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On twists and surgeries generating exotic smooth structures

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1 Background

In this note, we summarize the author's paper [14], explaining the background of 4-dimensional topology.

One of the most central problems in 4-dimensional topology has been to classify smooth 4-manifolds. To do this, it is natural to fix their underlying homeomorphism types, and note that there exists no algorithm that classify homeomorphism types of closed orientable smooth 4-manifolds (cf. [9]). In contrast to other dimensions, this problem has been open for any single (smoothable) homeomorphism type. The following would be one of the main difficulties.

Problem 1.1. Given a closed oriented smooth 4-manifold, find all smooth 4-manifolds homeomorphic to the given one.

Indeed, this problem has been open for any 4-manifold. Since handlebody diagrams (Kirby diagrams) can represent all closed smooth 4-manifolds, one might think that diagrams solve this problem. However, it is very difficult to see whether a given diagram represents a *closed* 4-manifold, and moreover there are no known method for constructing all handlebodies homeomorphic to a given 4-manifold. The same difficulties occur for other diagrammatic methods as well.

A potential approach is a twisting operation, that is, removing a submanifold and regluing it differently. Let us recall the definition of a cork twist. A cork (C, τ) is a pair consisting of a compact contractible oriented smooth 4-manifold C and a smooth involution τ on the boundary such that τ extends to a self-homeomorphism of C, but cannot extend to any self-diffeomorphism of C ([1]). Due to the order of τ, such (C, τ) is often called of order 2. Removing an embedded C from a 4-manifold and regluing it via τ is called a cork twist. A well-known theorem states that for any exotic (i.e. homeomorphic but non-diffeomorphic) pair of simply connected closed oriented smooth 4-manifolds, one is obtained from the other by a cork twist ([5], [10]). This cork theorem thus gives us an important clue to Problem 1.1. However, since there is no known classification of compact contractible 4-manifolds, and many contractible 4-manifolds admit infinitely many embeddings into a 4-manifold, this cork theorem does not solve the problem. We also note that a cork twist does not always yield an exotic copy.

Recently higher order corks were constructed ([11], [4]), and surprisingly Gompf [8] discovered infinite order corks. It is thus natural to ask whether every exotic copy of a simply connected closed oriented smooth 4-manifold is obtained by twisting a fixed compact contractible submanifold via a power of a fixed self-diffeomorphism of the boundary. However, Tange [12] answered negatively, that is, he gave infinite families of pairwise exotic simply connected closed 4-manifolds such that, for any 4-manifold X, any contractible submanifold C, and any self-diffeomorphism f of ∂C, the families cannot be constructed from X by twisting C via powers of f, by showing a certain finiteness for Ozsváth-Szabó invariants of cork twisted 4-manifolds.

2 Main results

It would be natural to discuss more general twists. Indeed, it has been well known that, under a certain condition, logarithmic transforms (i.e. twists along $T^2 \times D^2$) in a 4-manifold can produce infinitely many exotic smooth structures (cf. [9], [7]). (However, many twisting operations including logarithmic transforms do not always produce exotic copies. Indeed, the resulting 4-manifolds are often diffeomorphic to the original manifolds.) So we discuss twists and more general surgeries along not necessarily contractible submanifolds. To contrast with the cork theorem, we will state the main results only for simply connected closed 4-manifolds. However, the corresponding results hold for non-simply connected 4-manifolds and non-closed 4-manifolds as well.

2.1 Nonexistence of twists generating all exotic smooth structures

We first discuss twists, using the following terminologies.

Definition 2.1. Let X be an oriented smooth 4-manifold, and let W be a compact (not necessarily connected) codimension zero submanifold. For a family of smooth oriented 4-manifolds, we say that the family is *generated from X by twisting W*, if each member is orientation preserving diffeomorphic to a 4-manifold obtained from X by removing the submanifold W and gluing it back via a (not necessarily orientation preserving) self-diffeomorphism of the boundary ∂W. In the case where the gluing map reverses the orientation, the newly glued piece is the orientation reversal $\overline{W}$ of W.

Definition 2.2. For an oriented smooth 4-manifold X, let $\mathcal{S}(X)$ be the set of all smooth structures on X, that is, $\mathcal{S}(X)$ is the set of all (diffeomorphism types of) oriented smooth 4-manifolds homeomorphic to X preserving the orientations.

This set was inspired from Tange's *galaxy* ([13]). As is well-known, $\mathcal{S}(X)$ is a countable set for any compact oriented 4-manifold X. We consider the following problem.

Problem 2.3. Does a given compact oriented smooth 4-manifold X admit a compact (not necessarily connected) codimension zero submanifold W such that $\mathcal{S}(X)$ is generated from X by twisting W?

This problem asks a generalization of the cork theorem, since we do not impose any restrictions on the topology of W and on the gluing map. If the answer is affirmative, then we obtain a useful approach to Problem 1.1, since $\mathcal{S}(X)$ is generated by just a single submanifold in this case. However, we gave a partial negative answer under a mild assumption on $b_1(\partial W)$.

Theorem 2.4 ([14])**.** *For each positive integer n, there exists a simply connected closed oriented smooth 4-manifold X such that, for any compact (not necessarily connected) codimension zero submanifold W satisfying $b_1(\partial W) < n$, the set $\mathcal{S}(X)$ cannot be generated from X by twisting W. Furthermore, there exist infinitely many pairwise non-homeomorphic such 4-manifolds.*

For example, the elliptic surface $E(n+1)$ satisfies the condition of this theorem. We note that the aforementioned Tange's result follows from this result, since the boundary of any compact contractible 4-manifold is a (connected) homology 3-sphere and thus satisfies $b_1 = 0$. Our proof is completely different from Tange's one.

