数理解析研究所講究録 第2090巻 2018年 132-143

A Compactness Theorem for Variational Inequalities of Parabolic Type

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Abstract. This paper is concerned with the weak solvability for fully nonlinear parabolic variational inequalities with time dependent convex constraints. As a possible approach to such problems, there is for instance the fixed point method of the Schauder type with appropriate compactness theorems. However, there has not been prepared any compactness theorem up to date that enables us the application of the fixed point method to variational inequalities of prabolic type. We have to start establishing a new compactness theorem for a wide class of prabolic variational inequalities.

1. Introduction

We consider a variational problem of quasi-linear parabolic type:

$$\begin{aligned}&\int_Q \{u_{1,t}(u_1-\xi_1)+u_{2,t}(u_2-\xi_2)\}dxdt\\&+\int_Q\{a_1(x,t,u)\nabla u_1\cdot\nabla(u_1-\xi_1)+a_2(x,t,u)\nabla u_2\cdot\nabla(u_2-\xi_2)\}dxdt \qquad (1.1)\\&\leq \int_Q\{f_1(u_1-\xi_1)+f_2(u_2-\xi_2)\}dxdt,\end{aligned}$$

$$\forall \xi := [\xi_1.\xi_2] \in L^2(0,T;H_0^1(\Omega)\times H_0^1(\Omega)) \text{ with } \xi(t)\in K(t) \text{ a.e. } t\in[0,T], \qquad (1.2)$$

$$u(x,0)=u_0(x) \ \text{ in } \Omega, \quad u=0 \text{ on } \Sigma, \qquad (1.3)$$

where Ω is a bounded domain in $\mathbf{R}^N$, $Q := \Omega\times(0,T)$, $0<T<\infty$, $\Gamma := \partial\Omega$, $\Sigma := \Gamma\times(0,T)$, $u := [u_1,u_2]$ and the diffusion coefficients $a_i(x,t,u)$, $i=1,2$, are strictly positive, bounded and continuous in $(x,t,u)\in\overline{Q}\times\mathbf{R}^2$ as well as a constraint set $K(t)$ is convex and closed in $H_0^1(\Omega)\times H_0^1(\Omega)$ satisfying some smoothness assumption in $t\in[0,T]$.

Functions $f := [f_1, f_2]$ and u_0 are prescribed in $L^2(Q) \times L^2(Q)$ and $K(0)$, respectively, as the data. Our claim is to construct a solution u of (1.1)-(1.3) in a weak sense such that

$$u \in C([0,T]; L^2(\Omega) \times L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega) \times H_0^1(\Omega)), \quad u(t) \in K(t), \text{ a.e. } t \in [0,T].$$

In the case without constraint, namely $K(t) = H_0^1(\Omega) \times H_0^1(\Omega)$, our problem is the usual initial-boundary value problem for parabolic quasi-linear system of PDEs:

$$u_{1,t} - \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\left(a_1(x,t,u)\frac{\partial u_1}{\partial x_k}\right) = f_1(x,t) \text{ in } Q,$$

$$u_{2,t} - \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\left(a_2(x,t,u)\frac{\partial u_2}{\partial x_k}\right) = f_2(x,t) \text{ in } Q.$$

For the solvability a huge number of results have been established (cf. [1, 19]), for instance, the Leray-Schauder principle together with some compactness theorems, such as [2, 22].

In connection with quasi-linear variational inequalities, the concept of nonlinear monotone mappings was generalized to several classes of nonlinear mappings of monotone type, for instance, semimonotone [20], pseudomonotone [3, 8, 14], and furthermore $L-$pseudomonotone mappings [4]. Especially the last class is available for parabolic variational inequalities and its simplified form is mentioned as follows: Given a linear maximal monotone mapping L from $D(L)$ in a reflexive Banach space X into its dual space X^* and a single-valued bounded mapping $A : D(A) = X \to X^*$, we say that A is L-pseudomonotome, if the following statement holds:

$$\begin{cases} \text{if } w_n \to w \text{ weakly in } X,\ \exists \ell_n^* \in Lw_n \text{ such that } \liminf_{n\to\infty} |\ell_n^*|_{X^*} < \infty, \\ Aw_n \to h \text{ weakly in } X^* \text{ and } \lim_{n\to\infty}\langle Aw_n, w_n\rangle \le \langle h, w\rangle, \text{ then } Aw = h. \end{cases}$$

Under some coerciveness assumption, it was proved in [5] that the range of $L + A$ is the whole of X^*. In this theory the linearity of L is crucial and it seems difficult to remove it. In a typical application of this theory to parabolic problems the linear maximal monotone L is the time-derivative $\frac{d}{dt}$.

