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Dirichlet series with periodic coefficients in function fields

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1. Introduction

Sarvadaman Chowla [7] proved that if p is an odd prime, then $\cot(2\pi j/p)$ $(j=1,\ldots,(p-1)/2)$ are linearly independent over the field of rational numbers. This result follows from the non-vanishing of the Dirichlet L-series $L(s,\chi)$ at $s=1$, when χ is a Dirichlet character with $\chi(-1)=-1$. For another proof of Chowla's theorem, we refer the reader to [1, 2, 10, 12]. We note that Chowla's result was generalized in [13, 16].

In [6], which was written in 1964, Chowla raised the following question:

We assume that p is a prime and f is a non-zero function defined on the integers with integer values and period p. Then, does the infinite series

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}$$

never vanish?

The Chowla question is valid whenever the series converges, which is equivalent to the condition $\sum_{a=1}^{p} f(a)=0$. Concerning this question, Baker, Birch, and Wirsing [4] proved the following:

Theorem 1 (Baker–Birch–Wirsing [4, 13]). *Let m be a positive integer and f a non-zero function defined on the integers with algebraic values and period m such that*

(i) *$f(r)=0$ if $1<\gcd(r,m)<m$.*
(ii) *The m-th cyclotomic polynomial Ψ_m is irreducible over $\mathbb{Q}(f(1),\ldots,f(m))$.*

Then,

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}\neq 0.$$

Let $\mathbb{C}_\infty$ be the completion of an algebraic closure of the field $\mathbb{F}_q((T^{-1}))$. The Carlitz exponential function $e(z)$, which is defined over $\mathbb{C}_\infty$, is given by

$$e(z)=z+\sum_{n=1}^{\infty}\frac{z^{q^n}}{(T^{q^n}-T^{q^{n-1}})\cdots(T^{q^n}-T)}. \tag{1.1}$$

Then, its reciprocal $c(z):=e(z)^{-1}$ is analogous to $\cot z$. In this report, using $c(z)$, we establish an analog of Chowla's theorem over function fields. As an application, we give an analog of the Baker–Birch–Wirsing theorem about the non-vanishing of Dirichlet series with periodic coefficients at $s=1$ in the function field setup with a parity condition.

2. Some functions in function fields

Let $\mathbb{F}_q$ be the finite field with q elements, where q is a power of the prime number p. Let $A = \mathbb{F}_q[T]$ and $K = \mathbb{F}_q(T)$. Let $K_\infty = \mathbb{F}_q((T^{-1}))$ be the completion of K at $\infty = (T^{-1})$, and let $\mathbb{C}_\infty$ be the completion of an algebraic closure $\overline{K}$ of K_∞. For a ring R, R^* denotes the unit group of R.

2.1. **The Carlitz exponential.** We denote by $A\{\tau\}$ the twisted polynomial ring whose multiplication is defined by $\tau a = a^q \tau$ $(a \in A)$. The $\mathbb{F}_q$-linear ring homomorphism $\rho : A \to A\{\tau\}$, defined by $1 \mapsto \tau^0$ and $T \mapsto \rho_T = T\tau^0 + \tau$, is called the Carlitz A-module. With each $N \in A \setminus \{0\}$, ρ associates an additive polynomial $\rho_N(x)$ given by $\rho_N(x) := \rho_N(\tau)(x) \in A[x]$. This is called the Carlitz N-polynomial. For $N \in A \setminus \{0\}$, let $\rho[N] = \{\alpha \in \mathbb{C}_\infty \mid \rho_N(\alpha) = 0\}$ be the set of Carlitz N-torsion points. The set $\rho[N]$ is a cyclic A-module and its generator (as a Carlitz A-module) is called the primitive Carlitz N-torsion point. The minimal polynomial $\Phi_N(x)$ of any primitive N-torsion point over K is called the Carlitz N-th cyclotomic polynomial. The polynomials $\rho_N(x)$ and $\Phi_N(x)$ have degrees $q^{\deg N}$ and $\varphi(N)$, respectively, where $\varphi(N) := \#(A/NA)^*$. For details on these polynomials, we refer the reader to [3]. For the primitive Carlitz N-torsion point λ_N, let $K_N = K(\lambda_N)$ be the field generated over K by adjoining λ_N. If $\sigma \in \mathrm{Gal}(K_N/K)$, then $\sigma(\lambda_N)$ is another primitive Carlitz N-torsion point. Hence, there exists $a \in A$ with $\gcd(a, N) = 1$ such that $\sigma(\lambda_N) = \rho_a(\lambda_N)$. The correspondence $\sigma \mapsto a$ induces the isomorphism $\mathrm{Gal}(K_N/K) \xrightarrow{\sim} (A/NA)^*$ (see [15, Theorem 12.8]).

