K u^*(a|\xi|_p^2)^{\frac{1}{2}}, \quad \xi \in \mathcal{E}.$$

Conversely, let a \mathbb{C} -valued function F on \mathcal{E} satisfies the above two conditions (a)(b). Then there exists a unique $\Phi \in \mathcal{F}_\alpha(\mathcal{E})^$ such that $F = S\Phi$. Moreover, for any $q > p$ with $ae^2 \|A^{-(q-p)}\|_{HS}^2 < 1$, we have the norm estimate*

$$\|\Phi\|_{-q,1/\alpha} \leq K(1 - ae^2 \|A^{-(q-p)}\|_{HS}^2)^{-\frac{1}{2}}.$$

Theorem 4.9. *Suppose that $u \in C_{+1/2}$ satisfies (U1)(U2)(U3). The S -transform $F = S\varphi$ of a test function $\varphi \in \mathcal{F}_\alpha(\mathcal{E})$ satisfies the conditions:*

- (a) *For any $\xi, \eta \in \mathcal{E}$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*
- (b) *For any $p \geq 0, a > 0$, there exists a constant $K > 0$ such that*

$$|F(\xi)| \leq K u(a|\xi|_{-p}^2)^{\frac{1}{2}}, \quad \xi \in \mathcal{E}.$$

Conversely, let a \mathbb{C} -valued function F on \mathcal{E} satisfies the above two conditions (a)(b). Then there exists a unique $\varphi \in \mathcal{F}_\alpha(\mathcal{E})$ such that $F = S\varphi$. Moreover, for any given $a, p > 0$, choose $q \in [0, p)$ such that $ae^2 \|A^{-(p-q)}\|_{HS}^2 < 1$, then we have the norm estimate

$$\|\varphi\|_{q,\alpha} \leq K(1 - ae^2 \|A^{-(p-q)}\|_{HS}^2)^{-\frac{1}{2}}.$$

Remark 4.10. Consult our papers [6][7] to see connections Gannoun *et al.* [13].

5 Generalization of Obata's Theorem

Obata [22][23] characterized the operator symbol of $\Xi \in \mathcal{L}(\mathcal{F}_1(\mathcal{E}), \mathcal{F}_1(\mathcal{E})^*)$ and Chung *et al.* [10] presented a simplified proof.

Definition 5.1. For any $\Xi \in \mathcal{L}(\mathcal{F}_\alpha(\mathcal{E}), \mathcal{F}_\alpha(\mathcal{E})^*)$, the operator symbol $\widehat{\Xi}$ of Ξ is defined by

$$\widehat{\Xi}(\xi, \eta) = \langle \langle \Xi e(\xi), e(\eta) \rangle \rangle, \quad \xi, \eta \in \mathcal{E}.$$

The operator symbol is an operator version of the S -transform. Therefore, one can generalize the characterization theorem for the operator symbol as follows.

Theorem 5.2. *Suppose that $u \in C_{+1/2}$ satisfies (U1)(U2)(U3). The symbol $G = \widehat{\Xi}$ of $\Xi \in \mathcal{L}(\mathcal{F}_\alpha(\mathcal{E}), \mathcal{F}_\alpha(\mathcal{E})^*)$ satisfies the conditions:*

- (a) *For any $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathcal{E}$, the function $G(z\xi_1 + \eta_1, w\xi_2 + \eta_2)$ is an entire function of $(z, w) \in \mathbb{C} \times \mathbb{C}$.*
- (b) *There exist constants $K > 0, a > 0, p \geq 0$ such that*

$$|G(\xi, \eta)| \leq Ku^*(a(|\xi|_p^2 + |\eta|_p^2))^{\frac{1}{2}}, \quad \xi, \eta \in \mathcal{E}.$$

Conversely, suppose a \mathbb{C} -valued function G on $\mathcal{E} \times \mathcal{E}$ satisfies the above two conditions (a)(b). Then there exists a unique $\Xi \in \mathcal{L}(\mathcal{F}_\alpha(\mathcal{E}), \mathcal{F}_\alpha(\mathcal{E})^)$ such that $G = \widehat{\Xi}$.*

Proof. Due to Theorem 4.5, there exist constants $c_1, c_2 > 0$ such that

$$u^*(s)u^*(t) \leq u^*(c_1(s+t)) \leq u^*(c_2s)u^*(c_2t), \quad s, t \geq 0. \quad (5.1)$$

Thanks to this inequality (5.1), one can apply the idea of Chung *et al.* [10]. It means that the proof can be done by applying Theorem 4.8 two times. □

Remark 5.3. It is not difficult to generalize further and state Theorem 4.8, Theorem 4.9 and Theorem 5.2 in a unified manner as of Ji-Obata [15]. It is because essential properties what we need for norm estimates related with a CKS space can be derived from conditions (A1)*(A2)(A3)(A4) on $\{\alpha(n)\}$ and (U1)(U2)(U3) on $u \in C_{+,1/2}$.

