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LONG RANGE SCATTERING FOR THE KLEIN-GORDON EQUATION WITH THE CRITICAL NONLINEARITY

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1. Introduction

This note is a survey of the papers [22, 23] which is a joint work with Satoshi Masaki (Osaka university). In this note, we consider the final state problem for the Klein-Gordon equation with a critical nonlinearity in space dimensions $d = 1, 2, 3$:

$$(1.1) \qquad \begin{cases} (\Box + 1)u = \lambda |u|^{2/d} u & t \in \mathbb{R},\ x \in \mathbb{R}^d, \\ u - u_{\mathrm{ap}} \to 0 & \text{in } L^2 \ \text{ as } t \to +\infty, \end{cases}$$

where $\Box = \partial_t^2 - \Delta$ is d'Alembertian, $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is an unknown function, $u_{\mathrm{ap}} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a given function, and λ is a non-zero real constant. To explain why we consider this problem, we briefly review known results on the global existence and long time behavior of solutions to the nonlinear Klein-Gordon equation

$$(1.2) \qquad (\Box + 1)u = \lambda |u|^{p-1} u, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d,$$

where $p > 1$ and $\lambda \in \mathbb{R} \backslash \{0\}$. The point-wise decay of a solution to the linear Klein-Gordon equation is $O(t^{-d/2})$ as $t \to \infty$, so the linear scattering theory indicates that the power $p = 1 + 2/d$ will be a borderline between the short and the long range scattering theories. This formal observation was firstly justified by Glassey [6], Matsumura [24] and Georgiev and Yordanov [4] for $p \leqslant 1 + 2/d$. More precisely, they proved that if $1 < p \leqslant 1 + 2/d$, then the equation (1.2) has no non-trivial solution which scatter to a solution to the linear Klein-Gordon equation as $t \to \infty$. On the other hand, Hayashi and Naumkin [10] proved that if $p > 1 + 2/d$ and $d = 1, 2$, then a small solution to (1.2) scatters to a solution to the linear Klein-Gordon equation. See also Georgiev and Lecente [3] for earlier results. The readers are referred to [7, 15, 28, 29, 30, 32] for the small data scattering when $d \geqslant 3$ and p is large.

From the above results we see that for the case where $p \leqslant 1 + 2/d$, the long time behavior of solution to (1.2) is different from that of the linear Klein-Gordon equation. So, we are interested in the long time behavior of solution to (1.2) for $p \leqslant 1 + 2/d$. For the critical case $p = 1 + 2/d$ and $d = 1$, Georgiev and Yordanov [4] studied point-wise decay of a solution to the initial value problem. Delort [1] obtained an asymptotic behavior of a global solution to (1.2) under the assumption that the support of the initial data is compact. See also Lindblad and Soffer [17] for an alternative proof of [1]. The compact support assumption in [1] was later removed by Hayashi and Naumkin in [8] by using the vector field approach by Klainerman [15].

Recently, the authors [22, 23] considered (1.2) with $p = 1 + 2/d$ and $d = 2, 3$ and specified an asymptotic profile u_{ap} that allows a unique solution u which converges to u_{ap} as $t \to \infty$.

To state the main theorems in [22, 23] precisely, we introduce an asymptotic profile u_{ap} which we work with. To this end, we first recall that a solution to the linear Klein-Gordon equation

$$\begin{cases} (\Box + 1)v = 0 & t \in \mathbb{R},\ x \in \mathbb{R}^d, \\ v(0,x) = \phi_0(x), \quad \partial_t v(0,x) = \phi_1(x) & x \in \mathbb{R}^d \end{cases}$$

behaves like

$$v = t^{-\frac{d}{2}} \mathbf{1}_{\{|x|<t\}}(t,x) A_1(\mu) \cos(\alpha - \beta) + o(t^{-\frac{d}{2}}),$$

in L^∞ as $t \to \infty$, where $\mathbf{1}_\Omega(t,x)$ is the characteristic function supported on $\Omega \subset \mathbb{R}^{1+d}$, $\mu = x/\sqrt{t^2 - |x|^2}$,

