数理解析研究所講究録 第2095巻 2018年 29-33

Banach limits with values in $\mathcal{B}(\mathcal{H})$

Ryotaro Tanaka

1 Introduction

This note is a survey on [11]. A Banach limit is a special linear functional on the bounded sequence space ℓ_∞ introduced as a generalization of the "limit" functional. The precise definition is as follows.

Definition 1.1. A bounded linear functional φ on ℓ_∞ is called a Banach limit if

(i) φ is positive, that is, if $\alpha_n \geq 0$ for each $n \in \mathbb{N}$ then $\varphi((\alpha_n)_{n\in\mathbb{N}}) \geq 0$;

(ii) $\varphi((\alpha_{n+1})_{n\in\mathbb{N}}) = \varphi((\alpha_n)_{n\in\mathbb{N}})$ for each $(\alpha_n)_{n\in\mathbb{N}} \in \ell_\infty$; and

(iii) $\varphi((\alpha_n)_{n\in\mathbb{N}}) = \lim_n \alpha_n$ whenever $(\alpha_n)_{n\in\mathbb{N}}$ is a convergent sequence.

In particular, the positivity of φ shows that $\|\varphi\| = \varphi(\mathbf{1}) = 1$. The existence of Banach limits is guaranteed by simple arguments using the Hahn-Banach theorem or ultrafilters on $\mathbb{N}$; and such functionals have various applications in functional analysis.

On the other hand, there are some ideas to generalize the notion of Banach limits to the case of vector sequences. Such "operators" were initially introduced by Deeds [4] for Hilbert spaces. The definition adopted here is found in [2, 5, 6] and bit different from the scalar case, but it is in fact equivalent to the original one for scalar sequences (that is, the case of $X = \mathbb{C}$ in the below).

Definition 1.2. Let X be a Banach space. Then a bounded linear operator T from $\ell_\infty(X)$ (the space of bounded sequences in X equipped with the supremum norm) into X is called a Banach limit on $\ell_\infty(X)$ if

(i) $\|T\| = 1$;

(ii) $T((x_{n+1})_{n\in\mathbb{N}}) = T((x_n)_{n\in\mathbb{N}})$ for each $(x_n)_{n\in\mathbb{N}} \in \ell_\infty(X)$; and

(iii) $T((x_n)_{n\in\mathbb{N}}) = \lim_n x_n$ whenever $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in X.

Unlike the scalar case, the first problem is their existence. Actually, there are Banach spaces with no Banach limits. A typical example is (real) c_0 ([2]). However, the dual Banach spaces always have Banach limits. Indeed, for a scalar-valued Banach limit φ, putting $\langle x, \overline{\varphi}((f_n)_{n\in\mathbb{N}})\rangle = \varphi((f_n(x))_{n\in\mathbb{N}})$ for each $(f_n)_{n\in\mathbb{N}} \in \ell_\infty(X^*)$ and each $x \in X$ yields an X^*-valued Banach limit $\overline{\varphi}$.

For the case of $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on a Hilbert space $\mathcal{H}$, one has another natural way to introduce Banach limits. For a scalar-valued Banach limit φ, let $\langle \overline{\varphi}_{\mathcal{H}}((A_n)_{n\in\mathbb{N}})x, y\rangle = \varphi((\langle A_n x, y\rangle)_{n\in\mathbb{N}})$. Then $\overline{\varphi}_{\mathcal{H}}$ is a Banach limit on $\ell_\infty(\mathcal{B}(\mathcal{H}))$.

We here emphasize that, though $\overline{\varphi}_{\mathcal{H}}$ is formally related to the Hilbert space $\mathcal{H}$, it is not essential. Indeed, if we consider $\mathcal{B}(\mathcal{H})$ as the dual space (of the trace class), then $\overline{\varphi}_{\mathcal{H}} = \overline{\varphi}$ since $\rho(\overline{\varphi}_{\mathcal{H}}((A_n)_{n\in\mathbb{N}})) = \varphi((\rho(A_n))_{n\in\mathbb{N}})$ holds for each ultraweakly continuous functionals ρ on $\mathcal{B}(\mathcal{H})$. Hence the property of $\overline{\varphi}_{\mathcal{H}}$ seems to depend on the operator algebraic structure of $\mathcal{B}(\mathcal{H})$ rather than the operator theoretic one.

