Conjecture about Regularity of Prefix Square Roots of Regular Languages

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Zsolt Fazekas, Robert Mercas, Daniel Reidenbach gave the conjecture in [2] which gives necessary and sufficient condition for the primitive prefix square root of a regular language $L$ to be regular. The author gives a counterexample of their conjecture and gives a new conjecture.

1. Preliminary

An alphabet $V$ is a finite and nonempty set of symbols, called letters. Every finite sequence of letters of $V$ is called a word over $V$. Words over $V$ together with the operation of concatenation with the empty word $\varepsilon$ form a free monoid $V^*$. We denote $V^+ = V^* - \{\varepsilon\}$.

Let $w = a_1a_2\cdots a_n$ where $a_1, a_2, \ldots, a_n \in V$. The length of a word $w$ is $n$ and denoted by $|w|$ and the length of the empty word $\varepsilon$ is $0$.

For a positive integer $p$,

$$V^{\leq p} = \{w \in V^* | |w| \leq p\},$$

$$V^p = \{w \in V^* | |w| = p\}.$$

For a word $w = xyz$ for $x, y, z \in V^*$, a prefix of $w$ is $x$, a factor of $w$ is $y$ and a suffix of $w$ is $z$.

For a word $w \in V^+$, the following operations are defined in [1]:

- prefix square reduction: $\square(w) = \{uv | w = uuv, \text{ for } u \in V^+, v \in V^*\}$
• suffix square reduction: $\square(w) = \{uv \mid w = vuu, \text{ for } u \in V^+, v \in V^+\}$

• prefix-suffix square reduction: $\square_2(w) = \square_1(w) \cup \square_2(w)$

For simplicity, we restrict the argument to prefix square reduction.

We define the bounded version for a fixed positive integer $p$:

• $p$-prefix square reduction: $p\square_1(w) = \{uv \mid w = uuv, \text{ for } u \in V^{\leq p}, v \in V^+\}$

For a language $L$, we have language: $\square_1(L) = \bigcup_{w \in L} \square_1(w)$.

The following languages are defined:

$\square^0_1(w) = \{w\}$,

$\square^k_{1}(w) = \square_1(\square^k_{1}(w))$ for any $k \geq 0$

$\square^\ast_1(w) = \bigcup_{k \geq 0} \square^k_{1}(w)$.

For a word $w$, the primitive prefix square root of $w$ is the set $\{u \mid u \in \square^\ast_1(w) \text{ and } \square_1(u) = u\}$ and it is denoted by $\sqrt{\square}_1 w$. The primitive bounded prefix square root of $w$ is the set $\{u \mid u \in p\square^\ast_1(w) \text{ and } p\square_1(u) = u\}$ and it is denoted by $p\sqrt{\square}_1 w$. For a language $L$, we define $\sqrt{\square}_1 L = \bigcup_{w \in L} \sqrt{\square}_1 w$ and $p\sqrt{\square}_1 L = \bigcup_{w \in L} p\sqrt{\square}_1 w$.

2. Conjectures

Zsolt Fazekas, Robert Mercas, Daniel Reidenbach gave the following conjecture in [2].

Conjecture (in [2]). Let $L$ be a regular language. The primitive prefix square root of $L$ is regular if and only if there exists some positive integer $p$ such that $\sqrt{\square}_1 L = p\sqrt{\square}_1 L$.

But, I give here the following counterexample and new conjecture.

Example. Let $L = aab^+aab^+c$ where $a, b, c \in V$. The language $L$ is regular. On the other hand, the primitive prefix square root of $L$ is $\sqrt{\square}_1 L = ab^+aab^+c \cup ab^+c$ and this language is regular.
But, there is no positive integer $p$ such that $\sqrt[p]{\psi^\square \overline{L}} = ^p\psi^\square \overline{L}$.

Now, we define a new term to describe our new conjecture: For a word $w$, if $xx$ is a non-trivial prefix of $w$ and $x$ is prefix square free, then we say that $xx$ is the \textit{minimal} prefix square of $w$.

\textbf{Conjecture.} Let $L$ be a regular language. The primitive prefix square root of $L$ is regular if and only if there exists positive integer $N$ such that, for every word $w \in \mathcal{D}^\ast (L)$, the length of the minimal prefix square of $w$ is smaller than $N$.

\textbf{References}

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