$$\begin{aligned} A_1(\mu) &= \sqrt{P_1^2(\mu) + Q_1^2(\mu)}, \\ P_1(\mu) &= \langle\mu\rangle^{\frac{d+2}{2}} \left\{ \cos\left(\frac{d\pi}{4}\right)\left(\operatorname{Re}\hat{\phi}_0(\mu) - \langle\mu\rangle^{-1} \operatorname{Im}\hat{\phi}_1(\mu)\right) \right. \\ &\qquad \left. - \sin\left(\frac{d\pi}{4}\right)\left(\operatorname{Im}\hat{\phi}_0(\mu) + \langle\mu\rangle^{-1} \operatorname{Re}\hat{\phi}_1(\mu)\right) \right\}, \\ Q_1(\mu) &= \langle\mu\rangle^{\frac{d+2}{2}} \left\{ \sin\left(\frac{d\pi}{4}\right)\left(\operatorname{Re}\hat{\phi}_0(\mu) - \langle\mu\rangle^{-1} \operatorname{Im}\hat{\phi}_1(\mu)\right) \right. \\ &\qquad \left. + \cos\left(\frac{d\pi}{4}\right)\left(\operatorname{Im}\hat{\phi}_0(\mu) + \langle\mu\rangle^{-1} \operatorname{Re}\hat{\phi}_1(\mu)\right) \right\}, \end{aligned}$$

$\alpha = \langle\mu\rangle^{-1} t$ and $\beta \in (0, 2\pi]$ is given by

$$\cos\beta = \frac{P_1}{A_1}, \quad \sin\beta = \frac{Q_1}{A_1},$$

see Hörmander's book [11] for instance. For given final state (ϕ_0, ϕ_1), we define an asymptotic profile u_{ap} by

$$u_{\mathrm{ap}}(t,x) := t^{-\frac{d}{2}} \mathbf{1}_{\{|x|<t\}}(t,x) A_1(\mu) \cos\left(\alpha + \Psi(\mu)\log t - \beta\right), \tag{1.3}$$

where the phase correction term is given by

$$\Psi(\mu) = \begin{cases} -\dfrac{3}{8}\lambda\langle\mu\rangle^{-1} A_1^2(\mu) & \text{if } d = 1, \\ -\dfrac{4\lambda}{3\pi}\langle\mu\rangle^{-1} A_1(\mu) & \text{if } d = 2, \\ -\dfrac{\Gamma(\frac{11}{6})}{\sqrt{\pi}\,\Gamma(\frac{7}{3})}\lambda\langle\mu\rangle^{-1} A_1(\mu)^{\frac{2}{3}} & \text{if } d = 3. \end{cases} \tag{1.4}$$

The final state (ϕ_0, ϕ_1) is taken from the function space Y defined by

$$\begin{aligned} Y &:= \{(\phi_0, \phi_1) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d);\ \|(\phi_0, \phi_1)\|_Y < \infty\}, \\ \|(\phi_0, \phi_1)\|_Y &:= \|\phi_0\|_{H_x^2} + \|x\phi_0\|_{H_x^3} + \|x^2\phi_0\|_{H_x^4} \\ &\quad + \|\phi_1\|_{H_x^1} + \|x\phi_1\|_{H_x^2} + \|x^2\phi_1\|_{H_x^3}. \end{aligned}$$

The main results in [22, 23] are summarized as follows.

Theorem 1.1. *Let $d = 1, 2, 3$. Then for $d/4 < \gamma < 1$, there exist a sufficiently large number $T \geqslant e$ and a sufficiently small number $\varepsilon > 0$ such that if $\|(\phi_0, \phi_1)\|_Y < \varepsilon$ then there exists a unique solution $u(t)$ for the equation (1.1) satisfying*

$$
\begin{aligned}
&u \in C([T, \infty); L^2_x), \\
&\sup_{t \geqslant T} t^{\gamma} \|u - u_{\mathrm{ap}}\|_{L^{\infty}((t,\infty); L^2_x)} < \infty,
\end{aligned}
\tag{1.5}
$$

where the asymptotic profile u_{ap} is defined by (1.3).

