On weakly \((\tilde{\rho}, \tilde{D})\)-separable polynomials in skew polynomial rings

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Abstract

Separable polynomials in skew polynomial rings were studied extensively by Y. Miyashita, T. Nagahara, S. Ikehata, and G. S. et al. In particular, Ikehata gave the characterization of \((\tilde{\rho}, \tilde{D})\)-separable polynomials in skew polynomial rings. In this article, we shall introduce the notion of weakly \((\tilde{\rho}, \tilde{D})\)-separable polynomials in skew polynomial rings, and we shall give a characterization of the \((\tilde{\rho}, \tilde{D})\)-separability and that of the weak \((\tilde{\rho}, \tilde{D})\)-separability.

1 Introduction and Preliminaries

Throughout this paper, \(A/B\) will represent a ring extension with common identity 1. Let \(M\) be an \(A\)-\(A\)-bimodule, and \(x, y\) arbitrary elements in \(A\). An additive map \(\delta : A \to M\) is called a \(B\)-derivation of \(A\) to \(M\) if \(\delta(xy) = \delta(x)y + x\delta(y)\) and \(\delta(\alpha) = 0\) for any \(\alpha \in B\). Moreover, \(\delta\) is called inner if \(\delta(x) = mx - xm\) for some fixed element \(m \in M\). We say that a ring extension \(A/B\) is separable if the \(A\)-\(A\)-homomorphism of \(A \otimes_B A\) onto \(A\) defined by \(a \otimes b \mapsto ab\) splits. It is well known that \(A/B\) is separable if and only if for any \(A\)-\(A\)-bimodule \(M\), every \(B\)-derivation of \(A\) to \(M\) is inner (cf. [1, Satz 4.2]). A ring extension \(A/B\) is said to be weakly separable if every \(B\)-derivation of \(A\) to \(A\) is inner. The notion of a weakly separable extension was introduced by N. Hamaguchi and A. Nakajima (cf. [2]). Obviously, a separable extension is weakly separable.

Let \(B\) be a ring, \(\rho\) an automorphism of \(B\), \(D\) a \(\rho\)-derivation of \(B\). \(B[X; \rho, D]\) will mean the skew polynomial ring in which the multiplication is given by \(\alpha X = X\rho(\alpha) + D(\alpha)\) for any \(\alpha \in B\). We set \(B[X; \rho] := B[X; \rho, 0]\) and \(B[X; D] := B[X; 1_A, D]\). By \(B[X; \rho, D]_{(0)}\) we denote the set of all monic polynomials \(g\) in \(B[X; \rho, D]\) such that \(gB[X; \rho, D] = B[X; \rho, D]g\). For a polynomial \(f \in B[X; \rho, D]_{(0)}\), the residue ring \(B[X; \rho, D]/fB[X; \rho, D]\) is a free ring extension of \(B\). We say that a polynomial \(f \in B[X; \rho, D]_{(0)}\) is separable (resp. weakly separable) in \(B[X; \rho, D]\) if \(B[X; \rho, D]/fB[X; \rho, D]\) is separable (resp. weakly separable) over \(B\).
Throughout this article, we assume that $\rho D = D \rho$, and let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 \in B[X; \rho, D]_{(0)} \cap B^\rho[X]$ and

$$f' = mX^{m-1} + (m-1)X^{m-2}a_{m-1} \cdots + Xa_2 + a_1 \ (\text{the derivative of } f),$$

$$Y_0 := X^{m-1} + X^{m-2}a_{m-1} \cdots + Xa_2 + a_1,$$

$$\ldots,$$

$$Y_j := X^{m-j-1} + X^{m-j-2}a_{m-1} \cdots + Xa_{j+2} + a_{j+1},$$

$$\ldots,$$

$$Y_{m-2} := X + a_{m-1},$$

$$Y_{m-1} := 1.$$

We shall use the following conventions:

