数理解析研究所講究録 第2099巻 2018年 36-41

Cluster Dehn twists in cluster modular groups

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1 Introduction

The *cluster algebra*, introduced by Fomin-Zelevinsky [FZ02], is a commutative associative algebra associated with a (possibly weighted) quiver Q without loops and 2-cycles. It is a subalgebra of the field of rational functions $\mathbb{C}(A_1, \ldots, A_N)$ obtained by applying a finite number of rational transformations called *cluster $\mathcal{A}$-transformations* to the initial variables $A_1, \ldots, A_N$. The cluster algebra is effectively used to investigate the function algebras of the double Bruhat cells of a reductive algebraic group and solve the classical *total positivity problem*. Almost simultaneously, a geometric counter part (and a "dualization") of the cluster algebra is formulated by Fock-Goncharov [FG09]. They associate a dual pair of schemes $\mathcal{A}_{|Q|}$ and $\mathcal{X}_{|Q|}$, each of which is equipped with a distinguished collection of birational toric charts whose transition functions are given by cluster $\mathcal{A}$- and $\mathcal{X}$-transformations. Here a cluster $\mathcal{X}$-transformation is another rational transformation. The pair $(\mathcal{A}_{|Q|}, \mathcal{X}_{|Q|})$ is called the *cluster ensemble* associated with Q. The cluster algebra lies in the function algebra of $\mathcal{A}_{|Q|}$.

The cluster transformations are induced by a *mutation*, which is a transformation of quivers. The *cluster modular group* $\Gamma_{|Q|}$ is, roughly speaking, the group of sequences of mutations which preserves a quiver. It acts on the cluster algebra and the cluster ensemble by compositions of cluster transformations.

Since the foundation, many fruitful connections between the cluster algebra/ensemble and a broad area of mathematics are found: higher Teichmüller theory [FG06], quantum groups [Ip16], integrable systems [GK13], and many others. In particular, the cluster modular group plays important roles in these theories: it gives a combinatorial description of the action of the mapping class group on higher Teichmüller spaces; the universal R-matrix of a quantum group is realized in the cluster modular group; it gives discrete flows in *cluster integrable systems*, and so on.

Example 1.1. A basic example is given by the quiver $Q = Q_\Delta$ associated with an ideal triangulation Δ of a marked surface Σ. In this case, the corresponding cluster ensemble describes a combinatorial structure of Teichmüller spaces related to Σ: the positive real part $\mathcal{A}_{|Q|}(\mathbb{R}_{>0})$ coincides with the *decorated Teichmüller space* introduced by Penner [Pen12] equipped with the lift of the Weil-Petersson form, $\mathcal{X}_{|Q|}(\mathbb{R}_{>0})$ coincides with the *enhanced Teichmüller space* [FG07] equipped with the Goldman Poisson structure. They are two types of extensions of the Teichmüller space of Σ, which is the universal cover of the Riemann moduli space of Σ. The cluster modular group $\Gamma_{|Q|}$ contains the *mapping class group* $MC(\Sigma)$ as a subgroup of finite index [BS15]. Namely, each element $\phi \in MC(\Sigma)$ is represented by a mutation sequence and the actions on these Teichmüller spaces are represented by the corresponding composition of cluster transformations.

From this example, we would be able to say that: *the cluster ensemble is a generalization of the Teichmüller space, and the cluster modular group is a generalization of the mapping class group.* Then our general question is the following:

Problem 1.2. *Is it possible to generalize a theorem for the Teichmüller spaces/mapping class groups to a theorem for the cluster ensembles/cluster modular groups?*

In this manuscript, we introduce *cluster Dehn twists* in the cluster modular group and show that they have similar properties as Dehn twists in the mapping class groups of surfaces.

2 Definitions

In this section, we recall basic notions of cluster ensemble following [FG09]. For our purpose, it is enough to consider the positive real part of the cluster ensemble which is a pair of contractible manifolds, rather than considering the corresponding schemes.

