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# Energy transfer model and large periodic boundary value problem for the quintic NLS

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## 1 Introduction

This note is based on a talk given at the conference on Nonlinear Wave and Dispersive Equations, Kyoto University. We consider a dynamics of mass density $|\widehat{u}(t,\xi)|$ and energy exchanges between a linear oscillator and a nonlinear interaction for the following defocusing quintic NLS equation:

$$i\partial_t u + \partial_x^2 u = |u|^4 u, \quad t \in \mathbb{R},\ x \in \mathbb{T}_L = [0, 2\pi L], \tag{1.1}$$

where $u = u(t,x) : \mathbb{R} \times \mathbb{T}_L \to \mathbb{C}$ is a complex-valued function and the spatial domain $\mathbb{T}_L$ is taken to be a torus of length $2\pi L > 0$, i.e., we assume the periodic boundary condition:

$$u(t, x + 2\pi L) = u(t, x).$$

The aim is to understand the dynamics of mass density $|\widehat{u}(t,\xi)|$, namely,

(i) how the wave energy is exchanged to another,

(ii) provide a demonstration of the conservative energy exchange of solutions between the modes initially excited.

The equation (1.1) possesses at least two conservation laws, the mass $M[u](t)$ and energy $E[u](t)$:

$$M[u](t) = \|u(t)\|_{L^2}^2,$$
$$E[u](t) = \int_{\mathbb{T}_L} \frac{1}{2}|\partial_x u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6\, dx = E[u](0).$$

These quantities impose the constraints on a dynamics of mass density of solutions.

We expect the following conjecture.

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**Conjecture 1.1.** *On bounded domain case: if the nonlinear Schrödinger equation is not integrable, then there exists a time global smooth solution $u(t)$ and some Sobolev exponent $s > 1$ such that*

$$\lim_{|t|\to\infty} \|u(t)\|_{H^s} = \infty.$$

*Remark* 1.1. The above conjecture means that the mass is shifted to high frequencies.

In fact, on $\mathbb{R}$ domain case, the local well-posedness for (1.1) was proved by Cazenave and Weissler [3] for data in $L^2$ (see also [8] and [12]). Notice that the time of existence time depends on the position of data and not only on its size. One can also prove the global well-posedness in $L^2$ provided that the initial data in $L^2$ is sufficiently small by using the above conservation laws. It is should be noted that the equation (1.1) is left invariant by the scaling

$$u \mapsto u_\lambda; \quad u(t,x) \mapsto u_\lambda(t,x) = \lambda^{1/2} u(\lambda^2 t, \lambda x), \qquad \lambda > 0,$$

which preserves the homogeneous Sobolev norm $\dot{H}^s(\mathbb{R})$ with $s = 0$. The global existence result for any data in $L^2$ was proved by Dodson [6]. In [6], the deep results on scattering behavior of solutions were also obtained. On the other hand, the associated focusing nonlinear Schrödinger equation (the minus sign applies to the nonlinear term) has a finite time blow-up solution [9].

## 2 Past works on the periodic boundary domain

When the spatial dimension is the two-dimensional torus $\mathbb{T}^2$, Bourgain [2] considered the cubic nonlinear Schrödinger equation in the defocusing case

$$i\partial_t u + \Delta u = |u|^2 u,$$

and obtained the apriori estimate of solutions

$$\|u(t)\|_{H^s} \lesssim \langle t \rangle^{2(s-1)+} \quad \text{for } u(0) \in H^s, \ s \geq 4.$$

In [5], Colliander, Keel, Staffilani, Takaoka and Tao constructed the solution satisfying that for any $s > 1$, $K \gg 1$, $0 < \sigma < 1$ there exist a solution $u(t)$ and a time $T > 0$ such that

$$\|u(0)\|_{H^s} \leq \sigma \quad \text{and} \quad \|u(T)\|_{H^s} \geq K.$$

Observe that the cubic nonlinear Schrödinger equation in two spatial dimensions is known as an example of invariant under the $L^2$-scaling:

$$u \mapsto u_\lambda = \lambda u(\lambda^2 t, \lambda x) \quad (\lambda > 0).$$

On the other hand, the quintic nonlinear Schrödinger equation in one dimension obeys scale invariance under the $L^2$-scaling. In [10], Grébert and L. Thomann examined the dynamics exhibited by the following Cauchy problem

$$i\partial_t u + \partial_x^2 u = \nu |u|^4 u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \tag{2.1}$$

for $\nu > 0$. More precisely, they proved the following theorem.