In this paper, we show that every $\mathcal{B}(\mathcal{H})$-valued Banach limit has the form $\overline{\varphi}_{\mathcal{H}}$ for some Banach limit φ on ℓ_∞. As an application, it is shown that the notion of almost convergence for sequences in $\mathcal{B}(\mathcal{H})$ cannot be characterized by using vector-valued Banach limits unless $\mathcal{H}$ is finite dimensional.

2 Banach limits with values in C^*-algebras

We first present the following proposition containing some basic properties of Banach limits with values in C^*-algebras.

Proposition 2.1. *Let $\mathfrak{A}$ be a C^*-algebra. Suppose that T is a Banach limit on $\ell_\infty(\mathfrak{A})$. Then the following hold:*

(i) *T is positive.*

(ii) *$T((ba_n)_{n\in\mathbb{N}}) = bT((a_n)_{n\in\mathbb{N}})$ and $T((a_nb)_{n\in\mathbb{N}}) = T((a_n)_{n\in\mathbb{N}})b$ hold for each $(a_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathfrak{A})$ and each $b \in \mathfrak{A}$.*

As an immediate consequence of the preceding proposition, we have a condition on C^*-algebras necessary for the existence of Banach limit with values in those algebras.

Corollary 2.2. *Let $\mathfrak{A}$ be a C^*-algebra. If there exists a Banach limit T on $\ell_\infty(\mathfrak{A})$, then $\mathfrak{A}$ is monotone σ-complete, that is, every norm bounded monotone increasing sequence in $\mathfrak{A}$ has a least upper bound.*

For a detailed investigation on monotone σ-complete C^*-algebras, the readers are referred to the monograph of Saitô and Wright [10].

Problem 2.3. What conditions on C^*-algebras $\mathfrak{A}$ are necessary and sufficient for the existence of $\mathfrak{A}$-valued Banach limits?

3 Banach limits on $\ell_\infty(\mathcal{B}(\mathcal{H}))$

We begin this section with the main result in this paper which shows that the set of vector-valued Banach limits with values in $\mathcal{B}(\mathcal{H})$ is in a one-to-one correspondence with that of usual complex-valued Banach limits.

Theorem 3.1. *Let $\mathcal{H}$ be a Hilbert space. If T is a Banach limit on $\ell_\infty(\mathcal{B}(\mathcal{H}))$, then $T = \overline{\varphi}_{\mathcal{H}}$ for some Banach limit φ on ℓ_∞.*

We here note that $\mathcal{B}(\mathcal{H})$-valued Banach limit $\overline{\varphi}_{\mathcal{H}}$ is restricted to any von Neumann algebra $\mathcal{R}$ acting on $\mathcal{H}$. Indeed, for each $(A_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathcal{R})$ and each $A' \in \mathcal{R}'$, we have

$$\begin{aligned}\langle \overline{\varphi}_{\mathcal{H}}((A_n)_{n\in\mathbb{N}})A'x, y\rangle &= \varphi((\langle A_nA'x, y\rangle)_{n\in\mathbb{N}})\\ &= \varphi((\langle A_nx, (A')^*y\rangle)_{n\in\mathbb{N}}) = \langle A'\overline{\varphi}_{\mathcal{H}}((A_n)_{n\in\mathbb{N}})x, y\rangle,\end{aligned}$$

which shows that $\overline{\varphi}_{\mathcal{H}}((A_n)_{n\in\mathbb{N}})A' = A'\overline{\varphi}_{\mathcal{H}}((A_n)_{n\in\mathbb{N}})$, that is, $\overline{\varphi}_{\mathcal{H}}((A_n)_{n\in\mathbb{N}}) \in \mathcal{R}'' = \mathcal{R}$ by the double commutant theorem.

The following results provide some special properties of Banach limits $\overline{\varphi}_{\mathcal{H}}$ with values in von Neumann algebras $\mathcal{R}$ acting on $\mathcal{H}$.

