Spectral Analysis of Infinite-dimensional Dirac Operators on an Abstract Boson-Fermion Fock Space

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Abstract
A review on spectral analysis of infinite dimensional Dirac type operators on an abstract boson-fermion Fock space is presented.

1 Introduction

For each pair \((\mathcal{H}, \mathcal{K})\) of complex Hilbert spaces, the tensor product Hilbert space

\[ \mathcal{F}(\mathcal{H}, \mathcal{K}) := \mathcal{F}_b(\mathcal{H}) \otimes \mathcal{F}_f(\mathcal{K}) \]

of the boson Fock space

\[ \mathcal{F}_b(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \bigotimes_{s}^{n} \mathcal{H} = \{ \psi = \{ \psi^{(n)} \}_{n=0}^{\infty} | \psi^{(n)} \in \bigotimes_{s}^{n} \mathcal{H}, \sum_{n=0}^{\infty} \| \psi^{(n)} \|^2 < \infty \} \]

over \(\mathcal{H}\) and the fermion Fock space

\[ \mathcal{F}_f(\mathcal{K}) := \bigoplus_{p=0}^{\infty} \bigwedge^{p} \mathcal{K} = \{ \phi = \{ \phi^{(p)} \}_{p=0}^{\infty} | \phi^{(p)} \in \bigwedge^{p} \mathcal{K}, \sum_{p=0}^{\infty} \| \phi^{(p)} \|^2 < \infty \} \]

over \(\mathcal{K}\) is defined, where \(\bigotimes_{s}^{n} \mathcal{H}\) denotes the \(n\)-fold symmetric tensor product of \(\mathcal{H}\) with \(\bigotimes_{0}^{0} \mathcal{H} := \mathbb{C}\), \(\bigwedge^{p} \mathcal{K}\) denotes the \(p\)-fold anti-symmetric tensor product of \(\mathcal{K}\) with \(\bigwedge^{0} \mathcal{K} := \mathbb{C}\) and, for a vector \(\Psi\) in a Hilbert space, \(\| \Psi \|\) denotes the norm of \(\Psi\). We call the Hilbert space \(\mathcal{F}(\mathcal{H}, \mathcal{K})\) the abstract boson-fermion Fock space over \((\mathcal{H}, \mathcal{K})\). In a previous paper [2], the author introduced a general class of infinite-dimensional
Dirac operators on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ and clarified general mathematical structures behind some supersymmetric quantum field models giving an abstract unification of them. In particular, a path (functional) integral representation of analytical index of an infinite dimensional Dirac operator was derived, which gives a kind of index theorem. But spectral analysis of the infinite dimensional Dirac operators is still missing. Only partial results are available [10]. In the present paper, we review some aspects of spectral analysis of infinite dimensional Dirac operators.

2 Preliminaries

We first recall basic objects and facts associated with Fock spaces. See [11] for more details.

In general, for a linear operator $A$ from a Hilbert space to a Hilbert space, we denote its domain by $D(A)$.

For each vector $f \in \mathcal{H}$, there is a unique densely defined closed linear operator $a(f)$ on $\mathcal{F}_b(\mathcal{H})$ such that its adjoint $a(f)^*$ takes the following form:

\[ D(a(f)^*) = \left\{ \psi \in \mathcal{F}_b(\mathcal{H}) \mid \sum_{n=1}^{\infty} \|\sqrt{n}S_n(f \otimes \psi^{(n-1)})\|^2 < \infty \right\}, \]
\[ (a(f)^*\psi)^{(0)} = 0, \quad (a(f)^*\psi)^{(n)} = \sqrt{n}S_n(f \otimes \psi^{(n-1)}), \quad n \geq 1, \quad \psi \in D(a(f)^*), \]

where $S_n$ denotes the symmetrization operator (symmetrizer) on the $n$-fold tensor product $\otimes^n \mathcal{H}$ of $\mathcal{H}$. The operator $a(f)$ (resp. $a(f)^*$) is called the boson annihilation (resp. creation) operator with test vector $f$.