**Theorem 2.1** (Grébert and Thomann [10]). *Let $k \in \mathbb{Z}\backslash\{0\}$ and $n \in \mathbb{Z}$. $\mathcal{A}$ is a set of the form $\mathcal{A} = \{a_2,\ a_1,\ b_2,\ b_1\}$ where $a_2 = n,\ a_1 = n+3k,\ b_2 = n+4k,\ b_1 = n+k$. There exist $T > 0,\ \lambda_0 > 0$ and a $2T$-periodic function $K_* : \mathbb{R} \mapsto (0,1)$ which satisfies $K_*(0) \leq 1/4$ and $K_*(T) \geq 3/4$ so that if $0 < \nu < \nu_0$, there exists a solution to (2.1) satisfying for all $0 \leq t \leq \nu^{-3/2}$*

$$u(t,x) = \sum_{j\in\mathcal{A}} u_j(t)e^{ijx} + \nu^{1/4}q_1(t,x) + \nu^{3/2}tq_2(t,x),$$

*with*

$$|u_{a_1}(t)|^2 = 2|u_{a_2}(t)|^2 = K_*(\nu t),$$
$$|u_{b_1}(t)|^2 = 2|u_{b_2}(t)|^2 = 1 - K_*(\nu t),$$

*and where for all $s \in \mathbb{R}$, $\|q_1(t,\cdot)\|_{H^s(\mathbb{T}_{\mathbb{Z}})} \leq C_s$ for all $t \in \mathbb{R}_+$, and $\|q_2(t,\cdot)\|_{H^s(\mathbb{T})} \leq C_s$ for all $0 \leq t \leq \nu^{-3/2}$.*

We now define the wavenumber set consisting of nonlinear resonance interactions in the equation (1.1).

**Definition 2.1** (Resonance interaction set). Let $n \in \mathbb{Z}$ and $k \in \mathbb{Z}\backslash\{0\}$ be fixed. For $j \in \mathbb{Z}$, we set $\alpha_{1,j},\ \alpha_{3,j},\ \alpha_{2,j}$ and $\alpha_{4,j}$ as follows:

$$\frac{\alpha_{1,j}}{2\pi} = n + 3k + \frac{j}{L},\ \frac{\alpha_{3,j}}{2\pi} = n + \frac{j}{L},\ \frac{\alpha_{2,j}}{2\pi} = n + k + \frac{j}{L},\ \frac{\alpha_{4,j}}{2\pi} = n + 4k + \frac{j}{L}.$$

With $\alpha_{m,j}$, we set

$$\mathcal{A}_m = \{\alpha_{m,j} \mid j \in \mathbb{Z},\ 0 \leq j \leq L\},$$

for $1 \leq m \leq 4$, and $\mathcal{C} = \cup_{m=1}^4 \mathcal{A}_m$.

Here we consider (1.1) on $\mathbb{T}_L$ instead of (2.1) on $\mathbb{T}$, and obtain the following theorem.

**Theorem 2.2.** *Let $n \in \mathbb{Z},\ k \in \mathbb{Z}\backslash\{0\},\ s \geq 1$ and large integer $L > 0$ be given. There exist a smooth global solution $u(t)$ and a time $T = O(L^{1/2-})$ s.t.*

$$u(t,x) = \sum_{m=1}^{4} u_{\mathcal{A}_m}(t,x) + e(t,x),$$

*where*

$$\|u_{\mathcal{A}_1}(t)\|_{L^2}^2 = 2\|u_{\mathcal{A}_3}(t)\|_{L^2}^2 = \frac{1}{2} - K(t),$$
$$\|u_{\mathcal{A}_2}(t)\|_{L^2}^2 = 2\|u_{\mathcal{A}_4}(t)\|_{L^2}^2 = \frac{1}{2} + K(t),$$