Our model problem (1.1)-(1.3) is formally written in the space $L^2(0,T; H^{-1}(\Omega) \times H^{-1}(\Omega))$ as

$$f \in Lu + A(u,u), \quad u(0) = u_0,$$

by taking as L the mapping $L := \frac{d}{dt} + \partial I_{K(t)}(\cdot) : D(L) \subset X := L^2(0,T; H_0^1(\Omega) \times H_0^1(\Omega)) \to X^* = L^2(0,T; H^{-1}(\Omega) \times H^{-1}(\Omega))$ and as A the mapping $A(v,u) : D(A) = X \to X^*$ given by

$$\langle A(v,u), [\xi,\eta]\rangle_{X^*,X} = \int_Q \{a_1(x,t,v)\nabla u_1 \cdot \nabla\xi_1 + a_2(x,t,v)\nabla u_2 \cdot \nabla\xi_2\} dxdt,$$

$$\text{for } u := [u_1,u_2], \ v = [v_1,v_2], \ \xi = [\xi_1,\xi_2] \in X,$$

where $\langle\cdot,\cdot\rangle_{X^*,X}$ stands for the duality between X^* and X. We see that L is maximal monotone from $D(L) \subset X$ into X^*, but L is nonlinear in general. Since 1970, it remains for us to set up an abstract approach to such a quasi-linear parabolic variational inequality as our model problem. In this paper we establish a new approach to parabolic variational inequalities with time-dependent constraints $\{K(t)\}$, based on a new compactness theorem derived from the total variation estimates for solutions of parabolic variational inequalities (cf. [13]).

There is a different approach to nonlinear variational inequalities of parabolic type with time-independent convex constraint in [1] in which the time-discretization method was employed and a compactness theorem was estsblished to ensure the strong convergence of time-discretized approximation schemes in time. This idea seems available to the case of time-dependent convex constraints.

2. Time-dependent convex sets

Throughout this paper, let H be a Hilbert space and V be a strictly convex reflexive Banach space such that V is dense in H and the injection from V into H is continuous. In this case, by identifying H with its dual space, we have:

$$V \subset H \subset V^* \text{ with continuous embeddings.}$$

For simplicity, we assume that the dual space V^* is strictly convex. Therefore the duality mapping F from V into V^* associated with gauge function $r \to |r|^{p-1}$ is singlevalued and demicontinuous from V into V^*, where p is a fixed number with $1 < p < \infty$.

For the sake of simplicity for notation, we write $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_{V^*,V}$

Let $K := \{K(t)\}_{t\in[0,T]}$ be a family of non-empty, closed and convex sets in V such that there are functions $\alpha \in W^{1,2}(0,T)$ and $\beta \in W^{1,1}(0,T)$ satisfying the following property: for any $s, t \in [0,T]$ and any $z \in K(s)$ there is $\tilde{z} \in K(t)$ such that

$$|\tilde{z} - z|_H \leq |\alpha(t) - \alpha(s)|(1 + |z|_V^{\frac{p}{2}}), \quad |\tilde{z}|_V^p - |z|_V^p \leq |\beta(t) - \beta(s)|(1 + |z|_V^p). \tag{2.1}$$

We denote by $\Phi(\alpha, \beta)$ the set of all such families $K := \{K(t)\}$, and put

$$\Phi_S := \bigcup_{\alpha \in W^{1,2}(0,T),\ \beta \in W^{1,1}(0,T)} \Phi(\alpha, \beta).$$

We call Φ_S the strong class of time-dependent convex sets.

