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# On a bifurcation problem for viscous compressible fluid between two rotating concentric cylinders

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## 1 Introduction

We consider a viscous fluid between two concentric cylinders. The inner cylinder is rotating with uniform speed $\omega$ and the outer one is at rest. If $\omega$ is sufficiently small, a laminar flow (Couette flow) is stable. When $\omega$ increases, beyond a certain value of $\omega$, a vortex flow pattern (Taylor vortex) appears. Mathematically, this phenomenon is formulated as a bifurcation problem. If the fluid is incompressible, the bifurcation of the Taylor vortex from the Couette flow was proved for the incompressible Navier-Stokes equations by Velte [12], Iudovich [3], Kirchgässner and Sorger [7] and etc. See the book [1] by Chossat and Iooss for the Taylor problem.

In this article we give a summary of the results in [6] on a bifurcation problem for the compressible Navier-Stokes equations.

A non-dimensional form of the governing equations is written as

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho \boldsymbol{v}) = 0, \\ \rho(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) - \nu \Delta \boldsymbol{v} - (\nu + \nu') \nabla \mathrm{div}\, \boldsymbol{v} + \frac{1}{\varepsilon^2} \nabla p(\rho) = \mathbf{0} \end{cases} \tag{1.1}$$

on a cylindrical domain $\Omega_\alpha$. Here $\rho$ and $\boldsymbol{v}$ are the unknown fluid density and velocity, respectively; $\nu > 0$ is a non-dimensional parameter proportional to $1/\omega$; $\varepsilon > 0$ is the Mach number; $p(\rho)$ is the pressure that is a smooth function of $\rho$ and satisfies $p'(1) = 1$; and the domain $\Omega_\alpha$ is given by

$$\Omega_\alpha = \{(r, \theta, z) : \frac{\eta}{1-\eta} < r < \frac{1}{1-\eta}, \quad \theta \in \mathbb{T}_{2\pi}, \quad z \in \mathbb{T}_{\frac{2\pi}{\alpha}}\}.$$

Here $(r, \theta, z)$ denotes the cylindrical coordinates; $0 < \eta < 1$, $\alpha > 0$ are given constants; and $\mathbb{T}_\beta = \mathbb{R}/\beta\mathbb{Z}$. We note that the periodic boundary condition in $z$ is included in the definition of $\Omega_\alpha$, namely, $\rho$ and $\boldsymbol{v}$ are $\frac{2\pi}{\alpha}$-periodic in $z$. The boundary conditions on $r = \frac{\eta}{1-\eta}, \frac{1}{1-\eta}$ are

$$v^\theta|_{r=\frac{\eta}{1-\eta}} = 1, \; v^\theta|_{r=\frac{1}{1-\eta}} = 0. \; v^r = v^z = 0 \text{ on } r = \tfrac{\eta}{1-\eta}, \tfrac{1}{1-\eta}, \tag{1.2}$$

Here $(v^r, v^\theta, v^z)$ are the $(r, \theta, z)$-components of $\boldsymbol{v} = v^r \boldsymbol{e}_r + v^\theta \boldsymbol{e}_\theta + v^z \boldsymbol{e}_z$, where $\boldsymbol{e}_r = {}^\top(\cos\theta, \sin\theta, 0)$, $\boldsymbol{e}_\theta = {}^\top(-\sin\theta, \cos\theta, 0)$ and $\boldsymbol{e}_z = {}^\top(0, 0, 1)$.

The problem (1.1)–(1.2) has a stationary solution (Couette flow) $u_{C,\varepsilon} = {}^\top(\rho_{C,\varepsilon}, \boldsymbol{v}_C)$:

$$\rho_{C,\varepsilon} = \rho_{C,\varepsilon}(r) = 1 + O(\varepsilon^2), \quad \boldsymbol{v}_C = v_C^\theta(r)\boldsymbol{e}_\theta.$$

Note that $\boldsymbol{v}_C$ represents the Couette flow for the incompressible Navier-Stokes equations:

$$\begin{cases} \operatorname{div} \boldsymbol{v} = 0, \\ \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nu \Delta \boldsymbol{v} + \nabla p = \boldsymbol{0} \end{cases} \tag{1.3}$$

on $\Omega_\alpha$ with the boundary condition (1.2).