$K(t) \approx \sin(\mathrm{Arctan}\frac{t}{4L^3})$ *for* $|t| \lesssim L^{1/2-}$,

$$\sup_{|t|\leq T} \|e(t,\cdot)\|_{H^s} \lesssim \frac{1}{L^{1/2-}}.$$

*where a constant $c_j$'s are independent of $L$.*

*Remark* 2.1. Putting $n = 0, \ s \geq 0, \ |k|^s \gg L^{\frac{5}{2}+\varepsilon}$, we have that there exists a solution $u(t)$ s.t. for some time $t_0 > 0$,

$$\|u(t_0)\|_{H^s}^2 - \|u(-t_0)\|_{H^s}^2 \approx \langle k\rangle^{2s}(1 + \frac{4^{2s}}{2} - 3^{2s})K(t_0).$$

If $s > 1$, then $\|u(-t_0)\|_{H^s} < \|u(t_0)\|_{H^s}$ (energy does not decrease). As opposites, we can construct solution whose energy will not increase for a long time.

# 3 Proof of Theorem 2.2

The proof of Theorem 2.2 consists of three steps:

1. construct resonant sets,
2. construct finite dimensional model with initial replacement,
3. construct approximation lemma (error estimates).

## 3.1 Resonant sets

We first set nonlinear resonant interaction sets as follows.

**Definition 3.1.** We say the set $\{(\xi_1, \xi_3, \xi_5), \ (\xi_2, \xi_4, \xi_6)\}$ is resonance, if and only if the following conditions hold;

(i) $\xi_1 + \xi_3 + \xi_5 = \xi_2 + \xi_4 + \xi_6$,

(ii) two of $\xi_1, \xi_3, \xi_5$ are elements of $\mathcal{A}_1$, that are $n + 3k + \frac{j_1}{L}, \ n + 3k + \frac{j_3}{L}$,

(iii) one of $\xi_1, \xi_3, \xi_5$ is element of $\mathcal{A}_2$, that is $n + \frac{j_5}{L}$,

(iv) two of $\xi_2, \xi_4, \xi_6$ are elements of $\mathcal{A}_3$, that are $n + k + \frac{j_2}{L}, \ n + k + \frac{j_4}{L}$,

(v) one of $\xi_2, \xi_4, \xi_6$ is element of $\mathcal{A}_4$, that is $n + 4k + \frac{j_6}{L}$,

(vi) $\{j_1, j_3\} = \{j_2, j_4\}$ and $j_5 = j_6 = \frac{j_1+j_3}{2} = \frac{j_2+j_4}{2}$.

From the relation in the above definition, we have the following lemma.

**Lemma 3.1.** *If* $\{(\xi_1, \xi_3, \xi_5), \ (\xi_2, \xi_4, \xi_6)\}$ *is resonance, then* $\phi(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) = 0$.

*Proof.* Since the equation $(n + 3k)^2 + (n + 3k)^2 + n^2 = (n + k)^2 + (n + k)^2 + (n + 4k)^2$, we calculate

$$\frac{1}{(2\pi)^2}\phi(\xi_1, \xi_2.\xi_3, \xi_4, \xi_5, \xi_6) = \frac{2k}{L}(3(j_1 + j_3) - (j_2 + j_4) - 4j_6) + \frac{1}{L^2}\phi(j_1, j_2, j_3, j_4, j_5, j_6),$$

which is equal to zero by $j_6 = \frac{j_1+j_3}{2} = \frac{j_2+j_4}{2}$ and $\phi(j_1, j_2, j_3, j_4, j_5, j_6) = 0$. $\square$

## 3.2 Finite dimensional model

Now suppose that $u$ is a smooth solution and let us start with the ansatz

$$u(t,x) = e^{-iGt} \int a_\xi(t) e^{ix\xi - it\xi^2} \, (d\xi)_L,$$

where

$$\phi(x) = \int e^{ix\xi} \widehat{\phi}(\xi) \, (d\xi)_L := \frac{1}{L} \sum_{\xi \in 2\pi\mathbb{Z}/L} e^{ix\xi} \widehat{\phi}(\xi).$$

We choose the parameter $G = -\frac{6M_0^2}{L^2}$ where $M_0 = \|u(0)\|_{L^2}^2$. By direct calculation, $a_\xi = a_\xi(t)$ satisfies

$$\begin{aligned} \dot{a}_\xi &= \left( -\frac{6M_0}{L^3}|a_\xi|^2 - \frac{3}{L^3} \int |a_{\xi'}|^4 (d\xi')_L + \frac{4}{L^4}|a_\xi|^4 \right) a_\xi \\ &+ \int^{*}_{\{\xi_1,\xi_3,\xi_5\} \neq \{\xi_2,\xi_4,\xi\}} a_{\xi_1} \overline{a_{\xi_2}} a_{\xi_3} \overline{a_{\xi_4}} a_{\xi_5} e^{it\phi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi)}, \end{aligned} \tag{3.1}$$

where $\phi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = -\xi_1^2 + \xi_2^2 - \xi_3^2 + \xi_4^2 - \xi_5^2 + \xi_6^2$.