- $B^\rho := \{ \alpha \in B \mid \rho(\alpha) = \alpha \}$
- $B^D := \{ \alpha \in B \mid D(\alpha) = 0 \}$
- $B^{\rho, D} := B^\rho \cap B^D$
- $C(B^{\rho, D}) := \{ \beta \in B^{\rho, D} \mid b\beta = \beta b \ (\forall b \in B^{\rho, D}) \}$ (the center of $B^{\rho, D}$)
- $A := B[X; \rho, D] / fB[X; \rho, D]$
- $x := X + fB[X; \rho, D] \in A$
- $f' := f' + fB[X; \rho, D] \in A$
- $y_j := Y_j + fB[X; \rho, D] \in A \ (0 \leq j \leq m - 1)$

$\rho$ : an automorphism of $A$ defined by $\rho \left( \sum_{j=0}^{m-1} x^j c_j \right) = \sum_{j=0}^{m-1} x^j \rho(c_j) \ (c_j \in B)$

$D$ : a $\rho$–derivation of $A$ defined by $D \left( \sum_{j=0}^{m-1} x^j c_j \right) = \sum_{j=0}^{m-1} x^j D(c_j) \ (c_j \in B)$

For any subsets $T \subset B$ and $S \subset A$, we set

$$J_{m-1}(T) := \{ z \in A \mid \rho^{m-1}(\alpha)z = z\alpha \ (\forall \alpha \in T) \},$$

$$V(T) := \{ z \in A \mid \alpha z = z\alpha \ (\forall \alpha \in T) \},$$

$$W(S) := \left\{ \sum_{j=0}^{m-1} y_j \omega \otimes x^j \omega \in S \right\},$$

$$(A \otimes_B A)^S := \{ \varepsilon \in A \otimes_B A \mid \varepsilon w = w\varepsilon \ (\forall w \in S) \},$$

$$S^\rho := \{ z \in S \mid \rho(z) = z \},$$

$$S^D := \{ z \in S \mid D(z) = 0 \},$$

$$S^{\rho, D} := S^\rho \cap S^D.$$

Note that $J_{m-1}(B') = V(B')$ for any subset $B'$ of $B^\rho$. 
We shall state some basic results which were already known.

**Lemma 1.1** ([7, Lemma 1.6]). \( f \) is in \( B[X; \rho, D]_{(0)} \) if and only if

1. \( a_i \rho^m(\alpha) = \sum_{j=i}^{m} \binom{j}{i} \rho^j D^{j-i}(\alpha)a_j \) \( (\alpha \in B, 0 \leq i \leq m-1, a_m = 1) \)

2. \( D(a_i) = a_{i-1} - \rho(a_{i-1}) - a_i(\rho(a_{m-1}) - a_{m-1}) \) \( (1 \leq i \leq m-1) \)

3. \( D(a_0) = a_0(\rho(a_{m-1}) - a_{m-1}) \)

**Lemma 1.2** ([7, Corollary 1.7]). If \( f \) is in \( B[X; \rho, D]_{(0)} \cap B^\rho[X] \) then \( f \) is in \( C(B^\rho \cup D)[X] \). Moreover,

\[ \alpha a_i = \sum_{j=i}^{m} (-1)^{j-i} \binom{j}{i} a_j \rho^{m-j} D^{j-i}(\alpha) \] \( (\alpha \in B, 0 \leq j \leq m, a_m = 1) \).

**Lemma 1.3** ([6, Theorem 2.2]). Let \( B \) be a commutative ring, and \( f(X) \) a monic polynomial in \( B[X] \). The following are equivalent.

1. \( f(X) \) is weakly separable in \( B[X] \).
2. \( f'(X) \) is a non-zero-divisor in \( B[X] \) modulo \( (f(X)) \), where \( f'(X) \) is a derivative of \( f(X) \).
3. \( \delta(f(X)) \) is a non-zero-divisor in \( B \), where \( \delta(f(X)) \) is a discriminant of \( f(X) \).

Now we consider the following \( A\)-\( A \)-homomorphisms:

\[ \mu : AA \otimes_B AA \rightarrow AA, \quad \mu(z \otimes w) = zw \]
\[ \xi : AA \otimes_B AA \rightarrow AA \otimes_B AA, \quad \xi(z \otimes w) = D(z) \otimes \rho(w) + z \otimes D(w) \]
\[ \eta : AA \otimes_B AA \rightarrow AA \otimes_B AA, \quad \eta(z \otimes w) = \rho(z) \otimes \rho(w) - z \otimes w \]

By making of the above mappings, S. Ikehata gave the following definition.