There is a distinguished vector

\[ \Omega_b := \{1, 0, 0, \cdots\} \in \mathcal{F}_b(\mathcal{H}), \]

called the boson Fock vacuum in $\mathcal{F}_b(\mathcal{H})$, which is vanished by the annihilation operator:

\[ a(f)\Omega_b = 0, \quad \forall f \in \mathcal{H}. \]

The set $\{a(f), a(f)^* | f \in \mathcal{H}\}$ of boson annihilation operators and boson creation operators obeys the canonical commutation relations (CCR) over $\mathcal{H}$:

\[ [a(f), a(g)^*] = \langle f, g \rangle_{\mathcal{H}}, \quad [a(f), a(g)] = 0, \quad f, g \in \mathcal{H} \]

on the bosonic finite particle subspace

\[ \mathcal{F}_{b,0}(\mathcal{H}) := \{\psi \in \mathcal{F}_b(\mathcal{H}) | \exists n_0 \in \mathbb{N} \text{ s.t. } \psi^{(n)} = 0, \forall n \geq n_0\}, \]

where $[X, Y] := XY - YX$ and $\langle \ , \ \rangle_{\mathcal{H}}$ denotes the inner product of $\mathcal{H}$ (linear in the second variable).
In general, for a subset $\mathcal{E}$ of a vector space, $\text{span}(\mathcal{E})$ or $\text{span} \mathcal{E}$ denotes the subspace generated by all the vectors of $\mathcal{E}$.

It is well known that, for each dense subspace $\mathcal{D}$ of $\mathcal{H}$, the subspace

$$\mathcal{F}_{b,\text{fin}}(\mathcal{D}) := \text{span}\{\Omega_b, a(f_1)^* \cdots a(f_n)^* \Omega_b | n \in \mathbb{N}, f_j \in \mathcal{D}, j = 1, \ldots, n\}$$

is dense in $\mathcal{F}_b(\mathcal{H})$. In fact, one has

$$\mathcal{F}_{b,\text{fin}}(\mathcal{D}) = \otimes^n \mathcal{D},$$

the algebraic $n$-fold symmetric tensor product of $\mathcal{D}$.

We next move on to the fermion Fock space $\mathcal{F}_f(\mathcal{K})$. For each $u \in \mathcal{K}$, there is a unique bounded linear operator $b(u)$ on $\mathcal{F}_f(\mathcal{K})$ such that $b(u)^*$ is given as follows:

$$(b(u)^* \phi)^{(0)} = 0, \quad (b(u)^* \phi)^{(p)} = \sqrt{p} A_p (f \otimes \phi^{(p-1)}), \quad p \geq 1, \quad \phi \in \mathcal{F}_f(\mathcal{K}),$$

where $A_p$ is the anti-symmetrization operator (anti-symmetrizer) on $\otimes^p \mathcal{K}$. The operator $b(u)$ (resp. $b(u)^*$) is called the fermion annihilation (resp. creation) operator with test vector $u$.

The vector

$$\Omega_f := \{1, 0, 0, \ldots \} \in \mathcal{F}_f(\mathcal{K})$$

is called the fermion Fock vacuum in $\mathcal{F}_f(\mathcal{K})$, which is vanished by $b(u)$:

$$b(u) \Omega_f = 0, \quad \forall u \in \mathcal{K}.$$

The set $\{b(u), b(u)^* | u \in \mathcal{K}\}$ obeys the canonical anti-commutation relations (CAR) over $\mathcal{K}$:

$$\{b(u), b(v)^*\} = \langle u, v \rangle_{\mathcal{K}}, \quad \{b(u), b(v)\} = 0, \quad u, v \in \mathcal{K},$$

where $\{X, Y\} := XY + YX$. It follows that

$$\|b(u)\| = \|u\|, \quad \|b(u)^*\| = \|u\|, \quad b(u)^2 = 0, \quad (b(u)^*)^2 = 0, \quad \forall u \in \mathcal{K},$$

where, for a bounded linear operator $T$ on a Hilbert space, $\|T\|$ denotes the operator norm of $T$.