Then plugging the observation in Lemma 3.1 into the equation (3.1), we have the resonant formula

$$\begin{aligned} i\dot{r}_\xi &= \left( -\frac{6M_0}{L^3}|r_\xi|^2 - \frac{3}{L^3} \int |r_{\xi'}|^4 (d\xi')_L + \frac{4}{L^4}|r_\xi|^4 \right) r_\xi \\ &+ \int_{\mathrm{res}(\xi)} r_{\xi_1} \overline{r_{\xi_2}} r_{\xi_3} \overline{r_{\xi_4}} r_{\xi_5}, \end{aligned} \tag{3.2}$$

where $\mathrm{res}(\xi)$ denotes the resonant modes with respect to $\xi_j$, $1 \le j \le 5$ such that the set $\{(\xi_1,\xi_3,\xi_5),(\xi_2,\xi_4,\xi)\}$ is resonance.

## 3.3 A priori estimates

Next, observe the conserved quantities for (3.1).

**Lemma 3.2.** *Let $\{r_\xi(t)\}$ be a global solution to recasted NLS (3.2). Then we have the relations;*

$$\frac{d}{dt} \sum_{\xi \in C} |r_\xi(t)|^2 = 0, \tag{3.3}$$

$$\frac{d}{dt} \sum_{\xi \in C} |\xi|^2 |r_\xi(t)|^2 = 0. \tag{3.4}$$

*Remark* 3.1. These corresponds to the mass and the energy conservations, respectively.

*Proof of Lemma 3.2.* It will be convenience to change the index $\xi$ by $\xi_6$. We first prove (3.3). Multiplying $\overline{r_{\xi_6}}$ to (3.2) and taking the imaginary part, we have that

$$\mathrm{Im}(i\dot{r}_{\xi_6}\overline{r_{\xi_6}}) = \frac{1}{L^4}\mathrm{Im}\sum_{\mathrm{res}(\xi_6)} r_{\xi_1}\overline{r_{\xi_2}}r_{\xi_3}\overline{r_{\xi_4}}r_{\xi_5}\overline{r_{\xi_6}}.$$

Note that the left-hand side will be $-\frac{1}{2}\frac{d}{dt}|r_{\xi_6}|^2$. Then after the summation over the resonant set, we arrive at the following;

$$-\frac{1}{2}\frac{d}{dt}\sum_{\xi_6\in\mathcal{C}}|r_{\xi_6}(t)|^2 = \frac{1}{2iL^4}\sum_{\xi_6\in\mathcal{C}}\sum_{\mathrm{res}(\xi_6)}\left(r_{\xi_1}\overline{r_{\xi_2}}r_{\xi_3}\overline{r_{\xi_4}}r_{\xi_5}\overline{r_{\xi_6}} - \overline{r_{\xi_1}}r_{\xi_2}\overline{r_{\xi_3}}r_{\xi_4}\overline{r_{\xi_5}}r_{\xi_6}\right),$$

which is zero, since by symmetry.

Next we prove (3.4). In a similar way to above, we have

$$\begin{aligned}\frac{d}{dt}\sum_{\xi\in\mathcal{C}}|\xi|^2|r_\xi(t)|^2 &= 2\mathrm{Re}\sum_{\xi_6\in\mathcal{C}}|\xi_6|^2\dot{r}_{\xi_6}\overline{r_{\xi_6}}\\ &= \frac{2}{L^4}\mathrm{Im}\sum_{\xi_6\in\mathcal{C}}\sum_{\mathrm{res}(\xi_6)}|\xi_6|^2 r_{\xi_1}\overline{r_{\xi_2}}r_{\xi_3}\overline{r_{\xi_4}}r_{\xi_5}\overline{r_{\xi_6}},\end{aligned}$$

which is deduced by

$$-\frac{1}{3iL^4}\sum_{\mathrm{res}}\left((\xi_1^2+\xi_3^2+\xi_5^2)-(\xi_2^2+\xi_4^2+\xi_6^2)\right) r_{\xi_1}\overline{r_{\xi_2}}r_{\xi_3}\overline{r_{\xi_4}}r_{\xi_5}\overline{r_{\xi_6}},$$

since by symmetrization. Th e last term is zero, since $(\xi_1^2+\xi_3^2+\xi_5^2)-(\xi_2^2+\xi_4^2+\xi_6^2)=0$ for the resonance interaction modes. □