**Definition 1.4** ([4, pp.119]). \( f \) is called \((\rho, D)\)-separable in \( B[X; \rho, D] \) if there exists an \( A\)-\( A \)-homomorphism \( \nu : A \rightarrow A \otimes_B A \) such that

\[ \mu \nu = 1_A, \quad \xi \nu = \nu D, \quad \eta \nu = \nu(\rho - 1_A). \]

Obviously, a \((\rho, D)\)-separable polynomial in \( B[X; \rho, D] \) is separable. In [4], S. Ikehata studied \((\rho, D)\)-separable polynomials in \( B[X; \rho, D] \) and he gave the following.
Lemma 1.5 ([4, Theorem 2.1]). The following are equivalent.

1. \( f \) is \((\rho, D)\)-separable in \( B[X; \rho, D] \).

2. There exists \( h \in J_{m-1}(B)^{\overline{\rho}, \overline{D}} \) such that \( f'h = hf' = 1 \).

3. \( f \) is separable in \( C(B^{\rho, D})[X] \).

Noting that Lemma 1.5 (3), we shall give the following definition as a generalization of \((\rho, D)\)-separable polynomials in \( B[X; \rho, D] \).

Definition 1.6. \( f \) is called weakly \((\rho, D)\)-separable in \( B[X; \rho, D] \) if \( f \) is weakly separable in \( C(B^{\rho, D})[X] \).

The purpose of this article is to give characterizations of weakly \((\rho, D)\)-separable in \( B[X; \rho, D] \). Moreover, we shall characterize the difference between the \((\rho, D)\)-separability and the weak \((\rho, D)\)-separability in \( B[X; \rho, D] \).

2 Main results

The conventions and notations employed in the preceding section will be used in this section. In particular, recall that \( \rho D = D\rho \) and let \( f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 \in B[X; \rho, D]_{(0)} \cap B^\rho[X] \). Note that \( f \) is in \( C(B^{\rho, D})[X] \) by Corollary 1.2. First we shall state the following.

Lemma 2.1. The following are equivalent.

1. \( f \) is weakly \((\rho, D)\)-separable in \( B[X; \rho, D] \).

2. \( f' \) is a non-zero-divisor in \( C(B^{\rho, D})[X]/fC(B^{\rho, D})[X] \cong V(B^{\rho, D})^{\overline{\rho}, \n\overline{D}} \).

3. \( \delta(f) \) is a non-zero-divisor in \( C(B^{\rho, D}) \), where \( \delta(f) \) is a discriminant of \( f \).

Proof. It is obvious by Lemma 1.3. \( \square \)

We recall that \( A\times A\)-homomorphism \( \mu : A \otimes_B A \to A \) defined by \( z \otimes w \mapsto zw \). Noting that \( \alpha f' = f'^{\rho m-1}(\alpha) \) for any \( \alpha \in B \), we can see that \( \mu \left( W(J_{m-1}(B)^{\overline{\rho}, \overline{D}}) \right) \subset V(B)^{\overline{\rho}, \overline{D}} \). In addition, it is easy to see that \( \mu \left( W(V(B^{\rho, D})^{\overline{\rho}, D^{-}}) \right) \subset V(B^{\rho, D})^{\overline{\rho}, \overline{D}} \). Then we shall state the following.

Theorem 2.2. \( f \) is \((\rho, D)\)-separable in \( B[X; \rho, D] \) if and only if the following \( A\times A\)-homomorphism is onto:

\[ \mu|_{W(J_{m-1}(B)^{\overline{\rho}, \overline{D}})} : W(J_{m-1}(B)^{\overline{\rho}, \overline{D}}) \to V(B)^{\overline{\rho}, \overline{D}} \]
Proof. Note that $\mu \left( \sum_{j=0}^{m-1} y_j h \otimes x^j \right) = f'h = hf'$ for any $h \in A^{\overline{\rho}, \overline{D}}$.