For each dense subspace $\mathcal{D}$ of $\mathcal{K}$, the subspace

$$\mathcal{F}_{f,\text{fin}}(\mathcal{D}) := \text{span}\{\Omega_f, b(u_1)^* \cdots b(u_p)^* \Omega_f | p \in \mathbb{N}, u_k \in \mathcal{D}, k = 1, \ldots, p\},$$

is dense in $\mathcal{F}_f(\mathcal{K})$. 
3 Exterior Differential Operators on the Boson-Fermion Fock Space

For a linear operator $L$ on a Hilbert space, we set

$$C^\infty(L) := \cap_{n=1}^\infty D(L^n),$$

the $C^\infty$-domain of $L$. If $L$ is self-adjoint, then $C^\infty(L)$ is dense.

Let $A$ be a densely defined closed linear operator from $\mathcal{H}$ to $\mathcal{K}$. Then, by von Neumann’s theorem, $A^*A$ and $AA^*$ are non-negative self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$ respectively and hence $C^\infty(A^*A)$ and $C^\infty(AA^*)$ are dense in $\mathcal{H}$ and $\mathcal{K}$ respectively. Therefore the algebraic tensor product

$$\mathcal{D}_A := \mathcal{F}_{b,\text{fin}}(C^\infty(A^*A)) \otimes \mathcal{F}_{f,\text{fin}}(C^\infty(AA^*))$$

is dense in the boson-fermion Fock space $\mathcal{F}(\mathcal{H}, \mathcal{K})$.

Proposition 3.1 There exists a unique densely defined closed linear operator $d_A$ on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ such that the following (i) and (ii) hold:

(i) $\mathcal{D}_A \subset D(d_A)$ and $\mathcal{D}_A$ is a core of $d_A$.

(ii) For each vector $\Psi \in \mathcal{D}_A$ of the form

$$\Psi = a(f_1)^* \cdots a(f_n)^* \Omega_b \otimes b(u_1)^* \cdots b(u_p)^* \Omega_f, \quad n, p \geq 0,$$

where $a(f_1)^* \cdots a(f_n)^* \Omega_b$ (resp. $b(u_1)^* \cdots b(u_p)^* \Omega_f$) with $n = 0$ (resp. $p = 0$) should read $\Omega_b$ (resp. $\Omega_f$), $d_A$ acts as

$$d_A \Psi = 0 \quad \text{for } n = 0,$$

$$d_A \Psi = \sum_{j=1}^n a(f_1)^* \cdots \overline{a(f_j)^*} \cdots a(f_n)^* \Omega_b \otimes b(Af_j)^* b(u_1)^* \cdots b(u_p)^* \Omega_f$$

for $n \geq 1$, where $\overline{a(f_j)^*}$ indicates the omission of $a(f_j)^*$. In particular, $d_A$ leaves $\mathcal{D}_A$ invariant.

Moreover, the following (iii)-(v) hold:

(iii) $\mathcal{D}_A \subset D(d_A^*)$ and $d_A^* \Psi = 0$ for $p = 0$,

$$d_A^* \Psi = \sum_{k=1}^p (-1)^{k-1} a(A^*u_k)^* a(f_1)^* \cdots a(f_n)^* \Omega_b \otimes b(u_1)^* \cdots \overline{b(u_k)^*} \cdots b(u_p)^* \Omega_f$$

for $p \geq 1$. In particular, $d_A^*$ leaves $\mathcal{D}_A$ invariant.
(iv) \( D(d_A^2) = D(d_A) \) and, for all \( \Psi \in D(d_A) \), \( d_A^2 \Psi = 0 \).

(v) Let \( B \) be a bounded linear operator from \( \mathcal{H} \) to \( \mathcal{K} \) with \( D(B) = \mathcal{H} \). Then, for all \( \Psi \in \mathscr{D}_A^\infty \) and \( \alpha, \beta \in \mathbb{C} \),

\[
\alpha d_A \Psi + \beta d_B \Psi = d_{\alpha A + \beta B} \Psi.
\]

We call the operator \( d_A \) the exterior differential operator on \( \mathscr{F}(\mathcal{H}, \mathcal{K}) \) associated with \( A \).