One can show that if $\nu \gg 1$ and $0 < \varepsilon \ll 1$, then $u_{C,\varepsilon}$ is asymptotically stable. In this article we are interested in what happens in the stability problem of the Couette flow $u_{C,\varepsilon}$ when $\nu$ decreases.

To study tthe stability problem of the Couette flow $u_{C,\varepsilon}$, we rewrite (1.1) into the equations for the perturbation of the Couette flow. We denote the perturbation by $u = {}^\top(\phi, \boldsymbol{w}) = {}^\top(\varepsilon^{-2}(\rho - \rho_{C,\varepsilon}), \boldsymbol{v} - \boldsymbol{v}_C)$. Since the Taylor vortex is axisymmetric, we consider the *axisymmetric perturbation* $u = {}^\top(\phi, \boldsymbol{w})$, where

$$\phi = \phi(r, z, t), \; \boldsymbol{w} = w^r(r, z, t)\boldsymbol{e}_r + w^\theta(r, z, t)\boldsymbol{e}_\theta + w^z(r, z)\boldsymbol{e}_z, .$$

i.e., $\phi$, $w^j$ $(j = r, \theta, z)$ do not depend on the variable $\theta$.

It then follows that $\operatorname{div}(\phi \boldsymbol{v}_C) = 0$, and, hence, the perturbation $u$ is governed by the following system of equations:

$$\begin{cases} \partial_t \phi + \frac{1}{\varepsilon^2} \operatorname{div}(\rho_{C,\varepsilon} \boldsymbol{w}) = -\operatorname{div}(\phi \boldsymbol{w}), \\ \partial_t \boldsymbol{w} - \frac{\nu}{\rho_{C,\varepsilon}} \Delta \boldsymbol{w} - \frac{\nu+\nu'}{\rho_{C,\varepsilon}} \nabla \operatorname{div} \boldsymbol{w} + \nabla \left( \frac{p'(\rho_{C,\varepsilon})}{\rho_{C,\varepsilon}} \phi \right) \\ \qquad + \boldsymbol{v}_C \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_C = \boldsymbol{g}(\phi, \boldsymbol{w}, \partial_x \phi, \partial_x \boldsymbol{w}, \partial_x^2 \boldsymbol{w}; \varepsilon, \nu). \end{cases} \tag{1.4}$$

Here $\boldsymbol{g} = -\boldsymbol{w} \cdot \nabla \boldsymbol{w} + \varepsilon^2 \tilde{\boldsymbol{g}}(\phi, \boldsymbol{w}, \partial_x \phi, \partial_x \boldsymbol{w}, \partial_x^2 \boldsymbol{w}; \varepsilon, \nu)$ denotes the nonlinear terms. Recall that the periodic boundary condition in $z$ is included in the definition of $\Omega_\alpha$: $\phi$ and $\boldsymbol{w}$ are $\frac{2\pi}{\alpha}$-periodic in $z$. The boundary conditions on $r = \frac{\eta}{1-\eta}, \frac{1}{1-\eta}$ are

$$w^r = w^\theta = w^z = 0 \text{ on } r = \tfrac{\eta}{1-\eta}, \tfrac{1}{1-\eta}, \tag{1.5}$$

Furthermore, we impose the condition

$$\int_{\Omega_\alpha} \phi \, dx = 0, \tag{1.6}$$

which naturally follows from the conservation of mass.

# 2 Results

In this section we state the stability and bifurcation results for the compressible problem (1.1)–(1.2) obtained in [6].

