数理解析研究所講究録 第2095巻 2018年 64-69

A note on some martingale spaces

大阪教育大学　教育学部　貞末 岳 (Gaku Sadasue)
Department of Mathematics,
Osaka Kyoiku University

1 Introduction

In this paper, we point out that a convexity inequality holds on martingale Morrey spaces. To do this, we introduce a modification of the notion of Banach function spaces. We show that martingale Morrey spaces are not necessarily Banach function spaces, but the modified notion can be applied to martingale Morrey spaces. This paper is an announcement of the author's recent results [10].

2 $\mathcal{B}$-Banach function space

In this section, we introduce the notion of $\mathcal{B}$-Banach function spaces. It is a modification of the notion of Banach function spaces in the sense of Bennett and Sharpley [1].

Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Let $\mathcal{M}^+$ be the set of all $[0, \infty]$-valued measurable functions on Ω. We denote by L_0 the set of all complex valued measurable functions on Ω.

Let $\mathcal{B} = \{B_n\}$ be a countable family of mutually disjoint measurable sets in Ω. We say $\mathcal{B} = \{B_n\}$ is a measurable partition if $\mu(B_n) < \infty$ for all n and $\bigcup_n B_n = \Omega$.

Definition 2.1. Let $\rho : \mathcal{M}^+ \to [0, \infty]$. Let $\mathcal{B} = \{B_n\}$ be a measurable partition. We say ρ is a $\mathcal{B}$-function norm if

2000 *Mathematics Subject Classification.* Primary 46E30, 60G46; Secondary 42B35, 26A33.
Key words and phrases. martingale, Banach function space, convexity inequality.
The author was supported by Grant-in-Aid for Scientific Research (C), No. 16K05203, Japan Society for the Promotion of Science.

(B.1) For each $a \geq 0, f, g \in \mathcal{M}^+$, $\rho(af) = a\rho(f)$, $\rho(f+g) \leq \rho(f) + \rho(g)$. Moreover, $\rho(f) = 0$ if and only if $f = 0$ μ-a.e.

(B.2) If $g \leq f$, then $\rho(g) \leq \rho(f)$.

(B.3) If $f_n \uparrow f$, then $\rho(f_n) \uparrow \rho(f)$.

(B.4) For each $B \in \mathcal{B}$, $\rho(\chi_B) < \infty$.

(B.5) For each $B \in \mathcal{B}$, there exists $C_B \in (0, \infty)$ such that

$$\int_B f(\omega)\, d\mu(\omega) \leq C_B \rho(f) \quad (f \in \mathcal{M}^+).$$

For a $\mathcal{B}$-function norm ρ, define

$$X = \{f \in L_0; \rho(|f|) < \infty\}$$

and

$$\|f\|_X = \rho(|f|) \quad (f \in L_0).$$

By the same way as in [1], we see that $\|\cdot\|_X$ is a norm on X and $(X, \|\cdot\|_X)$ is a Banach space. We call such X a $\mathcal{B}$-Banach function space.

Remark 2.1. On $\mathbb{R}^d$, Hakim and Sawano introduced the notion of Ball Banach function spaces in [2]. Our definition is a measure theoretic version of it.

Below, we see that $\mathcal{B}$-Banach function spaces satisfy the same fundamental properties of Banach function spaces. We first introduce the norm associate to ρ.

Proposition 2.1. *Let $\mathcal{B} = \{B_n\}$ be a measurable partition. Let ρ be a $\mathcal{B}$-function norm. For ρ, we define $\rho' : \mathcal{M}^+ \to [0, \infty]$ by*

$$\rho'(g) = \sup_{\rho(f) \leq 1} \int_\Omega f(\omega) g(\omega)\, d\mu(\omega).$$

Then, ρ' is also a $\mathcal{B}$-function norm.

The proof of Proposition 2.1 is obtained by a modification of the one in [1, Theorem 2.2].

We say ρ' defined above the associate norm of ρ. Further, let $X' = \{f \in L_0; \rho'(|f|) < \infty\}$ and let $\|f\|_{X'} = \rho'(|f|)$. We call X' the associate space of X.

Proposition 2.2. *Let $\mathcal{B} = \{B_n\}$ be a measurable partition. Let X be a $\mathcal{B}$-Banach function space and let X' be the associate space of X. Then, $X'' = X$.*

The proof of Proposition 2.2 is also obtained by a modification of the one in [1, Theorem 2.7].