Given $K := \{K(t)\} \in \Phi_S$, we consider the following time-dependent convex function

$$\varphi_K^t(z) := \frac{1}{p}|z|_V^p + I_{K(t)}(z),$$

where $I_{K(t)}(\cdot)$ is the indicator function of $K(t)$ on H. For each $t \in [0,T]$, $\varphi_K^t(\cdot)$ is proper, l.s.c. and strictly convex on H and on V. By the general theory on nonlinear evolution equations generated by time-dependent subdifferentials, condition (2.1) is sufficient in order that for any $u_0 \in \overline{K(0)}$ (the closure of $K(0)$ in H) and $f \in L^2(0,T;H)$ the Cauchy problem with real parameter $\lambda \in (0,1]$

$$u'(t) + \lambda \partial \varphi_K^t(u(t)) \ni f(t), \quad u(0) = u_0, \text{ in } H,$$

admits a unique solution u such that $u \in C([0,T];H) \cap L^p(0,T;V)$ with $u(0) = u_0$, $t^{\frac{1}{2}}u' \in L^2(0,T;H)$ and $t \to t\varphi_K^t(u(t))$ is bounded on $(0,T]$, where $\partial\varphi_K^t$ denotes the subdifferential of φ_K^t in H. In particular, if $u_0 \in K(0)$, then $u' \in L^2(0,T;H)$ and $t \to \varphi_K^t(u(t))$ is absolutely continuous on $[0,T]$.

Next, we introduce a weak class of time-dependent convex sets. Let $\mathcal{R}_0$ be a bounded, linear and self-adjoint operator in H as well as bounded and linear in V, and let σ_0 be a function in $W^{1,p'}(0,T;H) \cap C([0,T];V)$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then there exists an increasing continuous function $c_0(\varepsilon)$ of $\varepsilon \in (0,1]$ such that

$$|z + \varepsilon\mathcal{R}_0 z + \varepsilon\sigma_0(t)|_V^p \leq |z|_V^p + c_0(\varepsilon)\{1 + |z|_V^p + |\sigma_0(t)|_V^p\},$$

$$\forall t \in [0,T], \ \forall z \in V, \ \forall \varepsilon \in (0,1];$$

in fact, for instance, we can take $c_0(\varepsilon) = 12p(\|\mathcal{R}_0\| + \|\mathcal{R}_0\|^p + 1)\varepsilon$, where $\|\mathcal{R}_0\|$ denotes the operator norm of $\mathcal{R}_0$ in the space of all bounded linaer operators from V into itself.

Definition 2.1 (cf. [12]) Let $\mathcal{R}_0$ and σ_0 be as above. Then we define a class $\Phi_W := \Phi_W(\mathcal{R}_0, \sigma_0)$ by: $\{K(t)\} \in \Phi_W$ if and only if $K(t)$ is a closed and convex set in V for all $t \in [0,T]$ and there exists a sequence $\{K_n := \{K_n(t)\}\}_{n\in\mathbf{N}} \subset \Phi_S$ such that for any $\varepsilon \in (0, \varepsilon_0]$ there is a positive integer N_ε satisfying

$$(I + \varepsilon\mathcal{R}_0)K_n(t) + \varepsilon\sigma_0(t) \subset K(t), \ \ (I + \varepsilon\mathcal{R}_0)K(t) + \varepsilon\sigma_0(t) \subset K_n(t),$$

$$\forall t \in [0,T], \ \ \forall n \geq N_\varepsilon.$$

In this case, it is said that $\{K_n(t)\}$ converges to $\{K(t)\}$ as $n \to \infty$, which is denoted by "$K_n(t) \Longrightarrow K(t)$" on $[0,T]$ in this paper.

It is easy to see that Φ_W is strictly larger than Φ_S, in general. Now, given $\{K(t)\} \in \Phi_W$, we put

$$\mathcal{K} := \{v \in L^p(0,T;V) \mid v(t) \in K(t) \text{ for a.e. } t \in [0,T]\}$$

and

$$\mathcal{K}_0 := \{\eta \in \mathcal{K} \mid \eta' \in L^{p'}(0,T;V^*)\}.$$

Next, we introduce the time-derivative with constraint $K(t)$ and initial datum $u_0 \in \overline{K(0)}$.

Definition 2.2. Let $\{K(t)\} \in \Phi_W$ and $u_0 \in \overline{K(0)}$. Then we define an operator L_{u_0} whose graph $G(L_{u_0})$ is given in $L^p(0,T;V) \times L^{p'}(0,T;V^*)$, $\frac{1}{p} + \frac{1}{p'} = 1$ $(1 < p < \infty)$, as follows: $[u,f] \in G(L_{u_0})$ if and only if

$$f \in L^{p'}(0,T;V^*), \ u \in \mathcal{K}$$

and

$$\int_0^T \langle \eta' - f, u - \eta \rangle dt \leq \frac{1}{2}|u_0 - \eta(0)|_H^2, \ \ \forall \eta \in \mathcal{K}_0.$$

The most important property of L_{u_0} is given in the next theorem.