Let us consider the ansatz

$$r_\xi(t) = \sqrt{I_\xi(t)}e^{i\theta_\xi(t)}.$$

Once again, using this coordinate, we obtain the following lemma.

**Lemma 3.3.** *For $j\in[0,L]$, we have that*

$$\frac{d}{dt}\left(I_{n+\frac{j}{L}}(t) + I_{n+4k+\frac{j}{L}}(t)\right) = 0,$$

$$\frac{d}{dt}\left(I_{n+3k+\frac{j}{L}}(t) + I_{n+k+\frac{j}{L}}(t)\right) = 0,$$

Moreover, assuming additional constrains of $I_j(t)$ and $\theta_j(t)$, we have the following lemma.

**Lemma 3.4.** *Assume*

$$I_{n+\frac{j}{L}}(t) = I_{n+\frac{m}{L}}(t), \quad \theta_{n+\frac{j}{L}}(t) = \theta_{n+\frac{m}{L}}(t),$$

$$
\begin{aligned}
I_{n+3k+\frac{j}{L}}(t) &= I_{n+3k+\frac{m}{L}}(t), \quad \theta_{n+3k+\frac{j}{L}}(t) = \theta_{n+3k+\frac{m}{L}}(t),\\
I_{n+4k+\frac{j}{L}}(t) &= I_{n+4k+\frac{m}{L}}(t), \quad \theta_{n+4k+\frac{j}{L}}(t) = \theta_{n+4k+\frac{m}{L}}(t),\\
I_{n+k+\frac{j}{L}}(t) &= I_{n+k+\frac{m}{L}}(t), \quad \theta_{n+k+\frac{j}{L}}(t) = \theta_{n+k+\frac{m}{L}}(t)
\end{aligned}
$$

*for all $|j|, |m| \leq L$. Then*

$$
\frac{d}{dt}\sum\left(I_{n+3k+\frac{j}{L}}(t) - 2I_{n+\frac{j}{L}}(t)\right) = 0,
$$

$$
\frac{d}{dt}\sum\left(I_{n+k+\frac{j}{L}}(t) - 2I_{n+4k+\frac{j}{L}}(t)\right) = 0.
$$

*Proof of Lemmas 3.3 and 3.4.* The proof of Lemmas 3.3 and 3.4 is similar to the one of Lemma 3.2. □

Then by Lemmas 3.2, 3.3 and 3.4, it is reasonable that we recast the equation (3.2) into the following Toy model form:

$$
\dot{I}_{\mathcal{A}_2} = \frac{3!L^2 + 3(2L+1)}{(2L+1)L^4}\sqrt{I^2_{\mathcal{A}_2} I_{\mathcal{A}_4} I^2_{\mathcal{A}_1} I_{\mathcal{A}_3}}\sin\left(2\theta_{\mathcal{A}_2} + \theta_{\mathcal{A}_4} - 2\theta_{\mathcal{A}_1} - \theta_{\mathcal{A}_3}\right),
$$

$$
\dot{I}_{\mathcal{A}_1}(t) = \frac{3!(2L^2 + 2L+1)}{(2L+1)L^4}\sqrt{I^2_{\mathcal{A}_2} I_{\mathcal{A}_4} I^2_{\mathcal{A}_1} I_{\mathcal{A}_3}}\sin\left(2\theta_{\mathcal{A}_2} + \theta_{\mathcal{A}_4} - 2\theta_{\mathcal{A}_1} - \theta_{\mathcal{A}_3}\right),
$$

$$
\dot{I}_{\mathcal{A}_4} = \frac{3!L^2 + 3(2L+1)}{(2L+1)L^4}\sqrt{I^2_{\mathcal{A}_1} I_{\mathcal{A}_3} I^2_{\mathcal{A}_2} I_{\mathcal{A}_4}}\sin\left(2\theta_{\mathcal{A}_1} + \theta_{\mathcal{A}_3} - 2\theta_{\mathcal{A}_2} - \theta_{\mathcal{A}_4}\right)
$$