There exists a unique entire function $e(z)$ over $\mathbb{C}_\infty$ such that for each $a \in A$, we have $\rho_a(e(z)) = e(az)$ (see [9, Chapter 3]). The function $e(z)$ is called the Carlitz exponential. The function denoted by $e(z)$ in the Introduction is exactly the Carlitz exponential. Let L be the set of all zeros of $e(z)$. Then, L is a rank one free A-module (see [9, Corollary 3.2.9]). It is well known that $L = \overline{\pi} A$ is analogous to $\pi\mathbb{Z}$. Using L, $e(z)$ can be written as

$$e(z) = z \prod_{0 \neq l \in L} \left(1 - \frac{z}{l}\right). \tag{2.1}$$

From (1.1), it holds that $e'(z) = 1$.

2.2. **The cotangent function.** Let $c(z) := e(z)^{-1}$. Using (1.1) and (2.1), we have

$$c(z) = \frac{e'(z)}{e(z)} = \frac{1}{z} + \sum_{0 \neq l \in L} \frac{1}{z + l}. \tag{2.2}$$

The analogy between $c(z)$ and the usual cotangent function is that $\cot z$ is $\pi\mathbb{Z}$-periodic and is expressed by

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z + \pi n} + \frac{1}{z - \pi n} \right);$$

$c(z)$ is L-periodic and is expressed by (2.2).

For a positive integer n, let $P_n(z) = \sum_{l \in L} (z + l)^{-n}$. According to Goss [8], there exists a monic polynomial $G_n(X) \in K[X]$ of degree n such that $P_n(z) =$

$G_n\left(e(z)^{-1}\right)$. This is called the Goss polynomial. Let $\overline{\pi}$ be a generator of L. Petrov [14] proved that

$$D_{n-1}c(\overline{\pi}z) = (-\overline{\pi})^{n-1}P_n(\overline{\pi}z) = (-\overline{\pi})^{n-1}G_n(e(\overline{\pi}z)^{-1}), \tag{2.3}$$

where D_{n-1} is the $n-1$-th hyperdifferential operator in z that is discussed by Bosser and Pellarin in [5].

2.3. **Goss L-functions.** Let A_+ be the set of all monic elements in A and let $M \in A_+$ with $\deg M > 0$. A group homomorphism $\chi : (A/MA)^* \to \mathbb{C}_\infty^*$ is called a character modulo M. This can be extended to A by

$$\chi(a) = \begin{cases} \chi(a+MA) & \textit{if} \gcd(a,M)=1, \\ 0 & \textit{otherwise}. \end{cases}$$

For this χ, let $\overline{\chi}$ be the character defined by

$$\overline{\chi}(a) = \begin{cases} \chi\left((a+MA)^{-1}\right) & \textit{if} \gcd(a,M)=1, \\ 0 & \textit{otherwise}. \end{cases}$$

The Goss L-function of χ is defined by

$$L(s,\chi) = \sum_{a\in A+} \frac{\chi(a)}{a^s} \quad (s\in\mathbb{N}).$$

This can be thought of as an entire function on the Goss plane $S_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers. Moreover, it has the following Euler product expression

$$L(s,\chi) = \prod_{\substack{P\in A_+ \\ P:\text{irreducible}}} \left(1-\chi(P)P^{-s}\right)^{-1}.$$

From this, we have that

$$L(s,\chi) \neq 0 \qquad (s\in\mathbb{N}). \tag{2.4}$$

For more details, we refer the reader to the book [9, Chapter 8].

3. The main theorem

Let $M \in A_+$ with $\deg M > 0$ and let R_M be the subset of A_+ defined by

$$R_M = \{a \in A_+ \mid \deg a < \deg M, \gcd(a,M)=1\}.$$

The main theorem of this paper is the following.