Theorem 2.1. *Let $\{K(t)\} \in \Phi_W$ and $u_0 \in \overline{K(0)}$. Then L_{u_0} is maximal monotone from $D(L_{u_0}) \subset L^p(0,T;V)$ into $L^{p'}(0,T;V^*)$, and the domain $D(L_{u_0})$ is included in the set $\{u \in C([0,T];H) \cap \mathcal{K} \mid u(0) = u_0\}$.*

In the proof of Theorem 2.1 we observe the following characterization of L_{u_0}: $f \in L_{u_0}u$ if and only if $u \in \mathcal{K} \cap C([0,T];H)$ with $u(0) = u_0$, $f \in L^{p'}(0,T;V^*)$ and there exist sequences $\{K_n := \{K_n(t)\}\} \subset \Phi_S$, $\{u_n\}$ and $\{f_n\}$ such that $u_n \in \mathcal{K}_n := \{v \in L^p(0,T;V) \mid v(t) \in K_n(t)$ for a.e. $t \in [0,T]\}$, $u_n' \in L^{p'}(0,T;V^*)$ (hence $u_n \in C([0,T];H)$), $f_n \in L^{p'}(0,T;V^*)$ and

$$K_n(t) \Longrightarrow K(t) \text{ on } [0,T],$$

$$u_n \to u \text{ in } C([0,T];H) \text{ and weakly in } L^p(0,T;V),$$

$$\int_0^T \langle u_n' - f_n, u_n - v\rangle dt \le 0, \ \forall v \in \mathcal{K}_n, \ \forall n,$$

$$f_n \to f \text{ weakly in } L^{p'}(0,T;V^*), \quad \limsup_{n\to\infty} \int_0^T \langle f_n, u_n\rangle dt \le \int_0^T \langle f, u\rangle dt.$$

Summarizing the structure of operator L_{u_0}, we have the following theorem.

Theorem 2.2. *Let $\{K(t)\} \in \Phi_W$. Then we have:*

(a) Let $u_0 \in \overline{K(0)}$ and $f \in L_{u_0}u$. Then, for any s, $t \in [0,T]$ with $s \le t$,

$$\int_s^t \langle \eta' - f, u - \eta\rangle d\tau + \frac{1}{2}|u(t) - \eta(t)|_H^2 \le \frac{1}{2}|u(s) - \eta(s)|_H^2, \quad \forall \eta \in \mathcal{K}_0.$$

(b) Let $u_{i0} \in \overline{K(0)}$, and $f_i \in L_{u_{i0}}u_i$ for $i = 1, 2$. Then, for any s, $t \in [0,T]$ with $s \le t$,

$$\frac{1}{2}|u_1(t) - u_2(t)|_H^2 \le \frac{1}{2}|u_1(s) - u_2(s)|_H^2 + \int_s^t \langle f_1 - f_2, u_1 - u_2\rangle d\tau.$$

Remark 2.1. In Hilbert spaces similar operators to L_{u_0} we considered in the time-independent case $K = K(t)$ (cf. [6]) and it was generalized to the time-dependent case $K(t)$ (cf. [17]). In the Banach space set-up (cf. [16]), the similar results were discussed, too.

Remark 2.2. Theorem 2.1 gives a generalization of the results of [16, 17] in a class of weak variational inequalities. Morover it is expected to compose L_{u_0} for various constraint set $K(t)$ in a much wider class Φ_W than in this paper, for instance the class in [18].

3. A compactness theorem

In this section, let V, H and V^* be the same as in the previous section. In order to avoid some irrelevant abstract arguments we suppose that H, V and V^* are separable.

Also we assume that V is compactly embedded in H and introduce another separable and reflexive Banach space W such that W is a dense subspace of V embedded continuously in V. Hence the injection from W into H is compact. We denote by C_W an embedding constant from W into V and H, namely

$$|z|_V \le C_W |z|_W, \quad |z|_H \le C_W |z|_W, \quad \forall z \in W. \tag{3.1}$$

For any function $w : [0,T] \to W^*$, we denote the total variation of w by $\mathrm{Var}_{W^*}(w)$, which is defined by

$$\mathrm{Var}_{W^*}(w) := \sup_{\substack{\eta \in C_0^1(0,T;W), \\ |\eta|_{L^\infty(0,T;W)} \le 1}} \int_0^T \langle w, \eta' \rangle_{W^*,W} dt.$$

We refer to [7] or [10] for the fundamental properties of total variation functions.

