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Stable and Unstable Modes at a Saddle Point in Micro-Macro Duality

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Abstract

Longitudinal modes associated with local gauge invariance cause such difficulties as breakdown of Lorentz covariance in ***Coulomb mode*** and/or that of the probabilistic interpretation owing to "***indefinite metric***". The latter one can be viewed in the context of dynamics as a saddle point causing bifurcation into stable and unstable modes.

In the light of "***Cartan involution***" characterizing symmetric space structure, natural connections can be found between the problem caused by longitudinal modes in gauge theory and the roles played by ***heat baths*** in thermodynamics, from the viewpoint of mutual relations between ***finite-*** and ***infinite-dimensional*** realizations of ***noncompact Lie groups*** in the ligth of indefinite metric.

These problems can be treated consistently from the viewpoint of bifurcations into stable and unstable modes at a saddle points in Micro-Macro duality.

1 Hidden similarities between Coulomb modes and heat baths?

Common features among "gauge", "thermal" and "symmetric spaces" are related with "***indefinite metric***":

1) In gauge systems, transverse & ***longitudinal*** modes are contrasted by their "metric", positive for the former vs. negative the latter $\Longrightarrow$ such difficulties as breakdown of Lorentz covariance due to ***Coulomb mode*** and/or breakdown of probabilistic interpretation by negative probabilities due to ***indefinite metric***.

2) In ***thermal systems***, we know the important roles played by ***heat baths*** which have been totally neglected in the Gibbs formula $\omega_\beta(A) =$

*On the occasion to celebrate Prof. Obata's 60th birthday

$Tr(e^{-\beta H}A)/Tr(e^{-\beta H})$ for statistical mechanics. In spite of its familiarity among physicists, however, this formula cannot be extended to ***infinitely extended systems*** without mathematical reformulation based on ***Tomita-Takesaki modular theory***, by which ***heat baths*** are revived!!

1.1 Symmetric spaces associated with *non-compact Lie* groups

3) In the third case of ***symmetric spaces***, such additional qualifications are important as those arising from ***non-compact Lie*** groups.

In this situation, the feature of indefinite metric can be found for the Casimir invariant (of the 2nd order) of non-compact G with Lie algebra $\mathfrak{g} = Lie(G)$ and its maximally compact Lie subgroup denoted by H with $\mathfrak{h} = Lie(H)$. Then, $\mathfrak{h}$ has negative definite sign and the corresponding non-compact part $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ positive one:

compact directions H: with ***minus sign***

non-compact directions $M = G/H$: with ***plus sign***

which are closely related with the Cartan involution with $+$ assigned to $\mathfrak{h}$ and $-$ to $\mathfrak{m}$ in the symmetric space structure of the homogeneous space $M = G/H$:

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h},[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m},[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}.$$

While the signs $\pm$ in this case is opposite to the first and the second examples (in Local Gauge & StatMech), this is simply due to the different sign conventions adopted in QFT and in Lie algebras.

1.2 "Indefinite metric" in thermal Hamiltonian

For conceptual clarity, convenient to suppress mathematical details, but it is really interesting or even ironical to note that its ***mathematical reformulation*** can exhibit the ***physical aspects of heat baths*** hidden in the above physical formula as follows:

Thermal Hamiltonian H_β of the total system consisting of the system and of the heat bath shows up in such a symmetric form under its sign changes as

$$JH_\beta J = -H_\beta, \qquad H_\beta = H - JHJ = \pi_l(H) - \pi_r(H),$$

where J is the antilinear ***modular conjugation*** operator to interchange the algebra $\mathcal{M}$ of observables of the system and its commutant $\mathcal{M}'$ corresponding to the heat bath:

$$J\mathcal{M}J = \mathcal{M}' \;\; \& \;\; J\mathcal{M}'J = \mathcal{M}.$$

Note the minus sign $(-)$ in H_β, similarly to the longitudinal photons with "***negative metric***".