A quiver without loops and 2cycles is given by the data $Q = (I, \varepsilon)$, where I is a finite set (the set of vertices), $\varepsilon = (\varepsilon_{ij})_{i,j \in I}$ is a skew-symmetric matrix (the entry ε_{ij} gives $\#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\}$). Let us consider a tuple $(Q, (A_i)_{i \in I}, (X_i)_{i \in I})$, called a *seed*, where Q is a quiver and $(A_i)_{i \in I}$, $(X_i)_{i \in I}$ are two bunches of commutative variables parametrized by the vertex set I of Q. For a vertex $k \in I$, we define the *mutation*

$\mu_k : (Q, (A_i)_{i\in I}, (X_i)_{i\in I}) \to (Q', (A'_i)_{i\in I}, (X'_i)_{i\in I})$ by the following formulas:

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & i = k \text{ or } j = k, \\ \varepsilon_{ij} + \dfrac{|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|}{2} & \text{otherwise,} \end{cases} \tag{1}$$

$$A'_i = \begin{cases} A_k^{-1}\left(\displaystyle\prod_{j:\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{j:\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}}\right) & i = k, \\ A_i & i \neq k. \end{cases} \tag{2}$$

$$X'_i = \begin{cases} X_k^{-1} & i = k, \\ X_i(1 + X_k^{-\mathrm{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & i \neq k, \end{cases} \tag{3}$$

The *mutation class* $|Q|$ is the set of quivers which are obtained by finite sequences of mutations from Q. Henceforth, let us concentrate our attention to the $\mathcal{A}$-side. (The story for the $\mathcal{X}$-side goes similarly.) The partial data $(Q, (A_i)_{i\in I})$ of a seed is encoded in the following geometric object. Let $\mathcal{A}_Q$ be the positive Euclidean space $\mathbb{R}^I_{>0}$ equipped with coordinates $(A_i)_{i\in I}$ and a presymplectic structure $\omega_Q := \sum_{i,j\in I} \varepsilon_{ij} \mathrm{d}\log A_i \wedge \mathrm{d}\log A_j$. The *cluster $\mathcal{A}$-transformation* is the isomorphism $\mu_k^a : (\mathcal{A}_Q, \omega_Q) \to (\mathcal{A}_{Q'}, \omega_{Q'})$ such that the pull-back $(\mu_k^a)^* A'_i$ is given by the right-hand side of the formula (2).

The cluster $\mathcal{A}$-space $\mathcal{A}_{|Q|}(\mathbb{R}_{>0})$ is defined to be a presymplectic manifold which satisfies the following conditions.

1. For each quiver $Q' \in |Q|$, we have an isomorphism $A_{Q'} : \mathcal{A}_{|Q|}(\mathbb{R}_{>0}) \to (\mathcal{A}_{Q'}, \omega_{Q'})$.
2. If $Q'' = \mu_k(Q')$, then the corresponding coordinate transformation $A_{Q''} \circ A_{Q'}^{-1}$ coincides with the cluster transformation $\mu_k^a : (\mathcal{A}_{Q'}, \omega_{Q'}) \to (\mathcal{A}_{Q''}, \omega_{Q''})$.

It depends only on the mutation class $|Q|$ and it has the following natural symmetry group. A *mutation sequence* is a finite sequence ϕ of mutations and permutations on the set I. Let ϕ^a denote the corresponding composition of μ_k^a's and σ^a's. Here σ^a denotes the permutation of the coordinates corresponding to a permutation σ. We identify two mutation sequences ϕ_1 and ϕ_2 if $\phi_1^a = \phi_2^a$. The *cluster modular group* Γ_Q at Q is the group of mutation sequences from Q to Q, modulo this identification. If $Q' = \mu_k(Q)$, then the conjugation by μ_k gives a group isomorphism $\Gamma_{Q'} \cong \Gamma_Q$. Therefore we identify these groups via this isomorphism and denote the resulting abstract group by $\Gamma_{|Q|}$. When we fix a "basepoint" $Q' \in |Q|$, an element $\phi \in \Gamma_{|Q|}$ is represented by a mutation sequence. The action of $\Gamma_{|Q|}$ on $\mathcal{A}_{|Q|}(\mathbb{R}_{>0})$ is given by the natural action $\Gamma_{Q'} \to \mathrm{Aut}(\mathcal{A}_{Q'}, \omega_{Q'})$, $\phi \mapsto \phi^a$ for some $Q' \in |Q|$.

Remark 2.1. One can similarly define the cluster $\mathcal{X}$-space $\mathcal{X}_{|Q|}(\mathbb{R}_{>0})$ using the cluster $\mathcal{X}$-transformation and a Poisson structure $\Pi_Q := \sum_{i,j\in I} \varepsilon_{ij} X_i \frac{\partial}{\partial X_i} \wedge X_j \frac{\partial}{\partial X_j}$ instead of ω_Q.