This theorem shows that there exists no universal generator of smooth structures regarding twists.

Corollary 2.5 ([14])**.** *There exists no compact (not necessarily connected) oriented smooth 4-manifold W such that for any simply connected oriented closed smooth 4-manifold X, the set $\mathcal{S}(X)$ is generated from a smooth oriented 4-manifold by twisting a fixed embedded copy of W.*

2.2 Nonexistence of surgeries generating all exotic smooth structures

Next we discuss surgeries using the following terminology.

Definition 2.6. Let X be an oriented smooth 4-manifold, and let W be a compact (not necessarily connected) codimension zero submanifold. For a family of oriented smooth

4-manifolds, we say that the family is *generated from X by performing surgeries on W*, if each member is obtained from X by removing the submanifold W and gluing a compact oriented smooth 4-manifold whose boundary is diffeomorphic to ∂W preserving the orientations. Note that we do not fix the newly glued piece.

Clearly surgeries are much more general operations than twists. Since surgeries (e.g. Fintushel-Stern knot surgery [6]) can produce various infinite exotic families (cf. [9], [7]), we consider a surgery version of Problem 2.3.

Problem 2.7. Does a given compact oriented smooth 4-manifold X with $b_2 > 0$ admit a compact (not necessarily connected) codimension zero submanifold W with $b_2(W) < b_2(X)$ such that $\mathcal{S}(X)$ is generated from X by performing surgeries on W?

Without the condition $b_2(W) < b_2(X)$, this problem has a trivial affirmative answer. Indeed, for any compact codimension zero submanifold V of the 4-ball, it is easy to see that $W = X - \operatorname{int} V$ provides an affirmative answer, realizing any integer not less than $b_2(X)$ as $b_2(W)$. We thus need the b_2 condition. We gave a partial negative answer under a mild assumption on $b_2(W) + 3b_1(\partial W)$.

Theorem 2.8 ([14])**.** *For each positive integer n, there exists a simply connected closed oriented smooth 4-manifold X with $b_2 > n$ such that, for any compact (not necessarily connected) codimension zero submanifold W with $b_2(W) + 3b_1(\partial W) < n$, the set $\mathcal{S}(X)$ cannot be generated from X by performing surgeries on W. Furthermore, there exist infinitely many pairwise non-homeomorphic such 4-manifolds.*

For example, the elliptic surface $E(n + 1)$ satisfies the condition of this theorem as well. Similarly to the case of twists, this theorem shows the nonexistence of a universal generator for surgeries.

Corollary 2.9 ([14])**.** *There exists no compact oriented smooth 4-manifold W such that for any simply connected closed oriented smooth 4-manifold X, the set $\mathcal{S}(X)$ is generated from X by performing surgeries on a fixed embedded copy of W.*

2.3 Nonexistence of twists generating all exotic smooth structures by varying embeddings

We further discuss another generalization.

Problem 2.10. Does a given compact oriented smooth 4-manifold X admit a compact (not necessarily connected) oriented smooth 4-manifold W such that $\mathcal{S}(X)$ is generated from X by twisting an embedded copy of W and varying the embedding of W into X?

This problem is largely flexible than Problem 2.3, since we vary an embedding of W. Akbulut and the author ([2], [3]) earlier studied a related problem and showed that many order-2 corks can produce infinite families of pairwise exotic simply connected closed 4-manifolds by twisting corks and varying embeddings of corks. By contrast, we gave a partial negative answer to this problem for sufficiently large W, by applying (the proof of) Theorem 2.8.

Theorem 2.11 ([14]). *For each positive integer n, there exists a simply connected closed oriented smooth 4-manifold X with $b_2 = 12n + 10$ such that for any compact oriented smooth 4-manifold W with $b_2(W) - 4b_1(\partial W) > 11n + 10$, the set $\mathcal{S}(X)$ cannot be generated from X by twisting an embedded copy of W and varying the embedding of W into X.*

In the rest, we explain the outline of the proofs of these results. Let us recall that the minimal genus function $g_X : H_2(X;\mathbb{Z}) \to \mathbb{Z}$ of a smooth 4-manifold X is a function that sends a second homology class to the minimal genus of a smoothly embedded surface representing the homology class. This function gives useful informations of 4-manifolds, but it is very hard to distinguish two functions due to identifications of second homology groups and also difficult to determine the values. To avoid these difficulties, we introduced a new diffeomorphism invariant $G_X(n)$ $(\in \mathbb{Z})$ determined from the minimal genus function g_X, which we call the adjunction n-genus. Here n is a positive integer satisfying $n \leq b_2(X)$, and for each n, the value $G_X(n)$ is a diffeomorphism invariant. We showed that an infinite family of (not necessarily closed) 4-manifolds with pairwise distinct adjunction n-genera cannot be generated by twists and surgeries as in Theorems 2.4 and 2.8. By applying the adjunction inequalities, we also gave sufficient conditions that infinitely many 4-manifolds have pairwise distinct adjunction n-genera. Then, by constructing infinite families of pairwise homeomorphic 4-manifolds satisfying the sufficient conditions, we proved the main results.

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