Theorem 2. *Let n be a positive integer. Then, $D_{n-1}c\left(\overline{\pi}z\right)|_{z=b/M}$ $(b\in R_M)$ are linearly independent over K.*

Remark. 1. In [13, 16], the following result was proved: Let n and m be positive integers with $m > 2$. Let R be a set of $\phi(m)/2$ representatives mod m such that the union $\{R, -R\}$ is a complete set of residues prime to m, where ϕ is the Euler totient function. Then, $\left(\frac{d}{dz}\right)^{n-1}(\cot \pi z)|_{z=a/m}$ $(a\in R)$ are linearly independent over the field of rational numbers. Theorem 2 is an analog of this result. The case $n=1$ in Theorem 2 is an analog of Chowla's theorem, which was mentioned in the introduction.

2. Hasse [10] proved that for an odd prime p, $\tan(\pi j/p)$ $(j = 1, \dots, (p-1)/2)$ are linearly independent over the field of rational numbers. Even though the Carlitz exponential $e(z)$ is analogous to $\tan(z)$, an analog of Hasse's result for $e(z)$ does not hold. In fact, for an irreducible polynomial $P \in A_+$ with $\deg P > 1$, $e(\overline{\pi} b/P)$ $(b \in A_+, \deg b < \deg P)$ are linearly dependent over K.

As an application of the above theorem, we have the following, which is an analog of the Baker–Birch–Wirsing theorem (Theorem 1) under a certain condition.

Theorem 3. *Let n be a positive integer and $g : A \to \overline{K}$ a non-zero function, which is defined on A/MA and then extended to A, such that*

(i) $g(\zeta a) = \zeta^n g(a) \quad (a \in A, \zeta \in \mathbb{F}_q^*)$;
(ii) $g(a) = 0$ *if* $\gcd(a, M) > 1$;
(iii) *The Carlitz M-th cyclotomic polynomial Φ_M is irreducible over K_g, which is the field generated over K by adjoining $\{g(b) \mid b \in A/MA\}$.*

Then,

$$\sum_{a \in A_+} \frac{g(a)}{a^n} \neq 0.$$

Remark. Let k be a positive integer. Okada [13] proved the following: Let f be a non-zero function defined on the integers with algebraic values and period $m > 2$ such that

(i) f is even or odd according as k is even or odd;
(ii) $f(n) = 0$ if $\gcd(n, m) > 1$;
(iii) The m-th cyclotomic polynomial Ψ_m is irreducible over $\mathbb{Q}(f(1), \dots, f(m))$.

Then, $\sum_{n=1}^{\infty} \frac{f(n)}{n^k} \neq 0$.

Theorem 3 is an analog of this result.

From this theorem, we obtain the following.

Theorem 4. *Let n be a positive integer. Let $G = \{\chi : (A/MA)^* \to \mathbb{C}_\infty^*\}$ be the set of all characters modulo M. For $\Lambda = \{\chi \in G \mid \chi(\zeta) = \zeta^n \ (\zeta \in \mathbb{F}_q^*)\}$, let $\mathbb{F}_{q^r}$ be the finite field generated over $\mathbb{F}_q$ by adjoining $\{\chi(b) \mid \chi \in \Lambda, b \in (A/MA)^*\}$. If $\gcd(\varphi(M), r) = 1$, then the Goss L-functions $L(n, \chi)$ $(\chi \in \Lambda)$ are linearly independent over K.*

Remark. Let k and m be positive integers with $m > 2$ and $\gcd(m, \phi(m)) = 1$. Let Λ denote the set of all even or odd Dirichlet characters modulo m according as k is even or odd. Then, Okada [13] proved that the Dirichlet L-functions $L(k, \chi)$ $(\chi \in \Lambda)$ are linearly independent over the field of rational numbers. Theorem 4 is an analog of this result as well.

4. Outline of the proofs of Theorems 2, 3, and 4

4.1. **Proof of Theorem 2.** We will use two lemmas. The first is a form of the Frobenius determinant relation (see Lang [11, Chapter 21]).