Theorem 3.1. *Let $\{K(t)\} \in \Phi_W$, $u_0 \in \overline{K(0)}$ and assume that there is a positive number κ such that*

$$\kappa B_W(0) \subset K(t), \quad \forall t \in [0,T], \tag{3.2}$$

where $B_W(0) := \{w \in W \mid |w|_W \le 1\}$. Let M_0 be any positive number. Then

$$Z(M_0) := \left\{ u \in D(L_{u_0}) \;\middle|\; \begin{array}{l} |u|_{L^p(0,T;V)} \le M_0, \ \exists f \in L_{u_0} u \text{ such that} \\ \sup_{t \in [0,T]} \int_0^t \langle f, u \rangle d\tau \le M_0, \ |f|_{L^1(0,T;W^*)} \le M_0 \end{array} \right\} \tag{3.3}$$

is relatively compact in $L^p(0,T;H)$.

We begin with the following lemma that is crucial for the proof of Theorem 3.1.

Lemma 3.1. *Let $\{K(t)\} \in \Phi_W$, $u_0 \in \overline{K(0)}$ and assume (3.2) holds. Further let M_0 be any positive number. Then there exists a positive constant $C^* := C^*(\kappa, M_0, |u_0|_H)$, depending only on κ, M_0 and $|u_0|_H$, such that*

$$|u|_{C([0,T];H)} \le C^*, \quad \mathrm{Var}_{W^*}(u) \le C^*, \tag{3.4}$$

for all $u \in Z(M_0)$.

Proof. Let u be any element in $Z(M_0)$, and take a function $f \in L_{u_0} u$ such that

$$\sup_{t \in [0,T]} \int_0^t \langle f, u \rangle d\tau \le M_0, \ |f|_{L^1(0,T;W^*)} \le M_0. \tag{3.5}$$

By (a) of Theorem 2.2, we have

$$\frac{1}{2}|u(t) - \eta(t)|_H^2 + \int_0^t \langle \eta' - f, u - \eta \rangle d\tau \le \frac{1}{2}|u_0 - \eta(0)|_H^2, \quad \forall \eta \in \mathcal{K}_0, \ \forall t \in [0,T]. \tag{3.6}$$

Now, note that $0 \in \mathcal{K}_0$ by (3.2). Take $\eta \equiv 0$ in (3.6) to get

$$\frac{1}{2}|u(t)|_H^2 \le \int_0^t \langle f, u \rangle d\tau + \frac{1}{2}|u_0|_H^2, \quad \forall t \in [0,T].$$

Hence it follows from (3.5) that

$$|u(t)|_H \le \{|u_0|_H^2 + 2M_0\}^{\frac{1}{2}} \le |u_0|_H + \sqrt{2M_0}, \quad \forall t \in [0,T]. \tag{3.7}$$

Next, let η be any function in $C_0^1(0,T;W)$ satisfying $|\eta|_{L^\infty(0,T;W)} > 0$ and put $\tilde{\eta}(t) := \frac{\eta(t)}{|\eta|_{L^\infty(0,T;W)}}$. Then, by (3.2), $\pm\kappa\tilde{\eta} \in \mathcal{K}_0$, so that it follows from (3.6) that

$$\int_0^T \langle \pm\kappa\tilde{\eta}' - f, u \mp \kappa\tilde{\eta}\rangle dt \le \frac{1}{2}|u_0|_H^2,$$

whence

$$\left|\int_0^T \langle u, \tilde{\eta}'\rangle dt\right| \le \left|\int_0^T \langle f, \tilde{\eta}\rangle dt\right| + \frac{1}{\kappa}\int_0^T \langle f, u\rangle dt + \frac{1}{2\kappa}|u_0|_H^2.$$

Hence,

$$\left|\int_0^T \langle u, \tilde{\eta}'\rangle dt\right| \le |f|_{L^1(0,T;W^*)}|\tilde{\eta}|_{L^\infty(0,T;W)} + \frac{1}{\kappa}\int_0^T \langle f, u\rangle d\tau + \frac{1}{2\kappa}|u_0|_H^2$$