For a Banach space X, we denote by X^* the dual space of X. The following proposition is proved by a similar way in [1, Theorem 3.12]

Proposition 2.3. *Let $\mathcal{B} = \{B_n\}$ be a measurable partition. Let X be a $\mathcal{B}$-Banach function space. Let X_b be the closure of the set of all bounded functions in X supported on finitely many B in $\mathcal{B}$. Then, X_b is a norm fundamental subspace of $(X')^*$, that is,*

$$\|f\|_{X'} = \sup_{g \in X_b, \|g\|_X \leq 1} \left| \int_\Omega f(\omega) g(\omega) d\mu(\omega) \right|.$$

We apply this framework to obtain a convexity inequality for martingales. To explain this application, we introduce some notation.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_n\}_{n\geq 0}$ be a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. We suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms, where $B \in \mathcal{F}_n$ is called an atom (more precisely a $(\mathcal{F}_n, P)$-atom), if any $A \subset B$ with $A \in \mathcal{F}_n$ satisfies $P(A) = P(B)$ or $P(A) = 0$. Denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$. We also suppose that $(\Omega, \mathcal{F}, P)$ is non-atomic.

The expectation operator is denoted by E. Let $L_{p,\mathrm{loc}}$ be the set of all measurable functions such that $|f|^p \chi_B$ is integrable for all $B \in A(\mathcal{F}_0)$. If $\mathcal{F}_0 = \{\Omega, \emptyset\}$, then $L_{p,\mathrm{loc}} = L_p$. An $\mathcal{F}_n$-measurable function $g \in L_{1,\mathrm{loc}}$ is called the conditional expectation of $f \in L_{1,\mathrm{loc}}$ relative to $\mathcal{F}_n$ if

$$E[g\chi_B\chi_G] = E[f\chi_B\chi_G] \quad \text{for all} \quad B \in A(\mathcal{F}_0) \quad \text{and} \quad G \in \mathcal{F}_n.$$

We denote by $E_n f$ the conditional expectation of f relative to $\mathcal{F}_n$. We say a sequence $(f_n)_{n\geq 0}$ in $L_{1,\mathrm{loc}}$ is a martingale relative to $\{\mathcal{F}_n\}_{n\geq 0}$ if it is adapted to $\{\mathcal{F}_n\}_{n\geq 0}$ and satisfies $E_n[f_m] = f_n$ for every $n \leq m$.

For a martingale $f = (f_n)_{n\geq 0}$, define maximal functions by

$$M_n f = \sup_{0\leq m\leq n} |f_m|, \quad Mf = \sup_{n\geq 0} |f_n|.$$

For $f \in L_{1,loc}$, define sharp functions by

$$M^\sharp f = \sup_{n \geq 0} E_n[|f - f_{n-1}|] \quad (f \in L_{1,\mathrm{loc}}, f_{-1} = 0).$$

Let X be a $A(\mathcal{F}_0)$-Banach function space. Then, it is easy to see that $X \subset L_{1,\mathrm{loc}}$. Hence, by considering $f \in X$ as a martingale by $(E_n f)_{n \geq 0}$, we define $Mf = \sup_{n \geq 0} |E_n f|$.

We now state our main result.

Theorem 2.4. *Let X be a $A(\mathcal{F}_0)$-Banach function space. Suppose that there exists $C > 0$ such that*

$$\|Mf\|_X \leq C\|f\|_X \quad \textit{for all} \quad f \in X$$

and that

$$\|f\|_X \leq C\|M^\sharp f\|_X \quad \textit{for all} \quad f \in X.$$

Then, there exists $C' > 0$ such that

$$\left\| \sum_{n \geq 0} E_n h_n \right\|_X \leq C' \left\| \sum_{n \geq 0} h_n \right\|_X$$

for all sequence $(h_n)_{n \geq 0}$ of non-negative measurable functions.

The proof of Theorem 2.4 will be given in [10].

3 Application to martingale Morrey spaces.

In this section, we see that the notion of $\mathcal{B}$-Banach function spaces can be applied to martingale Morrey spaces. First, we explain notations.

We now recall the definition of martingale Morrey spaces.

Definition 3.1. Let $p \in [1, \infty)$ and $\lambda \in (-\infty, \infty)$. For $f \in L_{1,\mathrm{loc}}$, let

$$\|f\|_{L_{p,\lambda}} = \sup_{n \geq 0} \sup_{B \in A(\mathcal{F}_n)} \frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f|^p \, dP \right)^{1/p},$$

and define

$$L_{p,\lambda} = \{f \in L_{p,\mathrm{loc}} : \|f\|_{L_{p,\lambda}} < \infty\}.$$

It is easy to see that $L_{p,\lambda}$ are $A(\mathcal{F}_0)$-Banach function spaces. However, martingale Morrey spaces are not necessarily Banach function spaces in the sense of Bennett-Sharpley. We show this fact by giving an example.