Lemma 5. *Let G be a finite abelian group and H a subgroup. Let $\lambda : H \to \mathbb{C}_\infty^*$ be a character of H and Λ the set of all characters of G given by*

$$\Lambda = \{\chi : G \to \mathbb{C}_\infty^* \mid \chi|_H = \lambda\}.$$

Then, for a $\mathbb{C}_\infty$-valued function f on G with

$$f(ah) = \lambda(h)f(a) \quad (a \in G, h \in H),$$

we have

$$\det_{b,c\in R} f\left(b^{-1}c\right) = \prod_{\chi\in\Lambda}\left(\sum_{a\in R}\overline{\chi}(a)f(a)\right),$$

where R is a complete set of representatives of G/H.

The second lemma connects $D_{n-1}c(\overline{\pi}z)|_{z=b/M}$ ($b \in R_M$) with the Goss L-function.

Lemma 6. *Let n be a positive integer and let $M \in A_+$ with $\deg M > 0$. Let f be any $\mathbb{C}_\infty$-valued function on A/MA satisfying:*

(i) $f(\zeta a) = \zeta^n f(a)$ $(\zeta \in \mathbb{F}_q^*)$;

(ii) $f(a) = 0$ *if* $\gcd(a, M) > 1$.

Then, we have

$$\sum_{a\in A_+}\frac{f(a)}{a^n} = \left(\frac{\overline{\pi}}{M}\right)^n \sum_{b\in R_M} f(b)P_n\left(\frac{\overline{\pi}b}{M}\right).$$

We now prove Theorem 2. By (2.3), it suffices to prove that $P_n(\overline{\pi}b/M)$ ($b \in R_M$) are linearly independent over K. We assume that

$$\sum_{b\in R_M} c_b P_n\left(\frac{\overline{\pi}b}{M}\right) = 0 \quad (c_b \in K). \tag{4.1}$$

For $a \in (A/MA)^*$, there exists $\overline{a} \in (A/MA)^*$ such that $a\overline{a} \equiv 1 \pmod{M}$. Noting that the Goss polynomial $G_n(X)$ belongs to $K[X]$, we map (4.1) by $\sigma_{\overline{a}} \in \mathrm{Gal}(K_M/K)$ corresponding to $\overline{a} \in (A/MA)^*$. Then, we obtain

$$\sum_{b\in R_M} c_b G_n\left(\sigma_{\overline{a}}\left(e(\overline{\pi}b/M)\right)^{-1}\right) = \sum_{b\in R_M} c_b P_n\left(\frac{\overline{\pi}\,\overline{a}b}{M}\right) = 0.$$

Using Lemmas 5 and 6, we see that

$$\begin{aligned}\det_{b,c\in R_M} P_n\left(\frac{\overline{\pi}\overline{b}c}{M}\right) &= \prod_{\chi\in\Lambda}\left(\sum_{a\in R_M}\overline{\chi}(a)P_n\left(\frac{\overline{\pi}a}{M}\right)\right)\\ &= \left(\frac{M}{\overline{\pi}}\right)^{n\varphi(M)/(q-1)} \prod_{\chi\in\Lambda} L(n,\overline{\chi}),\end{aligned}$$

where Λ is the set of all characters of $(A/MA)^*$ given by $\Lambda = \{\chi : (A/MA)^* \to \mathbb{C}_\infty^* \mid \chi(\zeta) = \zeta^{-n}\ (\zeta \in \mathbb{F}_q^*)\}$. Therefore, it follows from (2.4) that $c_b = 0$ for $b \in R_M$.

4.2. **Proof of Theorem 3.** For the primitive Carlitz M-torsion point λ_M, we have $\Phi_M(\lambda_M) = 0$. Since Φ_M is defined over K, for any $\sigma \in \mathrm{Gal}(K_M/K)$, we have $\Phi_M(\sigma(\lambda_M)) = 0$. Using the isomorphism $\mathrm{Gal}(K_M/K) \cong (A/MA)^*$, the set of all roots of Φ_M is $\{\rho_a(\lambda_M) \mid a \in (A/MA)^*\}$. Combining (iii) with $K_M := K(\lambda_M)$, we obtain

$$[K_M K_g : K_g] = [K_g(\lambda_M) : K_g] = \varphi(M) = [K_M : K].$$

Hence, K_M and K_g are linearly disjoint over K. By Theorem 2, $P_n(\overline{\pi} b/M)$ ($b \in R_M$), which belong to K_M, are linearly independent over K. Therefore, they are linearly independent over K_g as well. Since g is non-zero, we have $g(b) \neq 0$ for some $b \in R_M$. Hence, using Lemma 6, $\sum_{a \in A_+} g(a)/a^n = \left(\frac{\overline{\pi}}{M}\right)^n \sum_{b \in R_M} g(b) P_n(\overline{\pi} b/M) \neq 0$. This proves Theorem 3.