### 4 Infinite Dimensional Dirac Operators

The Dirac operator on \( \mathscr{F}(\mathcal{H}, \mathcal{K}) \) associated with \( A \) is defined by

\[ Q_A := d_A + d_A^* \]

**Theorem 4.1** The operator \( Q_A \) is self-adjoint and unbounded from above and below.

The Laplace-Beltrami-de Rham operator on \( \mathscr{F}(\mathcal{H}, \mathcal{K}) \) associated with \( A \) is defined by

\[ \Delta_A := d_A^* d_A + d_A d_A^* \]

**Theorem 4.2** \( \Delta_A = Q_A^2 \).

### 5 Supersymmetric Structure

Let

\[
\mathcal{F}_+ := \mathcal{F}_b(\mathcal{H}) \otimes (\oplus_{p=0}^{\infty} \wedge^{2p} \mathcal{K}) \quad \text{(even forms)},
\]

\[
\mathcal{F}_- := \mathcal{F}_b(\mathcal{H}) \otimes (\oplus_{p=0}^{\infty} \wedge^{2p+1} \mathcal{K}) \quad \text{(odd forms)}.
\]

Then we have the orthogonal decomposition

\[
\mathcal{F}(\mathcal{H}, \mathcal{K}) = \mathcal{F}_+ \oplus \mathcal{F}_-.
\]

Let \( P_\pm : \mathcal{F}(\mathcal{H}, \mathcal{K}) \to \mathcal{F}_\pm \) be the orthogonal projections. Then the operator

\[ \Gamma := P_+ - P_- \]

is unitary, self-adjoint and the grading operator for the above orthogonal decomposition.
Proposition 5.1 (anti-commutativity) Operator equality $Q_A \Gamma = -\Gamma Q_A$ holds.

Corollary 5.2 (spectral symmetry) The spectrum $\sigma(Q_A)$ of $Q_A$ is reflection symmetric with respect to the origin of $\mathbb{R}$: $\sigma(Q_A) = \sigma(-Q_A)$.

The quadruple $\text{SQFT}_A := (\mathcal{F}(\mathcal{H}, \mathcal{K}), Q_A, \Delta_A, \Gamma)$ is a supersymmetric quantum theory in the abstract sense [1], where $Q_A$ is a self-adjoint supercharge, $\Delta_A$ is the supersymmetric Hamiltonian and $\Gamma$ is the state-sign operator. We remark that $\text{SQFT}_A$ gives a unification of some supersymmetric free quantum field models [2, 3, 4, 5, 6].

6 Relations with Second Quantization Operators

For each self-adjoint operator $S$ on $\mathcal{H}$, one can define the bosonic second quantization of $S$ by
\[
d\Gamma_b(S) := \bigoplus_{n=0}^{\infty} d\Gamma_b^{(n)}(S)\]
with
\[
d\Gamma_b^{(0)}(S) := 0, \quad d\Gamma_b^{(n)}(S) := \sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes S \otimes I \otimes \cdots \otimes I, \quad n \geq 1,
\]
where, for a closable operator $T$ on a Hilbert space, $\overline{T}$ denotes the closure of $T$. It follows that $d\Gamma_b(S)$ is self-adjoint. If $S \geq 0$, then $d\Gamma_b(S) \geq 0$. Moreover,
\[
0 \in \sigma_p(d\Gamma_b(S)), \quad \Omega_b \in \ker(d\Gamma_b(S)).
\]

