Non-smooth decomposition of homogeneous Triebel-Lizorkin spaces with applications to the Marcinkiewicz integral

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Abstract

The aim of this paper is to develop a theory of non-smooth decomposition in homogeneous Triebel-Lizorkin spaces. As a byproduct, we can recover the decomposition results for Hardy spaces as a special case The result extends what Frazier and Jawerth obtained in 1990. The result by Frazier and Jawerth covers only the limited range of the parameters but the result in this paper is valid for all admissible parameters for Triebel-Lizorkin spaces. As an application of the main results, we prove that the Marcinkiewicz operator is bounded. What is new in this paper is to reconstruct sequence spaces other than classical ℓ^p spaces.

1 Preparation and Main result

Definition 1. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbf{R}$. Let $\varphi \in C_c^\infty(\mathbf{R}^n)$ satisfy $\chi_{B(4)\setminus B(2)} \leq \varphi \leq \chi_{B(8)\setminus B(1)}$. The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbf{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbf{R}^n)/\mathcal{P}(\mathbf{R}^n)$ for which the quantity

$$\|f\|_{\dot{F}^s_{p,q}} \equiv \|\{2^{js}\varphi_j(D)f\}_{j\in\mathbf{Z}}\|_{L^p(l^q)}$$

is finite, where $\varphi_j(x) \equiv \varphi(2^{-j}x)$, $\mathcal{P}(\mathbf{R}^n)$ denotes the set of all polynomials on $\mathbf{R}^n$, and

$$\psi(D)f(x) \equiv \mathcal{F}^{-1}\psi * f(x) \quad (x \in \mathbf{R}^n)$$

for $\psi \in \mathcal{S}(\mathbf{R}^n)$ and $f \in \mathcal{S}'(\mathbf{R}^n)$ and $\|\{f_j\}_{j\in\mathbf{Z}}\|_{L^p(l^q)}$ stands for the vector-norm of a sequence $\{f_j\}_{j=-\infty}^{\infty}$ of mesurable functions: For $0 < p, q \leq \infty$

$$\|\{f_j\}_{j\in\mathbf{Z}}\|_{L^p(l^q)} \equiv \left(\int_{\mathbf{R}^n}\left(\sum_{j=-\infty}^{\infty}|f_j(x)|^q\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}$$

Remark 2. The space $\dot{F}^s_{p,q}(\mathbf{R}^n)$ realizes many function spaces. Indeed,

$$\dot{F}^0_{p,2}(\mathbf{R}^n) = L^p(\mathbf{R}^n) \text{ if } 1 < p < \infty, \dot{F}^0_{p,2}(\mathbf{R}^n) = H^p(\mathbf{R}^n) \text{ if } 0 < p \leq 1$$

with equivalence of quasi-norms, where $H^p(\mathbf{R}^n)$ stands for the Hardy Space. See [3, Theorem 6.1.2] for the first equivalence and [4, Theorem 2.2.9] for the second equivalence.

Definition 3. For $\nu \in \mathbf{Z}$ and $m = (m_1, m_2, \dots, m_n) \in \mathbf{Z}^n$, we define

$$Q_{\nu,m} \equiv \prod_{j=1}^{n} \left[\frac{m_j}{2^\nu}, \frac{m_j+1}{2^\nu}\right).$$

Denote by $\mathcal{D} = \mathcal{D}(\mathbf{R}^n)$ the set of such cubes. The elements in $\mathcal{D}(\mathbf{R}^n)$ are called dyadic cubes.

We adopt the definition by Grafakos; see [4, Definition 2.3.5].

Definition 4. Let $0 < p < \infty, 0 < q \leq \infty$ and $s \in \mathbf{R}$. We consider the set of sequences $\{r_Q\}_{Q\in\mathcal{D}} \subset \mathbf{C}$ such that the function

$$g^s_q(\{r_Q\}_{Q\in\mathcal{D}}; x) \equiv \left(\sum_{Q\in\mathcal{D}} (|Q|^{-\frac{s}{n}} |r_Q| \chi_Q(x))^q\right)^{\frac{1}{q}} \quad (x \in \mathbf{R}^n)$$

is in $L^p(\mathbf{R}^n)$. For such sequences $r = \{r_Q\}_{Q\in\mathcal{D}}$ set $\|r\|_{\dot{\mathbf{f}}^s_{p,q}} \equiv \|g^s_q(r)\|_{L^p}$. A sequence $r = \{r_Q\}_{Q\in\mathcal{D}}$ is said to belong to $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$ if $\|r\|_{\dot{\mathbf{f}}^s_{p,q}} < \infty$.