Note that for the one dimensional case, Theorem 1.1 is proved by Hayashi and Naumkin [9] under weaker assumption on the final data. We also note that Lindblad and Soffer [16] showed existence of a modified wave operators for (1.2) for large final data in the case where $\lambda < 0$.

Remark 1.2. For the two and three dimensional cases, the coefficients of the phase function Ψ come from the first Fourier-cosine coefficients of a 2π-periodic function $|\cos\theta|^{2/d}\cos\theta$. See Sections 4 and 5 for the detail.

Remark 1.3. The global existence and asymptotic behavior of a solution to the Klein-Gordon equation with the cubic quasi-linear nonlinearity is studied by Moriyama [26], Katayama [12], and Sunagawa [33] in one space dimension. Concerning the Klein-Gordon equation with the quadratic nonlinearity in two dimensions, Ozawa, Tsutaya, and Tsutsumi [27] proved a global existence result and characterized the asymptotic behavior of a small solution to (1.2) with a smooth, quadratic, semi-linear nonlinearity, i.e., nonlinear term depends on $u, \partial_t u, \nabla u$. Delort, Fang, and Xue [2] extended Ozawa-Tsutaya-Tsutsumi's result to the case where the nonlinear term is quasi-linear. See also Kawahara and Sunagawa [14] and Katayama, Ozawa and Sunagawa [13] for related works.

The proof of Theorem 1.1 consists of two parts. As a first step, we solve a Cauchy problem at infinite initial time for the equation (1.1) for a given asymptotic profile which decays like a solution to the linear Klein-Gordon equation and approximately solves (1.1) for large time. Next, we construct an asymptotic profile satisfying those properties which is a crucial part of our proof. In Section 2 we solve a Cauchy problem at infinite initial time for the equation (1.1) in an abstract framework (Proposition 2.1). Then in Sections 3,4 and 5, we explain how to construct a function which satisfies the assumptions in Proposition 2.1 for the case $d = 1, 2$ and 3, respectively.

2. Abstract Cauchy problem

For $T > 0$, we define the function spaces X_T by

$$
\begin{aligned}
X_T &:= \{w \in C([T, \infty); L^2_x);\ \|w\|_{X_T} < \infty\}, \\
\|w\|_{X_T} &:= \sup_{t \geqslant T} t^{\gamma} (\|w\|_{L^{\infty}_t((t,\infty); H^{1/2}_x)} + \|w\|_{L^q((t,\infty); L^r_x)}),
\end{aligned}
$$

where $d/4 < \gamma < 1$ and

$$(q,r) = \begin{cases} (4,\infty) & \text{if } d=1, \\ (4,4) & \text{if } d=2, \\ (\frac{10}{3},\frac{10}{3}) & \text{if } d=3. \end{cases}$$

Proposition 2.1. *Let $d = 1,2,3$ and let $N(u) = \lambda|u|^{2/d}u$. Let γ be a constant such that $d/4 < \gamma < 1$. Then there exist a sufficiently large $T > 0$ and a sufficiently small $\eta > 0$ such that if $A(t,x)$ satisfies*

$$\|A(t)\|_{L^\infty_x} \leqslant \eta t^{-1}, \tag{2.1}$$

$$\|(\Box + 1)A(t) - N(A)(t)\|_{L^2_x} \leqslant \eta t^{-1-\gamma}, \tag{2.2}$$

then there exists a unique solution u for the equation (1.1) satisfying

$$u \in C([T,\infty); L^2_x),$$

$$\sup_{t\geqslant T} t^\gamma (\|u-A\|_{L^\infty((t,\infty);H^{1/2}_x)} + \|u-A\|_{L^q((t,\infty);L^r_x)}) < \infty. \tag{2.3}$$

By Proposition 2.1, once we find a function A satisfying (2.1) and (2.2), we can show the existence of a unique solution u to the equation (1.1) satisfying $u - A \in X_T$. In Sections 3,4 and 5, we construct a function A satisfying the conditions (2.1) and (2.2) for a given final state $(\phi_0, \phi_1) \in Y$.