1.3 Modular operator $\Delta_\beta = e^{-\beta H_\beta}$ & modular conjugation J

J and H_β are defined by the following formula:

$$Je^{-\beta H_\beta/2} A\Omega_\beta = A^*\Omega_\beta \quad \text{for } A \in \mathcal{M},$$

for GNS vector Ω_β corresponding to the KMS=Gibbs state ω_β: $\omega_\beta(A) = \langle \Omega_\beta, A\Omega_\beta \rangle$ $(A \in \mathcal{M})$.

If $e^{-\beta H}$ is of trace class (valid for a system with *discrete energy levels*), we have, $\omega_\beta(A) = \langle \Omega_\beta, A\Omega_\beta \rangle = Tr(e^{-\beta H}A)/Tr(e^{-\beta H})$, with $\Omega_\beta = e^{-\beta H/2}/\sqrt{Tr(e^{-\beta H})}$. Then $\mathcal{M}$ and $\mathcal{M}'$ act, respectively, from the left π_l and right π_r on the Hilbert space $L^2(\mathfrak{H}) \simeq \mathfrak{H} \otimes \overline{\mathfrak{H}}$ of square roots ξ of trace-class operators $\xi^*\xi \in L^1(\mathfrak{H})$:

$$\begin{aligned} &\pi_l(A)\pi_r(B)\xi = A\xi B, \\ &e^{-\beta H_\beta}\xi = e^{-\beta[\pi_l(H)-\pi_r(H)]}\xi = e^{-\beta H}\xi e^{\beta H}, \end{aligned}$$

where $\mathcal{M}$ and $\mathcal{M}'$ evidently commute *without involving negative metric*!

2 Similarities between gauge and thermal systems

Similarities among "gauge", "thermal" and "symmetric spaces":

In thermal case, physical meaning of the objects related to the system via the modular conjugation J is understood in relation with the thermal reservoir(s) or heat bath(s). Although the degrees of freedom related to J are the mirror image(s) $\mathcal{M}'$ of the system $\mathcal{M}$ described in general by some *non-commutative* operator algebra $\mathcal{M}$, the non-commutativity of $\mathcal{M}'$ is irrelevant to the physically meaningful correlation functions consisting solely of the elements belonging to $\mathcal{M}$.

This is consistent with the physical essence of thermodynamics where the thermal reservoir or heat bath plays anonymous roles to maintain (local) thermal equilibria, which can be accomplished by classical objects, too.

When the formulation is extended to thermal situations with multiple thermal reservoirs, this kind of situations are unchanged as far as local thermal equilibria are maintained in the neighbourhood of each reservoir.

Multiple-time KMS condition due to Prof. H. Araki can be related with such local thermal equilibrium in contact with multi-reservoir system:

$$\begin{aligned} &\phi(x_0\Delta^{z_1}_{\omega_1,\phi}x_1 \cdots \Delta^{z_j}_{\omega_j,\phi}x'_j \cdot x''_j\Delta^{z_{j+1}}_{\omega_{j+1},\phi}x_{j+1} \cdots \Delta^{z_n}_{\omega_n,\phi}x_n) \\ &= \phi(x''_j\Delta^{z_{j+1}}_{\omega_{j+1},\phi}x_{j+1} \cdots \Delta^{z_n}_{\omega_n,\phi}x_n\Delta_\phi^{1-\sum_{i=1}^n z_i}x_0\Delta^{z_1}_{\omega_1,\phi}x_1 \cdots \Delta^{z_j}_{\omega_j,\phi}x'_j). \end{aligned}$$

In relevance of this kind of localized situations, the case of a gauge system is not much different because of the local gauge invariance, if the longitudial modes showing no particle-like behaviours are described by classical

Coulomb modes. Therefore, the most relevant description of such situations as above can be implemented in a theoretical framework of the so-called quantum-classical compositie systems.

2.1 Symmetry breaking & emergence of sector-classifying space

Sector-classifying space emerges typically from spontaneous breakdown of symmetry of a dynamical system $\mathcal{X} \curvearrowleft G$ with action of a group G (without changing dynamics of the system = "spontaneous").