(1) Assume that $f$ is $(\rho, D)$-separable in $B[X; \rho, D]$. Then there exists $h \in J_{m-1}(B)^{\overline{\rho}, \overline{D}}$ such that $f'h = hf' = 1$ by Lemma 1.5 (2). For any $g \in V(B)^{\overline{\rho}, \overline{D}}$, we see that $hg = gh \in J_{m-1}(B)^{\overline{\rho}, \overline{D}}$ and $\mu \left( \sum_{j=0}^{m-1} y_j h g \otimes x^j \right) = f'h g = g$. Thus $\mu |_{W(J_{m-1}(B)^{\rho D})}$ is onto.

Conversely, assume that $\mu |_{W(J_{m-1}(B)^{\rho D})}$ is onto. Since $1 \in V(B)^{\overline{\rho}, \overline{D}}$, there exists $h \in J_{m-1}(B)^{\overline{\rho}, \overline{D}}$ such that $1 = \mu \left( \sum_{j=0}^{m-1} y_j h \otimes x^j \right) = f'h = hf'$. Therefore $f$ is $(\rho, D)$-separable by Lemma 1.5 (2).

(2) Assume that $f$ is weakly $(\rho, D)$-separable in $B[X; \rho, D]$. Then $f'$ is a non-zero-divisor in $V(B^{\rho D})^{\overline{\rho}, \overline{D}}$ by Lemma 2.1 (2). Let $\sum_{j=0}^{m-1} y_j h \otimes x^j$ be in $\text{Ker} \left( \mu |_{W(V(B^{\rho D})^{\overline{\rho}, \overline{D}})} \right)$ with $h \in V(B^{\rho D})^{\overline{\rho}, \overline{D}}$. Then we have $0 = f'h = hf'$. Since $f'$ is a non-zero-divisor in $V(B^{\rho D})^{\overline{\rho}, \overline{D}}$, we obtain $h = 0$ and hence $\text{Ker} \left( \mu |_{W(V(B^{\rho D})^{\overline{\rho}, \overline{D}})} \right) = \{0\}$. Thus $\mu |_{W(V(B^{\rho D})^{\overline{\rho}, \overline{D}})}$ is one-to-one.

Conversely, assume that $\mu |_{W(V(B^{\rho D})^{\overline{\rho}, \overline{D}})}$ is one-to-one. Let $hf' = 0$ for some $h \in V(B^{\rho D})^{\overline{\rho}, \overline{D}}$. This implies that $\mu \left( \sum_{j=0}^{m-1} y_j h \otimes x^j \right) = 0$. Since $\mu |_{W(V(B^{\rho D})^{\overline{\rho}, \overline{D}})}$ is one-to-one, we have $\sum_{j=0}^{m-1} y_j h \otimes x^j = 0$, namely, $h = 0$. Therefore $f'$ is a non-zero-divisor in $V(B^{\rho D})^{\overline{\rho}, \overline{D}}$, and hence $f$ is weakly $(\rho, D)$-separable by Lemma 2.1 (2).

Corollary 2.3. $f$ is $(\rho, D)$-separable in $B[X; \rho, D]$ if and only if $W(J_{m-1}(B)^{\overline{\rho}, \overline{D}}) \cong V(B)^{\overline{\rho}, \overline{D}}$ as an $A$-$A$-bimodule.

Proof. Note that $W(J_{m-1}(B)^{\overline{\rho}, \overline{D}}) \subset W(V(B^{\rho D})^{\overline{\rho}, \overline{D}})$ and $V(B)^{\overline{\rho}, \overline{D}} \subset V(B^{\rho D})^{\overline{\rho}, \overline{D}}$. If $f$ is $(\rho, D)$-separable in $B[X; \rho, D]$ then $f$ is also weakly $(\rho, D)$-separable, and so $\mu |_{W(J_{m-1}(B)^{\overline{\rho}, \overline{D}})}$ is one-to-one. Therefore $\mu |_{W(J_{m-1}(B)^{\overline{\rho}, \overline{D}})}$ is an isomorphism if and only if $f$ is $(\rho, D)$-separable in $B[X; \rho, D]$.

References


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