It is easy to obtain from the above inequality that

$$\left|\int_0^T \langle u, \eta'\rangle dt\right| \le \left(M_0 + \frac{1}{\kappa}M_0 + \frac{1}{2\kappa}|u_0|_H^2\right)|\eta|_{L^\infty(0,T;W)}$$

for all $\eta \in C_0^1(0,T;W)$. This shows that

$$\mathrm{Var}_{W^*}(u) \le M_0 + \frac{1}{\kappa}M_0 + \frac{1}{2\kappa}|u_0|_H^2.$$

By this inequality and (3.7), we obtain (3.4) with $C^* := |u_0|_H + \sqrt{2M_0} + M_0 + \frac{1}{\kappa}M_0 + \frac{1}{2\kappa}|u_0|_H^2$. □

Lemma 3.2. *Let M_1 be any positive number and let $\{u_n\}$ be any sequence of functions from $[0,T]$ into W^* such that $u_n \in L^p(0,T;V) \cap L^\infty(0,T;H)$*

$$|u_n|_{L^p(0,T;V)} \le M_1, \quad |u_n|_{L^\infty(0,T;H)} \le M_1, \quad \mathrm{Var}_{W^*}(u_n) \le M_1, \quad n = 1, 2, \cdots. \tag{3.8}$$

Then there are a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in L^p(0,T;V) \cap L^\infty(0,T;H)$ such that $u_{n_k}(t) \to u(t)$ weakly in H for every $t \in [0,T]$ as $k \to \infty$. Hence $u_{n_k}(t) \to u(t)$ in W^* for every $t \in [0,T]$ and $u_{n_k} \to u$ in $L^q(0,T;W^*)$ for every $q \in [1,\infty)$ as $k \to \infty$.

Proof. Since W is separable, there is a countable dense subset W_0 in W. Now, we consider a sequence of real valued functions $A_n(t,\xi) := (u_n(t),\xi)_H$ $(= \langle u_n(t),\xi\rangle_{W^*,W})$ on $[0,T]$ for each $\xi \in W_0$. Then we note from (3.8) that the total variation of $A_n(t,\xi)$ is bounded by $M_1|\xi|_W$. Hence from the Helly selection theorem (cf. [10 ; Section 5.2.3]) it follows that there is a subsequence $\{n_k\}$, depending on $\xi \in W_0$, such that $A_{n_k}(t,\xi)$ converges to a function $A_0(t,\xi)$ pointwise on $[0,T]$ and its total variation is not larger than $M_1|\xi|_W$.

Since W_0 is countable in W, by using extensively the above Helly selection theorem we can extract a subsequence, denoted by the same notation as $\{n_k\}$ again, and a function $A_0(t,\xi)$ on $[0,T]\times W_0$ such that

$$A_{n_k}(t,\xi)\to A_0(t,\xi) \text{ as } k\to\infty,\quad \forall t\in[0,T],\ \forall\xi\in W_0. \tag{3.9}$$

Furthermore, by density, this convergence (3.9) can be extended to all $\xi\in W$. Also, the functional $A_{n_k}(t,\xi)$ is linear in ξ and uniformly bounded by (3.1), i.e.

$$|A_{n_k}(t,\xi)|\le M_1|\xi|_H\le M_1C_W|\xi|_W,\quad \forall t\in[0,T],\ \forall\xi\in W.$$

This implies that $A_0(t,\xi)$ is linear and bounded in $\xi\in W$ and $|A_0(t,\xi)|\le M_1|\xi|_H$ for all $\xi\in W$ and $t\in[0,T]$. As a consequence, by the Riesz representation theorem, there is a function $u:[0,T]\to H$ with $|u(t)|_H\le M_1$ for all $t\in[0,T]$ such that

$$A_0(t,\xi)=(u(t),\xi)_H,\quad \forall\xi\in H,\ \forall t\in[0,T].$$

Now it is clear by (3.9) that $u_{n_k}(t)\to u(t)$ weakly in H for $t\in[0,T]$ as $k\to\infty$. Finally, by the compactness of the injection from H into W^*, we see that $u_{n_k}(t)\to u(t)$ in W^* for $t\in[0,T]$ and hence $u_{n_k}\to u$ in $L^q(0,T;W^*)$ for all $q\in[1,\infty)$ as $k\to\infty$. □

Proof of Theorem 3.1. We first note from Lemma 3.1 and (3.3) that

$$Z(M_0)\subset\mathcal{X}:=\{u \mid |u|_{L^p(0,T;V)}\le M_0,\ |u|_{L^\infty(0,T;H)}\le C^*,\ \mathrm{Var}_{W^*}(u)\le C^*\},$$

where M_0 and C^* are the same constants as in Lemma 3.1. Therefore it is enough to prove the compactness of $\mathcal{X}$ in $L^p(0,T;H)$; note that $\mathcal{X}$ is closed and convex in $L^p(0,T;V)$.