Similarly, for each self-adjoint operator $T$ on $\mathcal{K}$, one can define the fermionic second quantization of $T$ by
\[
d\Gamma_f(T) := \bigoplus_{p=0}^{\infty} d\Gamma_f^{(p)}(T)\]
with
\[
d\Gamma_f^{(0)}(T) := 0, \quad d\Gamma_f^{(p)}(T) := \sum_{j=1}^{p} I \otimes \cdots \otimes I \otimes T \otimes I \otimes \cdots \otimes I, \quad p \geq 1.
\]
It follows that $d\Gamma_f(T)$ is self-adjoint. If $T \geq 0$, then $d\Gamma_f(T) \geq 0$. Moreover,
\[
0 \in \sigma_p(d\Gamma_f(T)), \quad \Omega_f \in \ker(d\Gamma_f(T)).
\]
As we have already mentioned, the operator $A$ yields the non-negative self-adjoint operators $A^*A$ and $AA^*$. Therefore $A^*A$ (resp. $AA^*$) may be a one-particle Hamiltonian for a boson (resp. fermion). Then the Hamiltonian of a non-interacting system consisting of such bosons and fermions is given by

$$H(A) := d\Gamma_b(A^*A) \otimes I + I \otimes d\Gamma_f(AA^*).$$

It follows that $H(A)$ is a non-negative self-adjoint operator acting in $\mathcal{F}(\mathcal{H}, \mathcal{K})$ and

$$0 \in \sigma_p(H(A)), \quad \Omega_b \otimes \Omega_f \in \ker H(A).$$

**Theorem 6.1** $H(A) = \Delta_A$. In particular, $H(A)$ is a supersymmetric Hamiltonian.

### 7 Spectra of $H(A)$ and $Q_A$

In what follows, we assume that $\mathcal{H}$ and $\mathcal{K}$ are separable. For a linear operator $T$ from a Hilbert space to a Hilbert space, we set

$$\text{nul } T := \dim \ker T \in \{0\} \cup \mathbb{N} \cup \{+\infty\}.$$

**Theorem 7.1**

$$\sigma(H(A)) = \{0\} \cup \left( \bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \lambda_j \big| \lambda_j \in \sigma(A^*A) \setminus \{0\}, j = 1, \cdots, n \right\} \right),$$

$$\sigma_p(H(A)) = \{0\} \cup \left( \bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \lambda_j \big| \lambda_j \in \sigma_p(A^*A) \setminus \{0\}, j = 1, \cdots, n \right\} \right).$$

**Theorem 7.2** The spectrum $\sigma(Q_A)$ and the point spectrum $\sigma_p(Q_A)$ of $Q_A$ are symmetric with respect to the origin and

$$\sigma(Q_A) = \{0\} \cup \left( \bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^{n} \lambda_j} \big| \lambda_j \in \sigma(A^*A) \setminus \{0\}, j = 1, \cdots, n \right\} \right),$$

$$\sigma_p(Q_A) = \{0\} \cup \left( \bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^{n} \lambda_j} \big| \lambda_j \in \sigma_p(A^*A) \setminus \{0\}, j = 1, \cdots, n \right\} \right).$$

with

$$\text{nul } (Q_A - \lambda) = \text{nul } (Q_A + \lambda), \quad \lambda \in \sigma_p(Q_A).$$
8 A Simple Perturbation

In this section, we consider a simple perturbation of $Q_A$ via a perturbation of $d_A$. Let
\[ g \in D(A) \setminus \{0\}, \quad v \in D(A^*) \setminus \{0\} \]
and
\[ d(\alpha) := d_A + \alpha a(g) \otimes b(v)^*. \]
with a constant $\alpha \in \mathbb{C}$ being a perturbation parameter. It is easy to see that $d(\alpha)$ is densely defined with $D(d(\alpha)) \supset \mathcal{D}_A^\infty$ and
\[ d(\alpha)^2 = 0 \quad \text{on } \mathcal{D}_A^\infty. \]
Moreover, $d(\alpha)^*$ is densely defined with $\mathcal{D}_A^\infty \subset D(d(\alpha)^*)$ and
\[ d(\alpha)^* = d_A^* + \alpha^* a(g)^* \otimes b(v) \quad \text{on } \mathcal{D}_A^\infty. \]
Hence $d(\alpha)$ is closable. We denote the closure of $d(\alpha) | \mathcal{D}_A^\infty$ by $\overline{d}(\alpha)$.