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References

- [1] L. Accardi, U.C. Ji, and K. Saitô, *Analytic characterization of infinite dimensional distributions*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **20** (2017), 1750007 (13 pages).
- [2] N. Asai, I. Kubo, and H.-H. Kuo, *Bell numbers, log-concavity, and log-convexity*, *Acta. Appl. Math.*, **63** (2000), 79–87.
- [3] N. Asai, I. Kubo, and H.-H. Kuo, *Characterization of test functions in CKS-space*, in: “Mathematical Physics and Stochastic Analysis: in honor of L. Streit”, S. Albeverio et al. (eds.) World Scientific, (2000), pp. 68–78.

- [4] N. Asai, I. Kubo, and H.-H. Kuo, *CKS-space in terms of growth functions*, in: “Quantum Information II”, T. Hida and K. Saitô. (eds.) World Scientific, (2000), pp. 17–27.
- [5] N. Asai, I. Kubo, and H.-H. Kuo, *Roles of log-concavity, log-convexity, and growth order in white noise analysis*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **4** (2001), 59–84.
- [6] N. Asai, I. Kubo, and H.-H. Kuo, *General characterization theorems and intrinsic topologies in white noise analysis*, *Hiroshima Math. J.*, **31** (2001), 299–330.
- [7] N. Asai, I. Kubo, and H.-H. Kuo, *Gaussian and Poisson white noises with related characterization theorems*, *Contemp. Math.*, **317** (2003), 45–56.
- [8] E.A. Bender and E.R. Canfield, *Log-concavity and related properties of the cycle index polynomials*, *J. Combin. Theory A*, **76** (1996), 57–70.
- [9] E.R. Canfield, *Engel’s inequality for Bell numbers*, *J. Combin. Theory A*, **72** (1995), 184–187.
- [10] D.M. Chung, T.S. Chung, and U.C. Ji, *A simple proof of analytic characterization theorem for operator symbols*, *Bull. Korean Math. Soc.*, **34** (1997), 421–436.
- [11] W. G. Cochran, H.-H. Kuo, and A. Sengupta, *A new class of white noise generalized functions*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **1** (1998), 43–67.
- [12] K. Engel, *On the average rank of an element in a filter of the partition lattice*, *J. Combin. Theory A*, **65** (1994), 67–78.
- [13] R. Gannoun, R. Hachaichi, H. Ouerdiane, and A. Rezgui, *Un théorème de dualité entre espaces de fonctions holomorphes à croissance exponentielle*, *J. Funct. Anal.*, **171** (2000), 1–14.
- [14] M. Grothaus, Y.G. Kondratiev, and L. Streit, *Complex Gaussian analysis and the Bargmann-Segal space*, *Methods Funct. Anal. Topol.*, **3** (1997), 46–64.
- [15] U.C. Ji, and N. Obata, *A unified characterization theorem in white noise theory*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **6** (2003), 167–178.
- [16] U.C. Ji, and N. Obata, *A role of Bargmann-Segal spaces in characterization and expansion of operators on Fock space*, *J. Math. Soc. Japan*, **56** (2004), 311–338.
- [17] Yu. G. Kondratiev and L. Streit, *Spaces of white noise distributions: Constructions, Descriptions, Applications. I*, *Reports on Math. Phys.*, **33** (1993), 341–366.
- [18] I. Kubo, H.-H. Kuo, and A. Sengupta, *White noise analysis on a new space of Hida distributions*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **2** (1999), 315–335.
- [19] I. Kubo and S. Takenaka, *Calculus on Gaussian white noise I, II, III, IV*, *Proc. Japan Acad.*, **56A** (1980), 376–380, **56A** (1980), 411–416, **57A** (1981), 433–437, **58A** (1982), 186–189.
- [20] H.-H. Kuo, J. Potthoff, and L. Streit, *A characterization of white noise test functionals*, *Nagoya Math. J.*, **121** (1991), 185–194.
- [21] Y.-J. Lee, *Analytic version of test functionals, Fourier transform, and a characterization of measures in white noise calculus*, *J. Funct. Anal.*, **100** (1991), 359–380.

- [22] N. Obata, *An analytic characterization of symbols of operators on white noise functionals*, *J. Math. Soc. Japan*, **45** (1993), 422–445.
- [23] N. Obata, “*White Noise Calculus and Fock Space*”, Lecture Notes in Math. **1577**, Springer-Verlag, 1994.
- [24] J. Potthoff and L. Streit, *A characterization of Hida distributions*, *J. Funct. Anal.*, **101** (1991), 212–229.