Let us give an outline of proof for Proposition 2.1. To prove this proposition, we use the following inhomogeneous Strichartz estimates associated with the Klein-Gordon equation. Let

$$\mathcal{G}[g](t) := \int_t^\infty \sin((t-\tau)\sqrt{1-\Delta})(1-\Delta)^{-1/2} g(\tau)d\tau. \tag{2.4}$$

Lemma 2.2. *Let $2 \leqslant r < (2d)/(d-2)$ and $2/q + d/r = d/2$. Then we have*

$$\begin{aligned} \|\mathcal{G}[g]\|_{L^q_t([T,\infty),L^r_x)} &\leqslant C\|(1-\Delta)^{\frac{d}{4}-\frac{d+2}{2r}} g\|_{L^{q'}_t([T,\infty),L^{r'}_x)}, \\ \|\mathcal{G}[g]\|_{L^\infty_t([T,\infty),L^2_x)} &\leqslant C\|(1-\Delta)^{\frac{d-2}{8}-\frac{d+2}{4r}} g\|_{L^{q'}_t([T,\infty),L^{r'}_x)}, \\ \|\mathcal{G}[g]\|_{L^q_t([T,\infty),L^r_x)} &\leqslant C\|(1-\Delta)^{\frac{d-2}{8}-\frac{d+2}{4r}} g\|_{L^1_t([T,\infty),L^2_x)}. \end{aligned}$$

Proof of Lemma 2.2. The above inequalities follow from the L^p-L^q estimate for the solution to the Klein-Gordon equation by [18] and the duality argument by [34]. Since the proof is now standard, we omit the detail. □

Outline of the proof of Proposition 2.1. We put $v = u - A$ and $F = (\Box + 1)A - N(A)$. Then the equation (1.1) is equivalent to

$$(\Box + 1)v = N(v+A) - N(A) - F. \tag{2.5}$$

The associate integral equation to the equation (2.5) is

$$v = \mathcal{G}[\{N(v+A) - N(A)\} - F], \tag{2.6}$$

where $\mathcal{G}$ is given by (2.4). We show the existence of a unique solution v to the equation (2.6) in X_T for sufficiently large $T > 0$ and sufficiently small

$\eta > 0$ by the contraction argument. To this end, we define the nonlinear operator Φ by

$$\Phi v := \mathcal{G}[\{N(v+A) - N(A)\} - F]$$

for $v \in \widetilde{X}_T(\rho)$ and the function space $\widetilde{X}_T(\rho)$ by

$$\widetilde{X}_T(\rho) = \{w \in C([T,\infty); L^2_x);\ \|w\|_{X_T} \leqslant \rho\},$$

where $\rho > 0$ and $T > 0$. Note that $\widetilde{X}_T(\rho)$ is a complete metric space with the $\|\cdot\|_{X_T}$-metric. By using Lemma 2.2, we are able to show that for any $\rho > 0$, Φ is a contraction map on $\widetilde{X}_T(\rho)$ if $T > 0$ is sufficiently large and $\eta > 0$ is sufficiently small. Hence the Banach fixed point theorem yields Proposition 2.1. □

3. Outline of the proof of Theorem 1.1 Case: $d = 1$

In this section, we give an outline of the proof of Theorem 1.1 for $d = 1$ by using the argument by Delort [1]. We now explain how to construct the function $A = A(t,x)$ satisfying the conditions (2.1) and (2.2). It will turn out that $A = u_{\mathrm{ap}}$ does not work well, and so that we need further modification. The conclusion is that the choice $A := u_{\mathrm{ap}} + v_{\mathrm{ap}}$ works, where u_{ap} is the *first approximation* given by (1.3) and v_{ap} is *the second approximation* which is of the form

$$v_{\mathrm{ap}} := t^{-\frac{3}{2}}\mathbf{1}_{\{|x|<t\}}A_3(\mu)\cos\left(3(\alpha + \Psi(\mu)\log t - \beta)\right). \tag{3.1}$$

Here the phase function Ψ is the same as (1.4), and choice of A_3 will be specified later. Remark that $v_{\mathrm{ap}}(t) = O(t^{-1})$ in L^2_x. Toward the conclusion, we will observe (i) why the second approximation v_{ap} is required, and (ii) what is the appropriate choice of A_3. Hereafter, we consider the case $|x| < t$ only because u_{ap} and v_{ap} are identically zero in the region $|x| \geqslant t$.