**Lemma 8.1** For all $\Psi \in D(\overline{d}(\alpha))$, $\overline{d}(\alpha)\Psi$ is in $D(\overline{d}(\alpha))$ and
\[ \overline{d}(\alpha)^2 \Psi = 0. \]
Using the operator $\overline{d}(\alpha)$, one can define a perturbed Dirac operator:
\[ Q(\alpha) := \overline{d}(\alpha) + \overline{d}(\alpha)^*. \]
We note that
\[ Q(\alpha) = Q_A + V_{g,v}(\alpha) \quad \text{on } \mathcal{D}_A^\infty \]
with
\[ V_{g,v}(\alpha) := \alpha a(g) \otimes b(v)^* + \alpha^* a(g)^* \otimes b(v). \]

8.1 Self-adjointness of $Q(\alpha)$

Let $T_{g,v} : \mathcal{H} \to \mathcal{H}$ be defined by
\[ T_{g,v} f := \langle g, f \rangle v, \quad f \in \mathcal{H}. \]
It is obvious that $T_{g,v}$ is a bounded linear operator (a one-rank operator). Hence
\[ A(\alpha) := A + \alpha T_{g,v} \]
is a densely defined closed linear operator with $D(A(\alpha)) = D(A)$. 
Remark 8.2 Perturbations of a linear operator by one-rank or two-rank operators have been studied in various contexts. See, e.g. [12, 13] and references therein.

Lemma 8.3 (a key lemma) For all $\alpha \in \mathbb{C}$, the following operator equality holds:

$$\overline{d}(\alpha) = d_{A(\alpha)}.$$

Theorem 8.4

(i) For all $\alpha \in \mathbb{C}$, $Q(\alpha)$ is self-adjoint and

$$Q(\alpha) = Q_{A(\alpha)}.$$

(ii) For all $\alpha \in \mathbb{C}$, $Q(\alpha)$ is essentially self-adjoint on $\mathcal{D}_{A}^{\infty}$.

(iii) For all $\alpha \in \mathbb{C}$,

$$Q(\alpha) = Q_{A} + V_{g,v}(\alpha).$$

(iv) The operator $\Gamma$ leaves $D(Q(\alpha))$ invariant and

$$\Gamma Q(\alpha) + Q(\alpha) \Gamma = 0 \quad \text{on } D(Q(\alpha)).$$

(v) For all $\Psi \in \mathcal{D}_{A}^{\infty}$, the vector-valued function $\alpha \mapsto Q(\alpha)\Psi$ is strongly continuous on $\mathbb{C}$. Moreover, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(Q(\alpha) - z)^{-1}$ is strongly continuous in $\alpha \in \mathbb{C}$.

8.2 Spectra of $Q(\alpha)$

Theorem 8.5 For all $\alpha \in \mathbb{C}$, $\sigma(Q(\alpha))$ and $\sigma_{p}(Q(\alpha))$ are symmetric with respect to the origin and

$$\sigma(Q(\alpha)) = \{0\} \cup \left( \bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^{n} \lambda_{j}} \mid \lambda_{j} \in \sigma(A(\alpha)^{*}A(\alpha)) \setminus \{0\}, j = 1, \cdots, n \right\} \right),$$

$$\sigma_{p}(Q(\alpha)) = \{0\} \cup \left( \bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^{n} \lambda_{j}} \mid \lambda_{j} \in \sigma_{p}(A(\alpha)^{*}A(\alpha)) \setminus \{0\}, j = 1, \cdots, n \right\} \right)$$

with

$$\text{nul}(Q(\alpha) - \lambda) = \text{nul}(Q(\alpha) + \lambda), \quad \lambda \in \sigma_{p}(Q(\alpha)).$$

This theorem shows that the spectrum and the point spectrum of $Q(\alpha)$ are completely determined from those of $A(\alpha)^{*}A(\alpha) \setminus \{0\}$. 
8.3 Identification of the domain of $Q(\alpha)$

Recall that $|A| := (A^*A)^{1/2}$ acting in $\mathcal{H}$. It follows that $A$ is injective if and only if $|A|$ is injective.