Let $\{u_n\}$ be any sequence in the set $\mathcal{X}$. Then, by Lemma 3.2, there is a subsequence $\{u_{n_k}\}$ and a function $u\in L^\infty(0,T;H)$ such that $u_{n_k}(t)\to u(t)$ weakly in H for every $t\in[0,T]$ as $k\to\infty$. By the injection compactness from H into W^* we have that

$$u_{n_k}\to u \text{ in } L^p(0,T;W^*) \text{ as } k\to\infty. \tag{3.10}$$

and that $|u_{n_k}|_{L^p(0,T;V)}\le M_0$ and $|u|_{L^p(0,T;V)}\le M_0$.

Here we recall the Aubin lemma [3] (or [25; Lemma 5.1]): for each $\delta>0$ there is a positive constant C_δ such that

$$|z|_H^p\le\delta|z|_V^p+C_\delta|z|_{W^*}^p,\quad \forall z\in V.$$

By making use of this inequality for $z=u_{n_k}(t)-u(t)$, we get

$$\int_0^T|u_{n_k}(t)-u(t)|_H^p dt\le\delta(2M_0)^p+C_\delta\int_0^T|u_{n_k}(t)-u(t)|_{W^*}^p dt.$$

On account of (3.10), letting $k\to\infty$ gives that

$$\limsup_{k\to\infty}|u_{n_k}-u|_{L^p(0,T;H)}^p\le\delta(2M_0)^p.$$

Since $\delta > 0$ is arbitrary, we conclude that $u_{n_k} \to u$ in $L^p(0,T;H)$. □

Remark 3.1. In the case of $K(t) = W$ for all $t \in [0,T]$, $f \in L_{u_0}u$ implies that $u' = f \in L^{p'}(0,T;W^*)$. Therefore, Theorem 3.1 says that the set

$$\{u \mid |u|_{L^p(0,T;V)} \le M_0, \ |u'|_{L^{p'}(0,T;W^*)} \le M_0\}$$

is relatively compact in $L^p(0,T;H)$ for each finite positive constant M_0. This is nothing but a typical case of the Aubin compactness theorem [2]. Also, see [21] for various applications.

Remark 3.2. A compactness result of the Aubin type was extended in [15] to the case when $\{K(t)\} \in \Phi_S$ and $K(t)$ is a closed linear subspace of V for any $t \in [0,T]$. We refer to [9] for a further generalization to the Dubinskii's type, too.

4. Perturbations of semimonotone type

We assume that H, V and W be the same as in the previous section; V is dense in H with compact injection and W is dense in V with continuous injection.

Let $A(t,v,u)$ be a singlevalued mapping from $[0,T] \times H \times V$ into V^*, and assume that:

(a) (Boundedness) There are positive constants c_1, c_2 such that

$$|A(t,v,u)|_{V^*} \le c_1 |u|_V^{p-1} + c_2, \quad \forall v \in H, \ \forall u \in V, \ \forall t \in [0,T].$$

(b) (Coerciveness) There are positive constants c_3, c_4 such that

$$\langle A(t,v,u), u\rangle \ge c_3 |u|_V^p - c_4, \quad \forall v \in H, \ \forall u \in V, \ \forall t \in [0,T].$$

(c) (Semimonotonicity) For each $v \in H$ and $t \in [0,T]$, the mapping $u \to A(t,v,u)$ is demicontinuous from $D(A(t,v,\cdot)) = V$ into V^* and monotone, namely

$$\langle A(t,v,u_1) - A(t,v,u_2), u_1 - u_2\rangle \ge 0, \quad \forall u_1, \ u_2 \in V,$$

Moreover, for each $u \in V$ the mapping $(t,v) \to A(t,v,u)$ is continuous from $[0,T] \times H$ into V^*.

We have the following perturbation result of L_{u_0}.