Proposition 3.2. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and let φ be a Banach limit on ℓ_∞. Then $\rho(\overline{\varphi}_{\mathcal{H}}((A_n)_{n\in\mathbb{N}})) = \varphi((\rho(A_n))_{n\in\mathbb{N}})$ for each $(A_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathcal{R})$ and each $\rho \in \mathcal{R}_*$.*

Corollary 3.3. *Let $\mathcal{R}$ and $\mathcal{S}$ be von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Suppose that $\Phi : \mathcal{R} \to \mathcal{S}$ is a $*$-isomorphism, and that φ is a Banach limit on ℓ_∞. Then*

$$\Phi^{-1}(\overline{\varphi}_{\mathcal{K}}((\Phi(A_n))_{n\in\mathbb{N}})) = \overline{\varphi}_{\mathcal{H}}((A_n)_{n\in\mathbb{N}})$$

for each $(A_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathcal{R})$.

In what follows, for a von Neumann algebra $\mathcal{R}$ and a Banach limit φ on ℓ_∞, let $\overline{\varphi}$ denote the $\mathcal{R}$-valued Banach limit satisfying $\rho(\overline{\varphi}((A_n)_{n\in\mathbb{N}})) = \varphi((\rho(A_n))_{n\in\mathbb{N}})$ for each $(A_n)_{n\in\mathbb{N}} \in \ell_\infty(\mathcal{R})$ and each $\rho \in \mathcal{R}_*$. By Proposition 3.2, if $\mathcal{R}$ acts on a Hilbert space $\mathcal{H}$, then $\overline{\varphi} = \overline{\varphi}_{\mathcal{H}}$.

We wonder whether Banach limits with values in von Neumann algebras has a general form.

Problem 3.4. Can we classify, or determine a general form of Banach limits with values in von Neumann algebras?

The following problem might be the first step.

Problem 3.5. Let $\mathcal{R}$ be a von Neumann algebra, and let T be a Banach limit with values in $\mathcal{R}$. Suppose that T satisfies $T((\alpha_n I)_{n\in\mathbb{N}}) \in \mathbb{C}I$ for each $(\alpha_n)_{n\in\mathbb{N}} \in \ell_\infty$. Does T have the form $\overline{\varphi}$ for some Banach limit φ on ℓ_∞?

4 Almost convergence in $\mathcal{B}(\mathcal{H})$

The notion of almost convergence can be found in Lorentz [9] and Boos [3]; see also [1, 2, 5, 6]. A sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X is said to be *almost convergent* to an element $x \in X$ (called the almost limit of $(x_n)_{n\in\mathbb{N}}$) if

$$\lim_p \sup_{m\in\mathbb{N}} \|p^{-1} \sum_{j=0}^{p-1} x_{m+j} - x\| = 0.$$

In the scalar case, almost convergence is characterized by using scalar-valued Banach limits. Namely, if $(\alpha_n)_{n\in\mathbb{N}} \in \ell_\infty$ and $\alpha \in \mathbb{C}$, then α is the almost limit of $(\alpha_n)_{n\in\mathbb{N}}$ if and only if $\varphi((\alpha_n)_{n\in\mathbb{N}}) = \alpha$ for each Banach limit φ on ℓ_∞. However, in the case of vector sequences, only one-sided implication is known, that is, if x is the almost limit of $(x_n)_{n\in\mathbb{N}}$ then $T((x_n)_{n\in\mathbb{N}}) = x$ for each X-valued Banach limit. This can be shown by a simple argument essentially found in the proof of [2, Theorem 2]. Indeed, if S is the unilateral shift on $\ell_\infty(X)$ given by $S((x_n)_{n\in\mathbb{N}})) = (x_{n+1})_{n\in\mathbb{N}}$, then it follows from

$$\lim_p \sup_{m\in\mathbb{N}} \|p^{-1}\sum_{j=0}^{p-1} x_{m+j} - x\| = 0$$

that $p^{-1}\sum_{j=0}^{p-1} S^j((x_n)_{n\in\mathbb{N}}))$ converges to $(x, x, \ldots)$ in $\ell_\infty(X)$ as $p \to \infty$. Hence we have