Criterion for Symmetry Breaking: judged by non-triviality of ***central*** dynamical system $\mathfrak{Z}_\pi(\mathcal{X}) \curvearrowleft G$ arising from the original one $\mathcal{X} \curvearrowleft G$

I.e., symmetry G is ***broken in sectors*** $\in Sp(\mathfrak{Z})$ ***shifted non-trivially by central action*** of G.

The G-transitivity assumption with ***unbroken*** subgroup H in broken G leads to such a specific form of sector-classifying space as G/H.

$\Longrightarrow$ ***Classical geometric*** structure on G/H arises physically from ***emergence*** process via ***condensation*** of a family of ***degenerate vacua***, each of which is mutually distinguished by condensed values $\in Sp(\mathfrak{Z}) = G/H$.

In this way, ∞-number of low-energy quanta are condensed into geometry of classical Macro objects $\in G/H$.

In combination with sector structure $\hat{H}$ of unbroken symmetry H, total sector structure due to this symmetry breaking is described by a sector bundle $G \underset{H}{\times} \hat{H}$ with fiber $\hat{H}$ over base space G/H of "***degenerate vacua***" [IO03, IO04].

When this geometric structure is established, all the physical quantities are ***parametrized by condensed values*** $\in G/H \Longrightarrow$ "***logical extension***" of ***constants*** into ***sector-dependent functional objects*** (: origin of local gauge structures

3 Symmetric space structure of sector-classifying space

The homogeneous space $M = G/H$ arising from symmetry breaking is seen to be a ***symmetric space*** characterized by Cartan involution.

In terms of Lie structures on G, H, M denoted by $\mathfrak{g}, \mathfrak{h}, \mathfrak{m}$, the corresponding Lie algebraic quantities satisfying $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$. Then validity of $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ provides M with Cartan involution $\mathcal{I}$ to characterize a symmetric space whose eigenvalues are $\mathcal{I} \restriction_{\mathfrak{h}} = +1$ and $\mathcal{I} \restriction_{\mathfrak{m}} = -1$, respectively:

Note that $[\mathfrak{m}, \mathfrak{m}]$ is the ***holonomy*** term corresponding to an infinitesimal loop along ***broken direction*** $G/H = M = Sp(\mathfrak{Z})$ as ***inter-sectorial***

space. Namely, $[\mathfrak{m},\mathfrak{m}]$ describes the effect of ***broken*** G transformations generated by an infinitesimal loop on M starting from a point in M and going back to the same point. According to the Criterion of Symmetry Breaking in terms of ***non-trivial shift under central action*** of G, the absence of $\mathfrak{m}$-components in $[\mathfrak{m},\mathfrak{m}]$: $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$, follows from the identity of initial and final points of the loop. Thus, $M=G/H=Sp(3)$ is a symmetric space.

3.1 Examples

Example 1: ***Relativity*** controlled by Lorentz group:

Typical example of the above sort can be found in the case of Lorentz group $\mathcal{L}_+^{\uparrow} =: G$, rotation group $SO(3) =: H$, $G/H=M\cong\mathbb{R}^3$: symmetric space of Lorentz frames mutually connected by Lorentz boosts.

The relations $[\mathfrak{h},\mathfrak{h}]=\mathfrak{h}$, $[\mathfrak{h},\mathfrak{m}]=\mathfrak{m},[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$ with $\mathfrak{h}:=\{M_{ij};i,j=1,2,3,i<j\}$, $\mathfrak{m}:=\{M_{0i};i=1,2,3\}$ follow from the basic Lie algebra structure:

$[iM_{\mu\nu},iM_{\rho\sigma}]=-(\eta_{\nu\rho}iM_{\mu\sigma}-\eta_{\nu\sigma}iM_{\mu\rho}-\eta_{\mu\rho}iM_{\nu\sigma}+\eta_{\mu\sigma}iM_{\nu\rho})$.