Definition 5. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbf{R}$. A sequence $r = \{r_Q\}_{Q\in\mathcal{D}}$ is called an ∞-atom for $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$ with cube Q_0 if there exists a dyadic cube Q_0 such that

$$g^s_q(\{r_Q\}_{Q\in\mathcal{D}}; \cdot) \equiv \left(\sum_{Q\in\mathcal{D}} (|Q|^{-\frac{s}{n}} |r_Q| \chi_Q)^q\right)^{\frac{1}{q}} \leq \chi_{Q_0}.$$

Our first theorem is as follows:

Theorem 6. *Suppose that we are given parameters p, q, s, u satisfying*

$$0 < p < \infty, \quad 0 < q \leq \infty, \quad s \in \mathbf{R}, \quad 0 < u \leq \min(1, q).$$

1. *For any $t \in \dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$, there exists a decomposition*

$$t = \sum_{j=1}^{\infty} \lambda_j r_j,$$

where each r_j is an ∞-atom for $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$ with cube Q_j and $\{\lambda_j\}_{j=1}^\infty$ satisfies

$$\left\| \left(\sum_{j=1}^\infty |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{L^p} \leq C \|t\|_{\dot{\mathbf{f}}^s_{p,q}}.$$

2. *If a sequence $\{Q_j\}_{j=1}^\infty$ of cubes and a sequence $\{\lambda_j\}_{j=1}^\infty$ of complex numbers satisfy*

$$\left\| \left(\sum_{j=1}^\infty |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{L^p} < \infty,$$

then for any ∞-atoms r_j for $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$ with cube Q_j, the series $t = \sum_{j=1}^\infty \lambda_j r_j$ belongs to $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$.

The case of $s \in \mathbf{R}, 0 < p = u \leq 1$ and $p \leq q \leq \infty$ is proved in [2, Theorem 7.2]. In this case there is no condition on the position of the cubes since

$$\left\| \left(\sum_{j=1}^\infty |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\| = \left(\sum_{j=1}^\infty |\lambda_j|^p |Q_j| \right)^{\frac{1}{p}}.$$

Definition 7. Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbf{R}$ and $0 < v < \infty$. One says that a sequence $r = \{r_Q\}_{Q \in \mathcal{D}}$ is called a v-atom for $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$ with cube Q_0 if there exists a dyadic cube Q_0 such that

$$\operatorname{supp}(g^s_q(\{r_Q\}_{Q \in \mathcal{D}}; \cdot)) \subset Q_0, \quad \|g^s_q(\{r_Q\}_{Q \in \mathcal{D}}; \cdot)\|_{L^v} \leq |Q_0|^{\frac{1}{v}}.$$

We can refine the latter half of Theorem 6 as follows:

Theorem 8. *In addition to the assumption in Theorem 6, let $v \in (\max(1,p), \infty)$. If a sequence $\{Q_j\}_{j=1}^\infty$ of cubes and a sequence $\{\lambda_j\}_{j=1}^\infty$ of complex numbers satisfy (??), then for any v-atoms r_j with cube Q_j, the series t given by* (1) *belongs to $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$.*

The above results cover the ones in [2, Section 7]. What is new about this paper is the case where $p > \min(q, 1)$. The case when $p > 1$ and $q = 2$ is especially interesting because this yields the decomposition for $L^p(\mathbf{R}^n) = \dot{F}^0_{p,2}(\mathbf{R}^n)$.

Definition 9. Let $0 < p < \infty, 0 < q \leq \infty, s \in \mathbf{R}$. Let $\nu \in \mathbf{Z}$ and $m \in \mathbf{Z}^n$. Suppose that the integers $K, L \in \mathbf{Z}$ satisfy $K \geq 0$ and $L \geq -1$. A function $a \in C^K(\mathbf{R}^n)$ is said to be a *smooth (K,L)-atom centered at $Q_{0,m}$* for $\dot{\mathbf{f}}^s_{p,q}(\mathbf{R}^n)$, if it is supported on $3\,Q_{0,m}$ and if it satisfies the differential inequality and the moment condition:

$$\|\partial^\alpha a\|_{L^\infty} \leq 2^{\nu|\alpha|}, \quad |\alpha| \leq K,$$

$$\int_{\mathbf{R}^n} x^\beta a(x)\,dx = 0, \quad |\beta| \leq L. \tag{1}$$

The case $L = -1$ is excluded in (1).