We first introduce notation used in this paper. For $1 \le p \le \infty$ we denote by $L^p(\Omega_\alpha)$ the usual Lebesgue space over $\Omega_\alpha$ and its norm is denoted by $\| \cdot \|_p$. The $m$th order $L^2$ Sobolev space over $\Omega_\alpha$ is denoted by $H^m(\Omega_\alpha)$, and its norm is denoted by $\| \cdot \|_{H^m}$. The inner product of $L^2(\Omega_\alpha)$ is denoted by $(\cdot, \cdot)$, i.e.,

$$(f, g) = \int_{\Omega_\alpha} f(x) \overline{g(x)} dx.$$

Here $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

We set

$$\begin{aligned} H_0^1(\Omega_\alpha) &= \text{the } H^1(\Omega_\alpha)\text{-closure of } C_0^\infty(\Omega_\alpha), \\ H^{-1}(\Omega_\alpha) &= \text{the dual space of } H_0^1(\Omega_\alpha). \end{aligned}$$

We define $L^2_*(\Omega_\alpha)$ and $H^k_*(\Omega_\alpha)$ by

$$L^2_*(\Omega_\alpha) = \{f \in L^2(\Omega_\alpha); \int_{\Omega_\alpha} f(x)dx = 0\},$$

$$H^k_*(\Omega_\alpha) = H^k(\Omega_\alpha) \cap L^2_*(\Omega_\alpha) \ \ (k \geq 1).$$

We set

$$L^2_\sigma(\Omega_\alpha) = \{\boldsymbol{v} \in L^2(\Omega_\alpha)^3; \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega_\alpha, \ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega_\alpha} = 0\}.$$

Here and in what follows, $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega_\alpha$. It is known that

$$(L^2(\Omega_\alpha))^3 = L^2_\sigma(\Omega_\alpha) \oplus G^2(\Omega_\alpha),$$

where $G^2(\Omega_\alpha) = \{\nabla p; \, p \in H^1_*(\Omega)\}$ is orthogonal complement of $L^2_\sigma(\Omega_\alpha)$.

The orthogonal projection $\mathbb{P}$ from $L^2(\Omega_\alpha)^3$ onto $L^2_\sigma(\Omega_\alpha)$ is called the Helmholtz projection.

Let $X$ be a function space consisting functions $u = {}^\top(\phi, \boldsymbol{w})$ on $\Omega_\alpha$, where $\phi$ and $\boldsymbol{w}$ are scalar and vector fields on $\Omega_\alpha$, respectively. We denote by $X_{sym}$ the set of functions in $X$ that satisfy the following symmetries:

- axisymmetry:

$$\phi = \phi(r, z), \ \boldsymbol{w} = w^r(r, z)\boldsymbol{e}_r + w^\theta(r, z)\boldsymbol{e}_\theta + w^z(r, z)\boldsymbol{e}_z,$$

- reflection symmetry with respect to $z = 0$:

$$\phi(r, -z) = \phi(r, z), \ w^j(r, -z) = w^j(r, z) \ (j = r, \theta), \ w^z(r, -z) = -w^z(r, z).$$

Similarly, for a function space $Y$ of vector fields on $\Omega_\alpha$, we denote by $Y_{sym}$ the set of vector fields in $Y$ with the above symmetries.

We denote the resolvent set of an operator $A$ by $\rho(A)$ and the spectrum of $A$ by $\sigma(A)$.

To state our results, we next introduce linearized operators around the Couette flow. We define the linearized operator

$$\mathbb{L}_\nu : L^2_{\sigma,sym}(\Omega_\alpha) \to L^2_{\sigma,sym}(\Omega_\alpha)$$

around the Couette flow for the incompressible problem by

$$\mathbb{L}_\nu \boldsymbol{v} = -\nu\mathbb{P}\Delta\boldsymbol{v} + \mathbb{P}(\boldsymbol{v}_C \cdot \nabla\boldsymbol{v} + \boldsymbol{v} \cdot \nabla\boldsymbol{v}_C)$$

for $\boldsymbol{w} \in D(\mathbb{L}_\nu)$ with domain $D(\mathbb{L}_\nu) = [H^2(\Omega_\alpha) \cap H_0^1(\Omega_\alpha)]^3 \cap L^2_{\sigma,sym}(\Omega_\alpha)$.