While both $\mathfrak{h}$ and $\mathfrak{m}$ are taken as unbroken in physics, such results as Borcher-Arveson thm (: affiliation of Poincaré generators to algebra of global observables in vacuum situation) & IO86 (: spontaneous breakdown of Lorentz boosts at $T\neq 0K$) indicate the ***speciality of unbroken*** $\mathfrak{m}$ ***in the vacuum situation***. In this sense, the symmetric space of Lorentz frames $M\cong\mathbb{R}^3$ with [boost, boost] = rotation, gives a typical example of symmetric space structure emerging from symmetry breaking.

Example 2: 2nd law of thermodynamics

Along this line, chiral symmetry with current algebra structure $[V,V]=V,[V,A]=A,[A,A]=V$ and

conformal symmetry also provide typical examples.

Physically most interesting example can be found in thermodynamics:

Corresponding to $\mathfrak{h}\hookrightarrow\mathfrak{g}\twoheadrightarrow\mathfrak{m}=\mathfrak{g}/\mathfrak{h}$, we find here exact sequence $\Delta'Q\hookrightarrow\Delta E=\Delta'Q+\Delta'W\twoheadrightarrow\Delta'W$ due to 1st law of thermodynamics.

With respect to Cartan involution with $+$ assigned to heat production $\Delta'Q$ and $-$ to macroscopic work $\Delta'W$, the holonomy $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$ corresponding to a loop in the space M of thermodynamic variables becomes just

Kelvin's version of 2nd law of thermodynamics:

namely, holonomy $[\mathfrak{m},\mathfrak{m}]$ in the cyclic process with $\Delta E=\Delta'Q+\Delta'W=0$, describes heat production $\Delta'Q\geq 0$: $-\Delta'W=-[\mathfrak{m},\mathfrak{m}]=\Delta'Q>0$ (from system to outside)

3.2 "Resolution" of indefinite metric into Macro condensates

We see that the "would-be" *indefinite metric* is resolved by means of an induced representation to extend representations of a subgroup H to that of the whole group G.

$\Longrightarrow$ Induced rep. of G with cpt. subgroup H embedded into a fiber bundle over a base space $M = G/H$ acted on by G via "Wigner rotations": here, non-compactness of G with indefinite metric is resolved into a ***classical*** geometric structure M to describe the condensate associated with the symmetry breaking from G to H!!

$\Longrightarrow$ ***Micro-Macro composite structure*** (with Micro acted on by H and Macro M) and its ***gauge structure*** with Clebsch parametrization to yield Yukawa potentials with gauge transformations and (co-)cycles associated with broken chiral symmetry (Sakuma-Ohtsu-Ojima, in preparation).

Due to probabilistic inconsistency, "constituent particles" with *negative* probability cannot be extracted from their Macro condensate $M = G/H$.

3.3 Adjunction between quotients & subobjects

Note the definition of G/H as a ***symmetric space*** (characterized by Lie structure $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$) is just equivalent to that of an ***$\mathfrak{h}$-valued connection*** on $M = G/H$, according to the following diagram ensuring bi-directional arrows to define a connection: $G/H \underset{\hookrightarrow}{\overset{\twoheadleftarrow}{\rightleftarrows}} G \underset{\twoheadrightarrow}{\overset{\hookleftarrow}{\rightleftarrows}} H$.

While the arrows $G/H \twoheadleftarrow G \hookleftarrow H$ from the right to the left simply define a homogeneous space G/H as a quotient of G w.r.t. its subgroup H,

the arrows $G/H \hookrightarrow G \twoheadrightarrow H$ in the opposite direction defines an ***$\mathfrak{h}$-valued connection*** on the manifold G/H.

From this viewpoint, the above "*resolution of indefinite metric* of non-compact symmetric space via ***classicalization***" can be seen to result in Verdier localization, through which a *quotient* object $M = G/H$ becomes adjoint to the *sub*group H.

3.4 Multiplicative system and triangulated categories

Thus we see the relation between resolution of indefinite metric due to a non-compact symmetric space $M = G/H$ of classifying space of sectors and triangulated category via Verdier localization by multiplicative system (e.g., K. Kato: "Triangulated Categories and Homological Algebras").