Definition 10. Let $0 < p < \infty, 0 < q \leq \infty, s \in \mathbf{R}$. We say that A is a non-smooth atom for $\dot{F}^s_{p,q}(\mathbf{R}^n)$ with cube $\tilde{Q}$ if there exists a cube $\tilde{Q}$ such that

$$A = \sum_{Q \subset \tilde{Q}} r_Q a_Q$$

where $r = \{r_Q\}_{Q \in \mathcal{D}}$ is an ∞-atom for $\dot{\mathbf{f}}^s_{p,q}$ and each a_Q is a smooth $(K, L)-$atom centered at Q.

The following theorem extends [4, Corollary 2.3.9].

Theorem 11. *Let* $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbf{R}$, $0 < u \leq \min(1, q)$, *and let*

$$\mathbf{Z} \ni L \geq \max(-1, [\sigma_{p,q} - s])$$

where $[\cdot]$ *denotes the Gauss sign,* $\sigma_p \equiv n\left(\frac{1}{\min(1,p)} - 1\right)$ *and* $\sigma_{p,q} \equiv \max(\sigma_p, \sigma_q)$. *Then we have the following.*

1. *Let* $f \in \dot{F}^s_{p,q}(\mathbf{R}^n)$. *Then we can write*

$$f = \sum_{j=1}^\infty \lambda_j A_j$$

in $\mathcal{S}'(\mathbf{R}^n)/\mathcal{P}(\mathbf{R}^n)$, *where* $\{A_j\}_{j=1}^\infty$ *is a sequence of non-smooth atoms and* $\{\lambda_j\}_{j=1}^\infty$ *and* $\{Q_j\}_{j=1}^\infty$ *satisfy the following condition:*

The estimate $\left\|\left(\sum_{j=1}^\infty |\lambda_j|^u \chi_{Q_j}\right)^{\frac{1}{u}}\right\|_{L^p} \leq C\,\|f\|_{\dot{F}^s_{p,q}}$ *holds and* $\mathrm{supp} A_j \subset 3Q_j$.

2. *Suppose that each* A_j *is a non-smooth atom with cube* Q_j *and the complex sequence* $\{\lambda_j\}_{j=1}^\infty$ *satisfies*

$$\left\|\left(\sum_{j=1}^\infty |\lambda_j|^u \chi_{Q_j}\right)^{\frac{1}{u}}\right\|_{L^p} < \infty.$$

Then by letting $f \equiv \sum_{j=1}^{\infty} \lambda_j A_j$, *the sum converges in* $\mathcal{S}'(\mathbf{R}^n)/\mathcal{P}(\mathbf{R}^n)$ *and satisfies*

$$\|f\|_{\dot{F}^s_{p,q}} \le C \left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{L^p}.$$

In Theorem 11 the case of $s \in \mathbf{R}, 0 < p = u \le 1$ and $p \le q \le \infty$ is [2, Theorem 7.4(ii)].

2 Application

Definition 12. Let $0 < \rho < n$, $1 < q < \infty$. The Marcinkiewicz operator is defined by

$$\mu_{\Omega,\rho,q} f(x) \equiv \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{B(t)} f(x-y) \frac{\Omega(y/|y|)}{|y|^{n-\rho}} dy \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

where we write $B(r) = \{|x| < r\} \subset \mathbf{R}^n$ for $r > 0$ here and below.

We suppose

$$\int_{S^{n-1}} \Omega(\omega) d\sigma(\omega) = 0, \ \Omega \in C^1(S^{n-1}),$$

where $S^{n-1} = \{|x| = 1\}$. According to [9, Theorem 1], we have

$$\|\mu_{\Omega,\rho,q} f\|_{L^p} \le C \|f\|_{\dot{F}^0_{p,q}}$$

if $1 < p < \infty$.

The following is an application of Theorem 11, and extends [9, Theorem 1].

Theorem 13. *The estimate* $\|\mu_{\Omega,\rho,q} f\|_{L^p} \le C \|f\|_{\dot{F}^0_{p,q}}$ *for all* $f \in \dot{F}^0_{p,q}$ *if*

$$\frac{nq}{nq+1} < p < \infty, 1 < q < \infty.$$

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