4.3. **Proof of Theorem 4.** We assume that

$$\sum_{\chi \in \Lambda} c_\chi L(n, \chi) = 0 \quad (c_\chi \in K). \tag{4.2}$$

Let $g = \sum_{\chi \in \Lambda} c_\chi \chi$. We note that this satisfies conditions (i) and (ii) in Theorem 3. Using the identity in Lemma 6, we have

$$\sum_{b \in R_M} g(b) P_n(\overline{\pi} b/M) = \left(\frac{M}{\overline{\pi}}\right)^n \sum_{a \in A_+} \frac{g(a)}{a^n} = 0. \tag{4.3}$$

Since $[K_M : K] = \varphi(M)$ and $[K\mathbb{F}_{q^r} : K] = r$ are coprime, K_M and $K\mathbb{F}_{q^r}$ are linearly disjoint over K. Hence, using Theorem 2, $P_n(\overline{\pi} b/M)$ ($b \in R_M$) are linearly independent over $K\mathbb{F}_{q^r}$. Combining (4.2) with $g(b) \in K\mathbb{F}_{q^r}$ ($b \in R_M$), we see that $g(b) = 0$ ($b \in R_M$). Namely, we obtain

$$\sum_{\chi \in \Lambda} c_\chi \chi(b) = 0 \quad (b \in R_M). \tag{4.4}$$

We set $d = \varphi(M)/(q-1)$. Letting $\Lambda = \{\chi_1, \dots, \chi_d\}$ and $R_M = \{b_1, \dots, b_d\}$, we see that the $d \times d$ matrix $(\chi_i(b_j))$ is invertible. Therefore, we conclude from (4.4) that $c_\chi = 0$ ($\chi \in \Lambda$).

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References

[1] R. Ayoub, On a theorem of S. Chowla, J. Number Theory **7** (1975), 105–107.

[2] R. Ayoub, On a theorem of Iwasawa, J. Number Theory **7** (1975), 108–120.

[3] S. Bae, The arithmetic of Carlitz polynomials, J. Korean Math. Soc. **35** (1998), 341–360.

[4] A. Baker, B.J. Birch, and E.A. Wirsing, On a theorem of Chowla, J. Number Theory **5** (1973), 224–236.

[5] V. Bosser and F. Pellarin, Hyperdifferential properties of Drinfeld quasi-modular forms, Int. Math. Res. Not. 2008 (2008), 56 pp.

[6] S. Chowla, A special infinite series, Norske Vid. Selsk. Forth. (Trondheim) **37** (1964), 85–87.

[7] S. Chowla, The nonexistence of nontrivial linear relations between the roots of a certain irreducible equation, J. Number Theory **2** (1970), 120–123.

[8] D. Goss, The algebraist's upper half-plane, Bull. Amer. Math. Soc. **2** (1980), 391–415.

[9] D. Goss, Basic Structures of Function Field Arithmetic, Springer, 1996.

[10] H. Hasse, On a question of S. Chowla, Acta Arithmetica **18** (1971), 275–280.
[11] S. Lang, Elliptic Functions (2nd edition), Springer-Verlag, 1987.
[12] T. Okada, On a theorem of S. Chowla, Hokkaido Math. J. **6** (1977), 66–68.
[13] T. Okada, On an extension of a theorem of S. Chowla, Acta Arithmetica **38** (1981), 341–345.
[14] A. Petrov, On hyperderivatives of single-cuspidal Drinfeld modular forms with A-expansions, J. Number Theory **149** (2015), 153–165.
[15] M. Rosen, Number Theory in Function Fields, Springer, 2002.
[16] K. Wang, On a theorem of S. Chowla, J. Number Theory **15** (1982), 1–4.

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