Proposition 3.1. *Let $(\Omega, \mathcal{F}, \mu) = ((0,1], \mathcal{L}, m)$ be the Lebesgue space. Let $A(\mathcal{F}_0) = \{(\frac{1}{k+1}, \frac{1}{k}] : k \geq 1\}$ and let $A(\mathcal{F}_n) = \{(\frac{l}{2^n(k+1)}, \frac{l+1}{2^n k}] : k \geq 1, 1 \leq l \leq 2^n\}$. Let $f = \sum_{k=1}^{\infty} k\chi_{(\frac{1}{k+1}, \frac{1}{k}]}$. Then, f belongs to $L_{1,-1}$ but does not belong to L_1. In particular, $L_{1,-1}$ is not a Banach function space in the sense of Bennett-Sharpley.*

To state our application, we recall two theorems. One is the boundedness of M on martingale Morrey spaces.

Theorem 3.2 ([6, 7]). *Let $1 < p < \infty$ and $\lambda < 0$. Then M is bounded from $L_{p,\lambda}$ to itself.*

The other is an inequality on sharp maximal functions. We say $\{\mathcal{F}_n\}_{n\geq 0}$ is regular if there exists a constant $R \geq 2$ such that

$$f_n \leq R f_{n-1} \tag{3.1}$$

holds for all nonnegative martingales $(f_n)_{n\geq 0}$.

Theorem 3.3 ([8]). *Assume that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular. Let $f \in L_{p,\mathrm{loc}}$. Let $1 \leq p < \infty$ and $\lambda < 0$. If $M^\sharp f \in L_{p,\lambda}$, then $f \in L_{p,\lambda}$ and*

$$\|f\|_{L_{p,\lambda}} \leq C\|M^\sharp f\|_{L_{p,\lambda}}, \tag{3.2}$$

where the constant C is independent of f.

Now we state our application of Theorem 2.4 to martingale Morrey spaces.

Theorem 3.4. *Assume that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular and that $\lim_{n\to\infty} \sup_{B\in A(\mathcal{F}_n)} P(B) = 0$. Let $p \in (1,\infty)$ and $-1/p \leq \lambda < 0$. Then, there exists $C > 0$ such that*

$$\left\| \sum_{n\geq 0} E_n h_n \right\|_{L_{p,\lambda}} \leq C \left\| \sum_{n\geq 0} h_n \right\|_{L_{p,\lambda}}$$

for all sequence $(h_n)_{n\geq 0}$ of non-negative measurable functions.

References

[1] C. Bennett and R. Sharpley, Interpolation of operators, Pure and Applied Mathematics Vol. 129, Academic Press, Boston, MA, 1988.

[2] D. I. Hakim and Y. Sawano, Interpolation of generalized Morrey spaces. Rev. Mat. Complut. 29 (2016), no. 2, 295–340.

[3] R. L. Long, Martingale spaces and inequalities, Peking University Press, Beijing, 1993. ISBN: 7-301-02069-4

[4] T. Miyamoto, E. Nakai and G. Sadasue, Martingale Orlicz-Hardy spaces, Math. Nachr. 285 (2012), no. 5-6, 670-686.

[5] E. Nakai, G. Sadasue and Y. Sawano, Martingale Morrey-Hardy and Campanato-Hardy Spaces, J. Funct. Spaces Appl. 2013 (2013), Article ID 690258, 14 pages. DOI:10.1155/2013/690258

[6] E. Nakai and G. Sadasue, Martingale Morrey-Campanato spaces and fractional integrals, J. Funct. Spaces Appl. 2012 (2012), Article ID 673929, 29 pages. DOI:10.1155/2012/673929

[7] E. Nakai and G. Sadasue, Characterizations of boundedness for generalized fractional integrals on martingale Morrey spaces, Math. Inequalities Appl. 20 (2017), No 4, 929–947. doi:10.7153/mia-2017-20-58

[8] E. Nakai and G. Sadasue, Commutators of fractional integrals on martingale Morrey spaces, submitted.

[9] J. Neveu, Discrete-parameter martingales, North-Holland, Amsterdam, 1975. ISBN 0720428106

[10] G. Sadasue, An inequality on martingale Morrey spaces, in preparation.

[11] F. Weisz, Martingale Hardy spaces and their applications in Fourier analysis, Lecture Notes in Mathematics, 1568, Springer-Verlag, Berlin, 1994. ISBN: 3-540-57623-1