In Tomita-Takesaki theory of statistical mechanics: Cyclic and separating property of GNS vector $\Omega_\beta \Longrightarrow$ Commutant $\mathcal{M}' = J\mathcal{M}J$ of von Neumann algebra $\mathcal{M}$ is a multiplicative system on $(\mathcal{M} \vee \mathcal{M}')$-module $\mathfrak{H}$.

$\Longrightarrow$ Treating Verdier-localized fractions A/B' of $A \in \mathcal{M}$ by $B' \in \mathcal{M}'$ will be equivalent to treating $\mathcal{M}$ with $\mathcal{M}'$ neglected.

In view of the common features related with indefinite metrics among Coulomb modes, Tomita-Takesaki modular structure, and non-compact Lie groups with associated symmetric spaces mentioned at the beginning, we are now interested in the problem to extend the above discussion on the classification problem related with the triangulated categories.

4 Classification problem in sector space by means of indefinite metric and/or triangulated categories

In order to extract meaningful classifying indices for physical and engineering problems, such full-fledged machineries in (co)homological theory are available as triangulated and derived categories (as pioneered in our collaboration concerning "natural intelligence" due to M.Naruse, H.Hori, et al.).

Because of the present interesting connection between indefinite metric and saddle point with stable and unstable flows, however, I would like to mention a possibility to utilize more geometrical contexts in relation with Morse theory based on Morse functions and homology complexes treated in a Japanese book by A. Hattori, Various Geometries, II), as follows:

4.1 Stable and unstable modes in Morse theory

For a given indefinite metric $\eta = (-1, \cdots, -1, +1, \cdots, +1)$ with -1 and $+1$ repeated r times and q times, respectively, the number r is called the *index* of η. Then, the function

$$f(x) := \langle x, \eta x \rangle = -x_1^2 - \cdots - x_r^2 + x_{r+1}^2 + \cdots + x_{r+q}^2$$

gives a typical example of *Morse functions* defined by the non-degeneracy condition on its Hessian matrix $\partial_i \partial_j f$ (at each critical point $p \in M$ s.t. $f(p) = 0$). Depending on such situations as ($q > 0$ with $r = 0$), ($r > 0$ with $q = 0$), ($q > 0$ and $r > 0$), the corresponding (critical) points are a peak, a bottom and a saddle point, respectively. The former and the middle cases belong, respectively, to stable and unstable submanifolds and the third one contains both stable and unstable ones.

In view of the (C^2-)density of Morse functions among C^∞-functions, sector-classifying space M can be decomposed stable and unstable submanifolds which are diffeomorphic to $\mathbb{R}^n$ (with some n).

By small perturbations, two stable and/or unstable submanifolds can be so deformed as to satisfy the Morse-Smale condition of the transversality.

Then, these submanifolds can be used for constructing a chain complex (due to Witten and Floer) on which a *homology theory* can be developed in a duality to de Rham cohomology in terms of exterior differential forms on M (see A. Hattori, Various Geometries, II [in Japanese]).

An alternative way may be to decompose M into *handlebodies*, which seems to allow us to avoid consideration on the Morse-Smale transversality condition and to lead not only to homological invariants but to homotopical one (see Y. Matsumoto, Foundations of Morse Theory [in Japanese]).

5 Why we need Morse theory?

If we stick to universal methods for extracting geometrical invariants for classification problems, better choice may be found in abstract methods based on triangulated and/or derived categories. However, parallelism between the unstable modes at a saddle point and the classical condensates can give us a vivid picture about the mutual relation between Micro and Macro from the viewpoint of singularities responsible for non-trivial homology. In fact, the important roles of unstable modes can be seen in Morse inequality: $(-1)^\lambda \sum(-1)^k m_k \geq (-1)^\lambda \sum(-1)^k b_k$ with arbitrary non-negative integer λ and with b_k and m_k being the Betti number and the number of critical points of a Morse function whose index is k, respectively.

Finally, in the context of "natural intelligence", Prof. K. Nakajima (at Tokyo Univ.) has attempted recently such an interesting example of efficient computing methods as "octopus-leg computer". Because of its *resemblance to handlebodies in shape,* something more seems to be suggested. (Under construction ...)

References

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