We first focus on the nonlinear part $N(u_{\mathrm{ap}}) = \lambda|u_{\mathrm{ap}}|^2u_{\mathrm{ap}}$. Since $N(u) = \lambda|u|^2u$ is polynomial in $(u,\overline{u})$, it is easy to pick up a *resonant part* from $N(u_{\mathrm{ap}})$. Indeed, we have

$$\begin{aligned} N(u_{\mathrm{ap}}) &= \lambda t^{-\frac{3}{2}}A_1(\mu)^3\cos^3(\alpha + \Phi(\mu)\log t - \beta) \\ &= \frac{3}{4}\lambda t^{-\frac{3}{2}}A_1^3(\mu)\cos(\alpha + \Phi(\mu)\log t - \beta) \\ &\quad + \frac{1}{4}\lambda t^{-\frac{3}{2}}A_1^3(\mu)\cos(3(\alpha + \Phi(\mu)\log t - \beta)) \\ &=: N_{\mathrm{r}}(u_{\mathrm{ap}}) + N_{\mathrm{nr}}(u_{\mathrm{ap}}). \end{aligned} \tag{3.2}$$

Since both of the resonant and non-resonant parts are $O(t^{-1})$ in L^2_x, we need to cancel out those terms by the linear part, otherwise (2.2) fails. Thanks to the phase correction Ψ, we have the desired cancellation of the resonant part. Namely, we have

$$(\Box + 1)u_{\mathrm{ap}} = N_{\mathrm{r}}(u_{\mathrm{ap}}) + O(t^{-2}(\log t)^2)$$

in L^2 as $t \to \infty$. We then add a *second approximation* v_{ap} of u, given in (3.1), in order to cancel the non-resonant term $N_{\mathrm{nr}}(u_{\mathrm{ap}})$ out. This is the reason why we need the second approximation v_{ap}.

To obtain the desired cancellation, we will choose suitable A_3. More precisely, we choose A_3 so that the leading term of $(\Box+1)v_{\rm ap}$ and $N_{\rm nr}(u_{\rm ap})$ coincide. By a computation, we have

$$(\Box+1)v_{\rm ap} = -8t^{-\frac{3}{2}}A_3(\mu)\cos\left(3(\alpha+\Phi(\mu)\log t-\beta)\right)+O(t^{-2}(\log t)^2)$$

in L^2 as $t\to\infty$. Hence, we obtain the specific choice

$$A_n(\mu) = -\frac{\lambda}{32}A_1^3(\mu). \tag{3.3}$$

With this choice, the leading term of $(\Box+1)v_{\rm ap}$ and $N_{\rm nr}(u_{\rm ap})$ successfully cancel out each other. Thus, we see that $A=u_{\rm ap}+v_{\rm ap}$ satisfies the conditions (2.1) and (2.2).

Notice that this kind of approximation was introduced in Hörmander [11] for the Klein-Gordon equation with *polynomial* nonlinearity in $(u,\overline{u})$. See also [25, 31] for the nonlinear Schrödinger equation with polynomial nonlinearity in $(u,\overline{u})$.

4. Outline of the proof of Theorem 1.1 Case: $d=2$

In this section, we give an outline of the proof of Theorem 1.1 for $d=2$ which is given by [22].