**Theorem 8.6** Suppose that $A$ is injective and $g \in D(|A|^{-1})$. Then, for all $|\alpha| < 1/(\|v\| \| |A|^{-1}g\|)$, $Q(\alpha)$ is self-adjoint with $D(Q(\alpha)) = D(Q_A)$ and

$$Q(\alpha) = Q_A + V_{g,v}(\alpha).$$

Moreover, $Q(\alpha)$ is essentially self-adjoint on any core for $Q_A$.

**Proof.** The essential part of the proof is to show that $V_{g,v}(\alpha)$ is $Q_A$-bounded with a relative upper bound $|\alpha| |v| |A|^{-1}g|$. Then one needs only to apply the Kato-Rellich theorem. For more details, see the proof of [10, Theorem 17].

9 Kernel of $Q(\alpha)$

We now investigate the kernel of $Q(\alpha)$. We need a classification for conditions on $\{A, g, v\}$:

(C.1) $A$ is injective, $v \in D(A^{-1})$ and $\langle g, A^{-1}v \rangle \neq 0$. In this case we introduce a constant

$$\alpha_0 := -\frac{1}{\langle g, A^{-1}v \rangle}. \quad (9.1)$$

(C.2) $A^*$ is injective, $g \in D(A^{*-1})$ and $\langle v, A^{*-1}g \rangle \neq 0$. In this case we introduce a constant

$$\beta_0 := -\frac{1}{\langle A^{*-1}g, v \rangle}. \quad (9.2)$$

(C.3) (a) $A$ is injective and $v \notin D(A^{-1})$ or (b) $A$ is injective and $v \in D(A^{-1})$ with $\langle g, A^{-1}v \rangle = 0$.

(C.4) (a) $A^*$ is injective and $g \notin D(A^{*-1})$ or (b) $A^*$ is injective $g \in D(A^{*-1})$ with $\langle v, A^{*-1}g \rangle = 0$.

We first consider the kernel of $A(\alpha)$ and $A(\alpha)^*$.

**Lemma 9.1**

(i) Suppose that (C.1) holds. Then

$$\ker A(\alpha) = \{0\}, \quad \alpha \neq \alpha_0,$$

$$\ker A(\alpha_0) = \{cA^{-1}v | c \in \mathbb{C}\}.$$
(ii) Suppose that (C.2) holds. Then
\[
\ker A(\alpha)^* = \{0\}, \quad \alpha \neq \beta_0, \\
\ker A(\beta_0)^* = \{cA^{*-1}g | c \in \mathbb{C}\}.
\]

(iii) Suppose that (C.3) holds. Then, for all \(\alpha \in \mathbb{C}\),
\[
\ker A(\alpha) = \{0\}.
\]

(iv) Suppose that (C.4) holds. Then, for all \(\alpha \in \mathbb{C}\),
\[
\ker A(\alpha)^* = \{0\}.
\]

**Theorem 9.2**

(i) Assume (C.1). Then
\[
\ker Q(\alpha_0) = \bigoplus_{n,p=0}^{\infty} \left[ (\otimes^n \{zA^{-1}v | z \in \mathbb{C}\}) \otimes \wedge^p (\ker A(\alpha_0)^*) \right],
\]
and hence \(\text{nul} \ Q(\alpha_0) = \infty\).
Moreover, for all \(\alpha \neq \alpha_0\),
\[
\ker Q(\alpha) = \bigoplus_{p=0}^{\infty} \mathbb{C} \otimes \wedge^p (\ker A(\alpha)^*).
\]

(ii) Assume (C.2). Then
\[
\ker Q(\beta_0) = \bigoplus_{n=0}^{\infty} \left\{ [\otimes^n \ker(A(\beta_0))] \otimes [\mathbb{C} \oplus \text{span}\{A^{*-1}g\}] \right\}, \\
\ker Q(\alpha) = \bigoplus_{n=0}^{\infty} \left[ \otimes^n \ker A(\alpha) \otimes \mathbb{C} \right], \quad \alpha \neq \beta_0.
\]