The linearized operator

$$L_{\varepsilon,\nu} : H^1_{*,sym}(\Omega_\alpha) \times L^2_{sym}(\Omega_\alpha)^3 \to H^1_{*,sym}(\Omega_\alpha) \times L^2_{sym}(\Omega_\alpha)^3$$

for the compressible problem (1.4)–(1.6) is defined by

$$L_{\varepsilon,\nu} u = \begin{pmatrix} 0 & \frac{1}{\varepsilon^2}\mathrm{div}\,(\rho_{C,\varepsilon}\,\cdot\,) \\ \nabla\left(\frac{p'(\rho_{C,\varepsilon})}{\rho_{C,\varepsilon}}\,\cdot\right) & -\frac{\nu}{\rho_{C,\varepsilon}}\Delta - \frac{\nu+\nu'}{\rho_{C,\varepsilon}}\nabla\mathrm{div} + \boldsymbol{v}_C\cdot\nabla + {}^\top(\nabla\boldsymbol{v}_C)\cdot \end{pmatrix} \begin{pmatrix} \phi \\ \boldsymbol{w} \end{pmatrix}$$

for $u = {}^\top(\phi, \boldsymbol{w}) \in D(L_{\varepsilon,\nu})$ with domain $D(L_{\varepsilon,\nu}) = H^1_{*,sym}(\Omega_\alpha) \times \left[H^2_{sym}(\Omega_\alpha) \cap H^1_{0,sym}(\Omega_\alpha)\right]^3$.

We make the following assumption on the spectrum of the linearized operator $\mathbb{L}_\nu$ for the incompressible problem.

**Assumption (A)**: There are constants $\nu_c > 0$, $\kappa_0 > 0$ and $\Lambda_0 > 0$ such that for $|\nu - \nu_c| \ll 1$,

$$\rho(-\mathbb{L}_\nu) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -\kappa_0|\mathrm{Im}\,\lambda|^2 - \Lambda_0\} \setminus \{\lambda(\nu)\}.$$

Here $\lambda(\nu) \in \mathbb{R}$ is a simple eigenvalue satisfying $\lambda(\nu_c) = 0$ and $\frac{d\lambda}{d\nu}(\nu_c) < 0$.

**Remark 1** (i) It was proved by Velte ([12]) and Iudovich ([3]) that $\lambda(\nu_c) = 0$, $\frac{d\lambda}{d\nu}(\nu_c) \neq 0$ (for a.e. $\alpha > 0$).

(ii) Numerical computations and experiments support the Assumption (A) for physically relevant values of $\alpha$. See, e.g. [1, 7].

Under Assumption (A), the bifurcation of the Taylor vortex for the incompressible problem (1.3), (1.2) can be proved by applying the standard bifurcation theory ([2]).

**Proposition 1** ([12, 3, 7]) *For each* $\nu = \nu(\delta)$ $(|\delta| \ll 1)$, *the problem* (1.3), (1.2) *has a nontrivial stationary solution* $\boldsymbol{v}_\delta$ (incompressible Taylor vortex) *such that*

$$\begin{aligned} \nu(\delta) &= \nu_c - a\delta^2 + O(\delta^4), \\ \boldsymbol{v}_\delta &= \boldsymbol{v}_C + \delta(\boldsymbol{w}^{(0)} + \delta\boldsymbol{w}_\delta^{(1)}). \end{aligned}$$

*Here $a$ is a constant; $\boldsymbol{w}^{(0)}$ is the eigenfunction for the zero eigenvalue of $-\mathbb{L}_\nu$.*

**Remark 2** (i) the bifurcation of the Taylor vortex from the Couette flow was proved for the incompressible Navier-Stokes equations by Velte [12], Iudovich [3], Kirchgässner and Sorger [7] and etc. See the book [1] by Chossat and Iooss for the Taylor problem.

(ii) Numerical computations and experiments support that the constant $a$ satisfies $a > 0$ for physically relevant values of $\alpha$. See, e.g. [1, 7].

For sufficiently small Mach number $\epsilon$, we have the following result on the spectrum of the linearized operator $L_{\varepsilon,\nu}$.