We now explain how to construct the function $A=A(t,x)$ satisfying the conditions (2.1) and (2.2). We choose $A:=u_{\rm ap}+v_{\rm ap}$, where $u_{\rm ap}$ is the *first approximation* given by (1.3) and $v_{\rm ap}$ is *the second approximation* which is of the form

$$v_{\rm ap} \;:=\; t^{-2}\mathbf{1}_{\{|x|<t\}}\sum_{n=2}^{\infty}A_n(\mu)\cos\left(n(\alpha+\Psi(\mu)\log t-\beta)\right). \tag{4.1}$$

Here the phase function Ψ is given by (1.4), and choice of A_n will be specified later. Remark that $v_{\rm ap}(t)=O(t^{-1})$ in L^2_x. Hereafter, we consider the case $|x|<t$ only because $u_{\rm ap}$ and $v_{\rm ap}$ are identically zero in the region $|x|\geqslant t$.

We first focus on the nonlinear part $N(u_{\rm ap}) = \lambda|u_{\rm ap}|u_{\rm ap}$. Unlike the one dimensional case, the nonlinear term $N(u)=\lambda|u|u$ is not polynomial in $(u,\overline{u})$, so it becomes difficult to pick up a *resonant part* from $N(u_{\rm ap})$. Taking a hint from our previous paper [21], we use the Fourier series expansion of $N(u_{\rm ap})$ to decompose $N(u_{\rm ap})$ into the resonant part and the rest, the *non-resonant part.* This decomposition is done as follows.

$$\begin{aligned}
(4.2)\; & N(u_{\rm ap}) \\
&= \lambda t^{-2}A_1(\mu)^2|\cos(\alpha+\Phi(\mu)\log t-\beta)|\cos(\alpha+\Phi(\mu)\log t-\beta)\\
&= \lambda t^{-2}A_1(\mu)^2\sum_{n\geqslant 1}c_n\cos(n(\alpha+\Phi(\mu)\log t-\beta))\\
&= c_1\lambda t^{-2}A_1(\mu)^2\cos(\alpha+\Phi(\mu)\log t-\beta)\\
&\quad+\sum_{n\geqslant 2}\lambda c_n t^{-2}A_1(\mu)^2\cos(n(\alpha+\Phi(\mu)\log t-\beta))\\
&=: N_{\rm r}(u_{\rm ap})+N_{\rm nr}(u_{\rm ap}),
\end{aligned}$$

where c_n is the n-th Fourier coefficients for the function $|\cos\theta|\cos\theta$:

$$c_n = \frac{1}{\pi}\int_0^{2\pi} |\cos\theta|\cos\theta\cos n\theta d\theta = \begin{cases} -\dfrac{8}{\pi}\dfrac{\sin(\frac{n}{2}\pi)}{n(n^2-4)} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

This kind of technique was also used in Sunagawa [33] to pick up the resonant term from the cubic nonlinearity in one space dimension. As we explained in Section 2, for the one dimensional case, the Fourier series for $N(u_{\mathrm{ap}})$ consists of only two terms. We would emphasize that, in our setting, the Fourier series consists of *infinitely many terms*, so we need to take care of the convergence of the Fourier series, which seems a new ingredient. Fortunately, it turns out that the nonlinearity $|u|u$ has enough smoothness to ensure the convergence of the Fourier series for $|u|u$. We mention similar but slightly different expansion of a nonlinearity into a infinite Fourier sires is used by the first author and Miyazaki [19] in the context of nonlinear Schrödinger equation.

Since both of the resonant and non-resonant parts are $O(t^{-1})$ in L^2_x, we need to cancel out those terms by the linear part, otherwise (2.2) fails. Thanks to the phase correction Ψ, we have the desired cancellation of the resonant part. Namely, we have

$$(\Box + 1)u_{\mathrm{ap}} = N_{\mathrm{r}}(u_{\mathrm{ap}}) + O(t^{-2}(\log t)^2), \qquad \text{in } L^2$$

as $t \to \infty$. We then add a *second approximation* v_{ap} of u, given in (4.1), in order to cancel the non-resonant term $N_{\mathrm{nr}}(u_{\mathrm{ap}})$ out.