$$
\dot{I}_{\mathcal{A}_2}(t) = \frac{3!(2L^2 + 2L+1)}{(2L+1)L^4}\sqrt{I^2_{\mathcal{A}_1} I_{\mathcal{A}_3} I^2_{\mathcal{A}_2} I_{\mathcal{A}_4}}\sin\left(2\theta_{\mathcal{A}_1} + \theta_{\mathcal{A}_3} - 2\theta_{\mathcal{A}_2} - \theta_{\mathcal{A}_4}\right),
$$

$$
\begin{aligned}
-\dot{\theta}_{\mathcal{A}_3} \quad = \quad & -\frac{6M_0}{L^3}I_{\mathcal{A}_1} - \frac{12(L+1)}{L^4}\left(I_{\mathcal{A}_3} + I_{\mathcal{A}_1} + I_{\mathcal{A}_4} + I_{\mathcal{A}_2}\right) + \frac{4}{L^4}I^2_{\mathcal{A}_13}\\
& + \frac{3!L^2 + 3(2L+1)}{(2L+1)L^4}\sqrt{\frac{I^2_{\mathcal{A}_2} I_{\mathcal{A}_4} I^2_{\mathcal{A}_1}}{I_{\mathcal{A}_3}}}\cos\left(2\theta_{\mathcal{A}_2} + \theta_{\mathcal{A}_4} - 2\theta_{\mathcal{A}_1} - \theta_{\mathcal{A}_3}\right),
\end{aligned}
$$

$$
\begin{aligned}
-\dot{\theta}_{\mathcal{A}_1} \quad = \quad & -\frac{6M_0}{L^3}I_{\mathcal{A}_1} - \frac{12(L+1)}{L^4}\left(I_{\mathcal{A}3} + I_{\mathcal{A}_1} + I_{\mathcal{A}_4} + I_{\mathcal{A}_2}\right) + \frac{4}{L^4}I^2_{\mathcal{A}_1}\\
& + \frac{3!(2L^2 + 2L+1)}{(2L+1)L^4}\sqrt{I^2_{\mathcal{A}_2} I_{\mathcal{A}_4} I_{\mathcal{A}_3}}\cos\left(2\theta_{\mathcal{A}_2} + \theta_{\mathcal{A}_4} - 2\theta_{\mathcal{A}_1} - \theta_{\mathcal{A}_3}\right),
\end{aligned}
$$

$$\begin{aligned}-\dot{\theta}_{\mathcal{A}_4} &= -\frac{6M_0}{L^3}I_{\mathcal{A}_4} - \frac{12(L+1)}{L^4}\left(I_{\mathcal{A}_3}+I_{\mathcal{A}_1}+I_{\mathcal{A}_4}+I_{\mathcal{A}_2}\right)+\frac{4}{L^4}I_{\mathcal{A}_4}^2\\ &+\frac{3!L^2+3(2L+1)}{(2L+1)L^4}\sqrt{\frac{I_{\mathcal{A}_1}^2 I_{\mathcal{A}_3} I_{\mathcal{A}_2}^2}{I_{\mathcal{A}_4}}}\cos\left(2\theta_{\mathcal{A}_2}+\theta_{\mathcal{A}_4}-2\theta_{\mathcal{A}_1}-\theta_{\mathcal{A}_3}\right)\end{aligned}$$

and

$$\begin{aligned}-\dot{\theta}_{\mathcal{A}_2} &= -\frac{6M_0}{L^3}I_{\mathcal{A}_2} - \frac{12(L+1)}{L^4}\left(I_{\mathcal{A}_3}+I_{\mathcal{A}_1}+I_{\mathcal{A}_4}+I_{\mathcal{A}_2}\right)+\frac{4}{L^4}I_{\mathcal{A}_2}^2\\ &+\frac{3!(2L^2+2L+1)}{(2L+1)L^4}\sqrt{I_{\mathcal{A}_1}^2 I_{\mathcal{A}_3} I_{\mathcal{A}_4}}\cos\left(2\theta_{\mathcal{A}_2}+\theta_{\mathcal{A}_4}-2\theta_{\mathcal{A}_1}-\theta_{\mathcal{A}_3}\right).\end{aligned}$$