(iii) Assume (C.3). Then, for all \(\alpha \in \mathbb{C}\),
\[
\ker Q(\alpha) = \bigoplus_{p=0}^{\infty} [\mathbb{C} \otimes \wedge^p (\ker(A(\alpha))^*)].
\]

(iv) Assume (C.4). Then, for all \(\alpha \in \mathbb{C}\),
\[
\ker Q(\alpha) = \bigoplus_{n=0}^{\infty} [\otimes^n \ker A(\alpha) \otimes \mathbb{C}].
\]

**Corollary 9.3**

(i) Assume (C.1) and (C.2). Then
\[
\ker Q(\alpha_0) = \text{span} \left( \left\{ a(A^{-1}v)^* \otimes b(A^{*-1}g)^* \Omega_j | n \geq 0, j = 0, 1 \right\} \right), \\
\ker Q(\alpha) = \{c\Omega_b \otimes \Omega_j | c \in \mathbb{C}\}, \quad \alpha \neq \alpha_0.
\]
(ii) Assume (C.1) and (C.4). Then

\[ \ker Q(\alpha) = \text{span}(\{a(A^{-1}v)^* n \Omega \otimes \Omega_f | n \geq 0\}), \]

\[ \ker Q(\alpha_0) = \{c \Omega_b \otimes \Omega_f | c \in \mathbb{C}\}, \quad \alpha \neq \alpha_0. \]

(iii) Assume (C.2) and (C.3). Then

\[ \ker Q(\beta_0) = \text{span}(\{\Omega_b \otimes b(A^{*-1}g)^* \Omega_f | j = 0, 1\}). \]

\[ \ker Q(\alpha) = \{c \Omega_b \otimes \Omega_f | c \in \mathbb{C}\}, \quad \alpha \neq \beta_0. \]

(iv) Assume (C.3) and (C.4). Then, for all \( \alpha \in \mathbb{C} \),

\[ \ker Q(\alpha) = \{c \Omega_b \otimes \Omega_f | c \in \mathbb{C}\}. \]

10 Non-zero Eigenvalues of \( Q(\alpha) \)

Hypothesis (A)

(i) \( \mathcal{H} = \mathcal{K} \);

(ii) \( A \) is an injective and nonnegative self-adjoint operator;

(iii) \( g = v \in D(A^{-1}) \).

Under Hypothesis (A), the constant \( \alpha_0 \) defined by (9.1) takes the form

\[ \alpha_0 = -\frac{1}{\langle v, A^{-1}v \rangle} < 0. \]

Theorem 10.1 Let Hypothesis (A) be satisfied and \( \alpha < \alpha_0 (< 0) \). Then, there exists a unique constant \( x_0(\alpha) < 0 \) such that \( \alpha \langle v, (x_0(\alpha) - A)^{-1} v \rangle = 1 \) and, for all \( n \in \{0\} \cup \mathbb{N} \),

\[ \pm \sqrt{n} x_0(\alpha) \in \sigma_p(Q(\alpha)). \]

with eigenvectors

\[ [Q(\alpha) \pm \sqrt{n} x_0(\alpha)] \{a(\phi_\alpha)^* n \Omega_b \otimes b(\phi_\alpha)^* \Omega_f \} \]

\[ \in \ker(Q(\alpha) \mp \sqrt{n} x_0(\alpha)) (n \geq p \geq 0), \]

where

\[ \phi_\alpha := (x_0(\alpha) - A)^{-1} v. \]

Moreover, \( x_0(\alpha) \), as a function of \( \alpha < \alpha_0 \), is strictly monotone increasing on \( (-\infty, \alpha_0) \) with \( \lim_{\alpha \to -\infty} x_0(\alpha) = -\infty \) and \( \lim_{\alpha \to \alpha_0} x_0(\alpha) = 0 \).

Note that Theorem 10.1 holds even if \( Q_A \) has no non-zero eigenvalues. This is an interesting phenomenon. Since the condition \( \alpha < \alpha_0 < 0 \) implies that \( |\alpha| > |\alpha_0| \), the phenomenon may be regarded as a strong coupling effect.
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References