Theorem 4.1. *Let $\mathcal{A} := \mathcal{A}(v,u)$ be an operator from $L^p(0,T;V)$ into $L^{p'}(0,T;V^*)$ given by*

$$[\mathcal{A}(\sqsubseteq,\sqcap)](t) := A(t,v(t),u(t)), \quad \forall v, \ u \in L^p(0,T;V).$$

Let $\{K(t)\} \in \Phi_W$ and $u_0 \in \overline{K(0)}$. Then, for any $f \in L^p(0,T;V^*)$ there exists a function $u \in D(L_{u_0})$ such that

$$f \in L_{u_0}u + \mathcal{A}(u,u).$$

The precise proof is refrred to [13].

(Application to the model problem (1.1)-(1.3))
We use our abstract theorems in the set-up

$$H := L^2(\Omega) \times L^2(\Omega),\ V := H_0^1(\Omega) \times H_0^1(\Omega),\ W := W_0^{1,q} \times W_0^{1,q},\ N < q < \infty.$$

Hence $V^* = H^{-1}(\Omega) \times H^{-1}(\Omega)$, $V \subset H \subset V^* \subset W^*$ and $W \subset C(\overline{\Omega}) \times C(\overline{\Omega})$. Let $\psi = \psi(x,t)$ be an obstacle function prescribed in $C(\overline{Q})$ so that $\psi \ge c_\psi$ on $\overline{Q}$ for a positive constant c_ψ, and define a constraint set $K(t)$ by

$$K(t) := \{[\xi,\eta] \in V \mid |\xi| + |\eta| \le \psi(\cdot,t) \text{ a.e. in } \Omega\},\quad \forall t \in [0,T].$$

In case ψ is in $C(\overline{Q})$, it is known (cf. [12] or [18]) that $\{K(t)\}$ belongs to the weak class Φ_W. Therefore, on account of Theorem 2.1, the maximal monotone mapping L_{u_0} is well defined for any given $u_0 := [u_{10}, u_{20}] \in \overline{K(0)}$. Since any function of $B_W(0)$ is uniformly bounded in $C(\overline{\Omega}) \times C(\overline{\Omega})$, it is easy to see that

$$\kappa B_W(0) \subset K(t),\quad \forall t \in [0,T]$$

for a certain positive constant $\kappa (< c_\psi)$, namely condition (3.2) is satisfied. Also, we define a nonlinear mapping $A(t,v,u) : [0,T] \times H \times V \to V^*$ by

$$\langle A(t,v,u), \xi\rangle := \int_\Omega \{a_1(x,t,v)\nabla u_1 \cdot \nabla \xi_1 + a_2(x,t,v)\nabla u_2 \cdot \nabla \xi_2\}dx,$$

$$v := [v_1, v_2] \in H,\ u := [u_1,u_2] \in V,\ \xi = [\xi_1,\xi_2] \in V,\ t \in [0,T],$$

where $a_1(x,t,v)$ and $a_2(x,t,v)$ are continuous functions on $\overline{\Omega} \times [0,T] \times \mathbf{R}^2$ and

$$c_* \le a_i(x,t,v) \le c^*,\quad \forall (x,t,v) \in \overline{\Omega} \times [0,T] \times \mathbf{R}^2,\ i = 1,2,$$

for positive constants c_*, c^*. Under the above assumptions, we easily check the conditions (a), (b) and (c). Accordingly we can apply Theorems 4.1 to solve our model problem for given data $u_0 := [u_{01}, u_{02}] \in \overline{K(0)}$ and $f = [f_1, f_2] \in L^2(0,T;V^*)$ in the form

$$f \in L_{u_0}u + \mathcal{A}(u,u).$$

This functional inclusion is written in the following weak variational form:

$$u := [u_1,u_2] \in \mathcal{K} \cap C([0,T];H),\ u(0) = u_0;$$

$$\begin{aligned}
&\int_Q \{\xi_{1,t}(u_1 - \xi_1) + \xi_{2,t}(u_2 - \xi_2)\}dxdt \\
&\qquad + \int_Q \{a_1(x,t,u)\nabla u_1 \cdot \nabla(u_1 - \xi_1) + a_2(x,t,u)\nabla u_2 \cdot \nabla(u_2 - \xi_2)\}dxdt \\
&\le \int_Q \{f_1(u_1 - \xi_1) + f_2(u_2 - \xi_2)\}dxdt + \frac{1}{2}\{|u_{10} - \xi_1(0)|^2_{L^2(\Omega)} + |u_{20} - \xi_2(0)|^2_{L^2(\Omega)}\},
\end{aligned}$$

$$\forall \xi = [\xi_1, \xi_2] \in \mathcal{K} \cap W^{1,2}(0,T;H).$$

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