To obtain the desired cancellation, we will choose suitable A_n. More precisely, we choose them so that the leading term of n-th term of $(\Box + 1)v_{\mathrm{ap}}$ and n-th term of the Fourier expansion of $N_{\mathrm{nr}}(u_{\mathrm{ap}})$ coincide. By a computation, we have

$$\begin{aligned}(\Box + 1)v_{\mathrm{ap}} &= t^{-2}\sum_{n=2}^{\infty}(1-n^2)A_n(\mu)\cos\left(n(\alpha + \Phi(\mu)\log t - \beta)\right) \\ &\quad + O(t^{-2}(\log t)^2), \qquad \text{in } L^2\end{aligned}$$

as $t \to \infty$. Hence, we obtain the specific choice

$$A_n(\mu) = \begin{cases} \dfrac{8\sin(\frac{n}{2}\pi)}{\pi n(n^2-1)(n^2-4)}\lambda A_1^2(\mu) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{4.3}$$

With this choice, the leading term of the n-th term of $(\Box + 1)v_{\mathrm{ap}}$ and the n-th term of the Fourier expansion for $N_{\mathrm{nr}}(u_{\mathrm{ap}})$ successfully cancel out each other. Further, it turns out that the error term can be handled thanks to fast decay of A_n in n. Remark that the coefficients of A_n is order $O(|n|^{-5})$ as $|n| \to \infty$. The decay rate of the Fourier coefficients reflects the smoothness of the nonlinearity $\lambda|u|u$. Thus, we see that $A = u_{\mathrm{ap}} + v_{\mathrm{ap}}$ satisfies the conditions (2.1) and (2.2).

5. Outline of the proof of Theorem 1.1 Case: $d = 3$

In this section, we give an outline of the proof of Theorem 1.1 for $d = 3$ which is given by [23]. In this case, the power becomes a fractional number, so the argument in the two dimensional case [22] is not directly applicable. To overcome this difficulty, we use the argument by Ginibre and Ozawa [5].

We now explain how to construct the function $A = A(t,x)$ satisfying the conditions (2.1) and (2.2). The conclusion is that the choice $A := \tilde{u}_{\rm ap} + \tilde{v}_{\rm ap}$ works, where $\tilde{u}_{\rm ap}$ is the *first approximation* given by

$$\tilde{u}_{\rm ap} \quad := \quad t^{-\frac{3}{2}}\mathbf{1}_{\{|x|<t\}}A_1(\mu)\cos(\alpha + \tilde{\Psi}(\mu)\log t - \beta),$$

where $\tilde{\Psi}$ is given by

$$\tilde{\Psi}(\mu) = \sqrt{A_1^2(\mu) + t^{-1}}$$

and $\tilde{v}_{\rm ap}$ is *the second approximation* which is of the form

$$\tilde{v}_{\rm ap} \quad := \quad t^{-\frac{5}{2}}\mathbf{1}_{\{|x|<t\}}\sum_{n=2}^{\infty} A_n(\mu)\cos\Big(n(\alpha + \tilde{\Psi}(\mu)\log t - \beta)\Big). \tag{5.1}$$

where choice of A_n will be specified later. Note that $\tilde{v}_{\rm ap}(t) = O(t^{-1})$ in L_x^2. Hereafter, we consider the case $|x| < t$ only because $\tilde{u}_{\rm ap}$ and $\tilde{v}_{\rm ap}$ are identically zero in the region $|x| \geqslant t$.

We first focus on the nonlinear part $N(\tilde{u}_{\rm ap}) = \lambda|\tilde{u}_{\rm ap}|^{2/3}\tilde{u}_{\rm ap}$. As is the case of $d = 2$, $N(u) = \lambda|u|^{2/3}u$ is not polynomial in $(u, \overline{u})$, so we use the Fourier series expansion of $N(\tilde{u}_{\rm ap})$ to decompose $N(\tilde{u}_{\rm ap})$ into the resonant part and the rest, the *non-resonant part*. This decomposition is done as follows.