For each mutation sequence ϕ, one can associate the corresponding compostion ϕ^x. It is known [Man14] that $\phi_1^a = \phi_2^a$ if and only if $\phi_1^x = \phi_2^x$. Hence the cluster modular group $\Gamma_{|Q|}$ acts on $\mathcal{X}_{|Q|}(\mathbb{R}_{>0})$ as well.

Definition 2.2. An element $\phi \in \Gamma_{|Q|}$ of infinite order is called a *cluster Dehn twist* if there exist integers $n, l \in \mathbb{Z}$, $l \neq 0$, such that $\phi^n = ((i\ j)\mu_j)^l$ for some vertices $i, j \in I$ of a quiver $Q' \in |Q|$. Here $(i\ j)$ denotes the transposition of i and j.

Example 2.3 (Dehn twists and half-twists)**.** Dehn twists and half-twists in the mapping class group of a marked surface are cluster Dehn twists. (Details are in the talk)

3 Parabolic dynamics

The cluster $\mathcal{A}$-space $\mathcal{A}_{|Q|}(\mathbb{R}_{>0})$ admits a natural compactification $\overline{\mathcal{A}}_{|Q|}$, which we call the *Fock-Goncharov compactification*. It is constructed by attaching the *projectivized tropical space* at infinity. Crucial properties are: $\overline{\mathcal{A}}_{|Q|}$ is homeomorphic to a closed ball B^I and the action of the cluster modular group extends to $\overline{\mathcal{A}}_{|Q|}$ continuously. Then the action of a cluster Dehn twist on $\overline{\mathcal{A}}_{|Q|}$ has the following dynamical property:

Theorem 3.1. *Let Q be a connected quiver with at least 3 vertices, $\phi \in \Gamma_{|Q|}$ be a cluster Dehn twist. Then there exists a unique boundary point $\ell \in \partial\overline{\mathcal{A}}_{|Q|}$ such that*

$$\lim_{n \to \infty} \phi^{\pm n}(g) = \ell$$

in $\overline{\mathcal{A}}_{|Q|}$, for all $g \in \mathcal{A}_{|Q|}(\mathbb{R}_{>0})$.

In other words, the action of ϕ has parabolic dynamics. Note that the Dehn twist $t_C \in MC(\Sigma)$ along a simple closed curve $C \in \Sigma$ has parabolic dynamics in the Thurston compactification of the Teichmüller space: indeed, the limit point ℓ is given by the element represented by C in the Thurston boundary, which is the space of measured geodesic laminations on Σ.

Remark 3.2 (Nielsen-Thurston classification)**.** In [Ish17], the elements of a cluster modular groups are classified into 3 types: periodic/cluster-reducible/cluster-pA (pseudo-Anosov). They are characterized by fixed point properties of the action on the Fock-Goncharov compactification $\overline{\mathcal{X}}_{|Q|}$ of the $\mathcal{X}$-space, modulo some technical conjectures on cluster algebras. In these terms, a cluster Dehn twist is a cluster-reducible element which is maximally reducible (cluster-reduction induces a 2-dimensional dynamics).

4 Generation of a cluster modular group by cluster Dehn twists

It is known that the mapping class group of a marked surface is virtually generated by Dehn twists and half-twists. Next we provide a generalization of this theorem. A quiver Q is of *finite mutation type* if the set $|Q|$ is finite. Q is of *finite type* if it is mutation-equivalent to a Dynkin quiver. It is known that the cluster modular group $\Gamma_{|Q|}$ is a finite group if Q is of finite type, and it is a infinite group if Q is of mutation finite type and not finite type.

Theorem 4.1 (Felikson-Shapiro-Tumarkin [FST12])**.** *Suppose a quiver Q is of finite mutation type and not finite type. Then either Q is associated with an ideal triangulation of a marked surface or mutation-equivalent to one of the following 8 quivers: $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$, X_6, X_7.*

Then we prove the following.

Theorem 4.2. *Let Q be a quiver of finite mutation type. If Q is not of type $\widetilde{E}_6^{(1,1)}$, $\widetilde{E}_7^{(1,1)}$ or $\widetilde{E}_8^{(1,1)}$, then the cluster modular group $\Gamma_{|Q|}$ is generated by cluster Dehn twists.*

We conjecture that for all quiver of finite mutation type, the corresponding cluster modular group is generated by cluster Dehn twists.

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