$$T((x_n)_{n\in\mathbb{N}}) = \lim_p T\left(p^{-1}\sum_{j=0}^{p-1} S^j((x_n)_{n\in\mathbb{N}}))\right) = x$$

for each Banach limit T on $\ell_\infty(X)$. On the other hand, whether the converse holds true is depend on case by case; see [4, 7, 8], for related results. A Banach space X is said to *verify the vector-valued version of the Lorentz theorem* if x is the almost limit of $(x_n)_{n\in\mathbb{N}}$ whenever $T((x_n)_{n\in\mathbb{N}}) = x$ for each X-valued Banach limit T.

As a consequence of Theorem 3.1, we have the following result which provides a natural Banach space that does not verify the vector-valued version of the Lorentz theorem.

Theorem 4.1. *Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{B}(\mathcal{H})$ verifies the vector-valued version of the Lorentz theorem if and only if $\mathcal{H}$ is finite dimensional.*

Remark 4.2. The convergence with respect to $\mathcal{B}(\mathcal{H})$-valued Banach limits is just the weak* version of almost convergence. Indeed, a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathcal{H})$ and an element $A \in \mathcal{B}(\mathcal{H})$ satisfies $T((A_n)_{n\in\mathbb{N}}) = A$ for each $\mathcal{B}(\mathcal{H})$-valued Banach limit if and only if $\overline{\varphi}((A_n)_{n\in\mathbb{N}}) = A$ for each Banach limit φ in ℓ_∞ by Theorem 3.1 and Lemma 3.2, which happens if and only if $\rho(A) = \varphi((\rho(A_n))_{n\in\mathbb{N}})$ for each $\rho \in \mathcal{B}(\mathcal{H})_*$ and each Banach limit φ in ℓ_∞. Finally, this last statement just means $\rho(A)$ is the almost limit of $(\rho(A_n))_{n\in\mathbb{N}}$ for each $\rho \in \mathcal{B}(\mathcal{H})_*$.

References

[1] A. Aizpuru, R. Armario, F. J. García-Pacheco and F. J. Pérez-Fernández, *Banach limits and uniform almost summability*, J. Math. Anal. Appl., **379** (2011), 82–90.

[2] R. Armario, F. J. García-Pacheco and F. J. Pérez-Fernández, *On vector-valued Banach limits*, Funct. Anal. Appl., **47** (2013), 315318.

[3] J. Boos, *Classical and modern methods in summability*, Oxford Math. Monogr., Oxford Univ. Press, Oxford, 2000.

[4] J. B. Deeds, *Summability of vector sequences*, Studia Math., **30** (1968), 361–372.

[5] F. J. García-Pacheco, *Extremal properties of the set of vector-valued Banach limits*, Open Math., **13** (2015), 757–767.

[6] F. J. García-Pacheco and F. J. Pérez-Fernández, *Fundamental aspects of vector-valued Banach limits*, Izv. Math., **80** (2016), 316–328.

[7] D. Hajduković, *Almost convergence of vector sequences*, Mat. Vesnik, **27** (1975), 245–249.

[8] J. C. Kurtz, *Almost convergent vector sequences*, Tôhoku Math. J., **22** (1970), 493–498.

[9] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math., **80** (1948), 167–190.

[10] K. Saitô and J. D. M. Wright, *Monotone complete C^*-algebras and generic dynamics*, Springer Monographs in Mathematics, Springer, London, 2015.

[11] R. Tanaka, *On vector-valued Banach limits with values in $\mathcal{B}(\mathcal{H})$*, to appear in Publ. Math. Debrecen.

Ryotaro Tanaka
Faculty of Mathematics,
Kyushu University,
Fukuoka 819-0395, Japan
E-mail: r-tanaka@math.kyushu-u.ac.jp
(Current address)
Faculty of Industrial Science and Technology,
Tokyo University of Science,
Hokkaido 049-3514, Japan
E-mail: r-tanaka@rs.tus.ac.jp