**Theorem 2** ([6]) *There are constants $\varepsilon_0 > 0$, $\Lambda_1 > 0$ and $\nu_1 > 0$ such that the following assertion holds true. For each $0 < \varepsilon \leq \varepsilon_0$ there exists a critical value $\nu_c(\varepsilon)$ with $\nu_c(\varepsilon) \to \nu_c$ as $\varepsilon \to 0$ such that if $|\nu - \nu_c| \leq \nu_1$, then $\rho(-L_{\varepsilon,\nu}) \supset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -\Lambda_1\} \setminus \{\lambda_\varepsilon(\nu)\}$, where $\lambda_\varepsilon(\nu) \in \mathbb{R}$ is a simple eigenvalue satisfying $\lambda_\varepsilon(\nu_c(\varepsilon)) = 0$ and $\frac{\partial \lambda_\varepsilon}{\partial \nu}(\nu_c(\varepsilon)) < 0$.*

In view of Theorem 2 one could expect a stationary bifurcation from the Couette flow at $\nu = \nu_c(\varepsilon)$. However, the standard bifurcation theory is not applicable since the nonlinearity is not Fréchet differentiable due to the derivative loss in the term $-\mathrm{div}\,(\phi \boldsymbol{w})$. Nevertheless, we have the following bifurcation result.

**Theorem 3** ([6]) *Let $0 < \varepsilon \leq \varepsilon_0$. Then for each $\nu = \nu_\varepsilon(\delta)$ ($|\delta| \ll 1$), the problem (1.4)–(1.6) has a nontrivial stationary solution $u_{\delta,\varepsilon}$* (compressible Taylor vortex) *such that*

$$\begin{aligned} \nu_\varepsilon(\delta) &= \nu_c(\varepsilon) - a_\varepsilon \delta^2 + O(\delta^3), \\ u_{\delta,\varepsilon} &= \delta(U_\varepsilon^{(0)} + \delta U_{\delta,\varepsilon}^{(1)}). \end{aligned}$$

*Here $a_\varepsilon = a + O(\varepsilon^2)$ with the constant $a$ in Proposition 1; $U_\varepsilon^{(0)}$ is the eigenfunction for the zero eigenvalue of $-L_{\varepsilon,\nu_c(\varepsilon)}$.*

Theorem 3 can be proved in a similar manner to the argument in [4].

Our proof of Theorem 1 is outlined as follows. One can show that if $0 < \varepsilon \ll 1$, then $\sigma(-L_{\varepsilon,\nu}) \cap \{\lambda; |\mathrm{Re}\,\lambda| \leq \Lambda_0\}$ is decomposed into two parts $S_1 \cup S_2$, where $S_1 = \sigma(-L_{\varepsilon,\nu}) \cap \{\lambda; |\lambda| \leq O(1)\}$ is the *incompressible part* that is obtained by a perturbation of the incompressible spectrum $\sigma(-\mathbb{L}_\nu)$; and $S_2 = \sigma(-L_{\varepsilon,\nu}) \cap \{\lambda; |\mathrm{Im}\,\lambda| = O(\varepsilon^{-1})\}$ is the *compressible part* that consists of the spectra for acoustic modes (sound waves with propagation speed $O(\varepsilon^{-1})$). Due to the assumption on $\sigma(-\mathbb{L}_\nu)$, one can show that $S_1 = \{\lambda_\varepsilon(\nu)\}$. Since we consider axisymmetric perturbations, we can prove $\operatorname{Re} S_2 \le -\Lambda_1 < 0$ by using an argument similar to the one in [5] for the stability problem of stationary solution of the artificial compressible system.

**Remark 3** For general perturbations (i.e., without axisymmetric assumption), one can show the above decomposition by $S_1$ and $S_2$ with $S_1 = \{\lambda_\varepsilon(\nu)\}$ for $0 < \varepsilon \ll 1$. But, it is still open whether $\operatorname{Re} S_2 \le -\Lambda_1$ holds for the case of general perturbations.

## References


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