$$\begin{aligned}
&(5.2)\ N(\tilde{u}_{\rm ap}) \\
&= \lambda t^{-\frac{5}{2}}A_1(\mu)^{\frac{5}{3}}|\cos(\alpha + \tilde{\Phi}(\mu)\log t - \beta)|^{\frac{2}{3}}\cos(\alpha + \tilde{\Phi}(\mu)\log t - \beta) \\
&= \lambda t^{-\frac{5}{2}}A_1(\mu)^{\frac{5}{3}}\sum_{n\geqslant 1} c_n\cos(n(\alpha + \tilde{\Phi}(\mu)\log t - \beta)) \\
&= \lambda t^{-\frac{5}{2}}A_1(\mu)^{\frac{5}{3}}c_1\cos(\alpha + \tilde{\Phi}(\mu)\log t - \beta) \\
&\quad + \sum_{n\geqslant 2}\lambda c_n t^{-\frac{5}{2}}A_1(\mu)^{\frac{5}{3}}\cos(n(\alpha + \tilde{\Phi}(\mu)\log t - \beta)) \\
&=: N_{\rm r}(\tilde{u}_{\rm ap}) + N_{\rm nr}(\tilde{u}_{\rm ap}),
\end{aligned}$$

where c_n are the Fourier coefficients for the function $|\cos\theta|^{2/3}\cos\theta$:

$$c_n = \frac{1}{\pi}\int_0^{2\pi}|\cos\theta|^{\frac{2}{3}}\cos\theta\cos n\theta d\theta.$$

Note that c_n are explicitly given by

$$\begin{cases} \dfrac{2(-1)^{\frac{n-1}{2}}\Gamma(\frac{11}{6})\Gamma(\frac{3n-5}{6})}{\sqrt{\pi}\Gamma(-\frac{1}{3})\Gamma(\frac{3n+11}{6})} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

see Masaki, Miyazaki and Uriya [20] for the detail. Since both of the resonant and non-resonant parts are $O(t^{-1})$ in L_x^2, we need to cancel out those terms by the linear part, otherwise (2.2) fails. Thanks to the phase correction $\tilde{\Psi}$, we have the desired cancellation of the resonant part. Namely, we have

$$(\Box + 1)\tilde{u}_{\rm ap} = N_{\rm r}(\tilde{u}_{\rm ap}) + O(t^{-\frac{11}{5}}(\log t)), \qquad \textit{in } L^2$$

as $t \to \infty$. We then add a *second approximation* $\tilde{v}_{\text{ap}}$ of u, given in (5.1), in order to cancel the non-resonant term $N_{\text{nr}}(\tilde{u}_{\text{ap}})$ out.

To obtain the desired cancellation, we will choose A_n appropriately. More precisely, we choose them so that the leading term of n-th term of $(\Box + 1)\tilde{v}_{\text{ap}}$ and n-th term of the Fourier expansion of $N_{\text{nr}}(\tilde{u}_{\text{ap}})$ coincide. By a computation, we have

$$\begin{aligned}(\Box + 1)\tilde{v}_{\text{ap}} &= t^{-\frac{5}{2}} \sum_{n=2}^{\infty} (1-n^2) A_n(\mu) \cos\left(n(\alpha + \Phi(\mu)\log t - \beta)\right) \\ &\quad + O(t^{-2}), \qquad \textit{in } L^2\end{aligned}$$

as $t \to \infty$. Hence, we obtain the specific choice

$$A_n(\mu) = \frac{c_n \lambda}{1-n^2} A_1^{\frac{5}{3}}(\mu). \tag{5.3}$$

With this choice, the leading term of the n-th term of $(\Box + 1)\tilde{v}_{\text{ap}}$ and the n-th term of the Fourier expansion for $N_{\text{nr}}(\tilde{u}_{\text{ap}})$ successfully cancel out each other. Further, it turns out that the error term can be handled thanks to fast decay of A_n in n. Remark that the coefficients of A_n is order $O(|n|^{-14/3})$ as $|n| \to \infty$. The decay rate of the Fourier coefficients reflects the smoothness of the nonlinearity $\lambda|u|^{2/3}u$. Thus, we see that $A = \tilde{u}_{\text{ap}} + \tilde{v}_{\text{ap}}$ satisfies the conditions (2.1) and (2.2).

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