*Remark* 3.2. A straightforward calculation show that this ODE system enjoys the conservation of the Hamiltonian following Hamiltonian

$$\begin{aligned}H &= -\frac{3M_0}{L^3}\left(I_{\mathcal{A}_1}^2+I_{\mathcal{A}_2}^2+I_{\mathcal{A}_3}^2+I_{\mathcal{A}_4}^2\right)-\frac{6(L+1)}{L^4}\left(I_{\mathcal{A}_1}+I_{\mathcal{A}_2}+I_{\mathcal{B}_1}+I_{\mathcal{B}_2}\right)^2\\ &+\frac{4}{3L^4}\left(I_{\mathcal{A}_1}^3+I_{\mathcal{A}_2}^3+I_{\mathcal{A}_3}^3+I_{\mathcal{A}_4}^3\right)\\ &+\frac{3!(2L^2+2L+1)}{(2L+1)L^4}I_{\mathcal{A}_1}I_{\mathcal{A}_2}I_{\mathcal{A}_3}^{\frac{1}{2}}I_{\mathcal{A}_4}^{\frac{1}{2}}\cos\left(2\theta_{\mathcal{A}_2}+\theta_{\mathcal{A}_4}-2\theta_{\mathcal{A}_1}-\theta_{\mathcal{A}_3}\right).\end{aligned}$$

Indeed, we see that

$$\begin{cases}\dot{\theta}_{\mathcal{C}} = -\frac{\partial H}{\partial I_{\mathcal{C}}},\\ \dot{I}_{\mathcal{C}} = \frac{\partial H}{\partial \theta_{\mathcal{C}}},\end{cases}$$

for $\mathcal{C}=\mathcal{A}_j$.

In virtue of $\frac{d}{dt}(2I_{\mathcal{A}_4}^k+I_{\mathcal{A}_3}^k-2I_{\mathcal{A}_2}^k-I_{\mathcal{A}_1}^k)=0$ for $k=1,2$, we obtain the particular dynamics of $I_{\mathcal{A}_j}$, $\theta_{\mathcal{A}_j}$.

**Proposition 3.1.** *There exists a solution* $(I_{\mathcal{A}_j},\theta_{\mathcal{A}_j})_{1\le j\le 4}$ *to (Toy model) s.t.*

$$2\theta_{\mathcal{A}_2}+\theta_{\mathcal{A}_4}-2\theta_{\mathcal{A}_1}-\theta_{\mathcal{A}_3}\approx\frac{\pi}{2}$$

*and*

$$I_{\mathcal{A}_1}(t)=2I_{\mathcal{A}_3}(t)=\frac{1}{2}-K(t),\ I_{\mathcal{A}_2}(t)=2I_{\mathcal{A}_4}(t)=\frac{1}{2}+K(t),$$

$K(t)\approx\sin(\mathrm{Arctan}\frac{t}{4L^3})$

*Proof of Proposition 3.1.* The proof uses the symplectic change of variables.

### 3.4 Approximation lemma

We show how the toy model approximates the original NLS.

**Proposition 3.2.** *Let* $\{a_\xi(t)\}_{\xi\in\mathbb{Z}/L}$ *and* $\{r_\xi(t)\}_{\xi\in\mathcal{C}}$ *be a solution to the Fourier transformed NLS equation and its resonant NLS, respectively, with* $a_\xi(0) = r_\xi(0)$ *for* $\xi \in \mathcal{C}$ *and* $\|a_\xi(0)\|_{\ell^2_s(\xi\notin\mathcal{C})} \lesssim \frac{1}{L^{1/2-\varepsilon}}$. *Then for* $s \geq 1$ *and* $|t| \leq cL^{1/2-\varepsilon}$,

$$\|a_\xi(t) - r_\xi(t)\|_{\ell^2_s} \lesssim \frac{1}{L^{1/2-\varepsilon}},$$

*where* $r_\xi(t) = 0$ *if* $\xi \notin \mathcal{C}$, *and* $\|f\|_{\ell^2_s} = \|\langle\xi\rangle^s f(\xi)\|_{\ell^2}$.

*Proof of Proposition 3.2.* The proof uses a priori estimates of $M[u]$, $E[u]$, energy argument, bootstrap argument and the perturbation argument. □

By Propositions 3.1 and 3.2, we conclude the proof of Theorem 2.2.

## References


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