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SIMPLIFIED BROKEN LEFSCHETZ FIBRATIONS AND TRISECTIONS OF SMOOTH 4–MANIFOLDS

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Abstract. We present a survey on explicit algorithms for simplifying the topology of indefinite fibrations on 4–manifolds, which include broken Lefschetz fibrations and indefinite generic maps, from the viewpoint of singularity theory. These algorithms allow us to give purely topological and constructive proofs of the existence of simplified broken Lefschetz fibrations on all closed oriented 4–manifolds, and a theorem of Auroux–Donaldson–Katzarkov on the existence of simplified broken Lefschetz pencils on near-symplectic 4–manifolds. We moreover show the existence of simplified trisections on all 4–manifolds and establish a correspondence between broken Lefschetz fibrations and trisections of 4–manifolds.

1. Introduction

This is a survey article summarizing results obtained in [5, 6], which are joint works with İnanç Baykur (University of Massachusetts).

Lefschetz fibrations on smooth 4–manifolds have their origin in complex surfaces: in a sense, they have been imported from algebraic geometry into the world of differential topology (see [23, 21]). Since then, they have been extensively studied: in particular, their close relation to symplectic structures on 4–manifolds revealed by Donaldson [7] and Gompf [15] accelerated the study. A disadvantage, however, of Lefschetz fibrations and symplectic structures is that even if a smooth 4–manifold satisfies obvious algebraic topological necessary conditions, it may not admit such structures.

Then, Auroux–Donaldson–Katzarkov [3] introduced the notions of broken Lefschetz fibrations and near-symplectic structures. The important point is that they allowed a closed 2–form corresponding to a near-symplectic structure to vanish along a 1–dimensional submanifold. The topological counter part for such structures is Lefschetz fibrations with indefinite fold singularities appearing along 1–dimensional submanifolds. Since then, several authors have shown that every closed oriented smooth 4–manifold admits a broken Lefschetz fibration.

As a broken Lefschetz fibration has indefinite folds as its singularities, its image might be quite complicated. In this article, we present some explicit procedures for simplifying the indefinite fold image by several "moves" that modify a given broken Lefschetz fibration by homotopy. Here, a broken Lefschetz fibration is simplified if all the fibers are connected, its fold locus is connected, and the restriction to the fold locus is an embedding. The moves

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that we use are closely related to codimension one generic singularities for smooth maps of 4–manifolds into 2–manifolds: in other words, they correspond to generic bifurcations for 1–parameter families of smooth maps. Therefore, the singularity theory plays an essential role here. The construction of such a homotopy will be given explicitly: as a result, we get an explicit algorithm for simplifying a broken Lefschetz fibration. As a consequence, we give a purely topological proof of some results obtained by Auroux–Donaldson–Katzarkov [3].

In [13], Gay–Kirby introduced the notion of a trisection decomposition of a 4–manifold: it is a decomposition into three 4–dimensional 1–handlebodies attached nicely along their boundaries. They pointed out that such decompositions are naturally obtained from certain generic smooth maps of 4–manifolds into $\mathbb{R}^2$. In the final section of this article, we apply our simplification process for broken Lefschetz fibrations to such generic maps and show that every 4–manifold admits a "simplified" trisection. In fact, given a simplified broken Lefschez fibration, we give an explicit procedure to get a smooth generic map into $\mathbb{R}^2$ which is simplified and which corresponds to a trisection decomposition. In partiular, the image of such a generic map has symmetry of order three. This can be compared to the work of Gay [10], who constructed trisection decompositions starting from Lefschetz pencils.

Trisection decompositions of 4–manifolds can be considered as a generalization of Heegaard decompositions of 3–manifolds. Our result implies that every 4–manifold admits a trisection decomposition that satisfies certain simplicity conditions. Such simplified trisection decompositions are to be further studied. Here, we should note that Hayano [16] has already given a characterization of trisection decompositions that come from simplified trisections, and has also classified genus–2 simplified trisections.

Throughout the paper, all manifolds and maps between them are smooth of class C^∞ unless otherwise specified.

2. Broken Lefschetz fibrations

Let us start by introducing some concepts which play important roles in this article.

Let M^4 and Σ^2 be closed connected oriented manifolds of dimensions 4 and 2, respectively, and $f : M^4 \to \Sigma^2$ a smooth map.

Definition 2.1. A *singularity* (or a *singular point*) of f is a point $x \in M^4$ such that the rank of the differential $df_x : T_xM^4 \to T_{f(x)}\Sigma^2$ is strictly smaller than 2. The terminology "singularity" is often used also for referring to (an appropriate equivalence class of) the map germ of f at a singular point x.

A singularity of f that has the normal form

$$(z, w) \mapsto zw$$

with respect to C^∞ complex coordinates compatible with the orientations, is called a *Lefschetz singularity*.

A smooth map $f : M^4 \to \Sigma^2$ is a *Lefschetz fibration* if it has only Lefschetz singularities.

Definition 2.2. A *Lefschetz pencil* is a smooth map $M^4 \setminus B \to S^2$ for a finite subset $B \neq \emptyset$ of M^4 with only Lefschetz singularities such that around each point of B it has complex local model

$$(z_1, z_2) \mapsto z_1/z_2 \in \mathbb{C} \cup \{\infty\} = S^2.$$

Thus, blowing-up the points in B, we get a Lefschetz fibration $M^4 \sharp(\sharp^{|B|}\overline{\mathbb{C}P^2}) \to S^2$, where $|B|$ is the number of points in B. In other words, blowing-down disjoint (-1)–sections for a Lefschetz fibration, we get a Lefschetz pencil.

By Donaldson [7] and Gompf [15], the existence of a Lefschetz pencil is equivalent to the existence of a symplectic structure. Here, a *symplectic structure* refers to a closed 2–form $\omega \in \Omega^2(M^4)$, $d\omega = 0$, that is non-degenerate, $\omega^2 > 0$; i.e. $\omega \wedge \omega$ gives a volume form of M^4 compatible with the orientation.

If a 4–manifold M^4 admits a symplectic structure, then we see easily that $b_2^+(M^4) > 0$, where $b_2^+(M^4)$ denotes the number of positive eigenvalues of the intersection form of M^4, $H_2(M^4; \mathbb{R}) \times H_2(M^4; \mathbb{R}) \to \mathbb{R}$. However, it is known that not every M^4 with $b_2^+(M^4) > 0$ admits a symplectic structure. We can say that symplectic structures are "hard".

In order to make it "soft", let us consider the following singularity.

Definition 2.3. A singularity that has the (real) normal form

$$(t, x_1, x_2, x_3) \mapsto (t, x_1^2 + x_2^2 - x_3^2)$$

is called an *indefinite fold singularity*. See also Definition 3.1 below.

Definition 2.4 (Auroux–Donaldson–Katzarkov [3])**.** A smooth map $f : M^4 \to \Sigma^2$ is a *broken Lefschetz fibration* if it has at most Lefschetz and indefinite fold singularities.

A *broken Lefschetz pencil* is similarly defined.

For a broken Lefschetz fibration f, we denote by Z_f the set of indefinite fold singularities. Note that Z_f is a closed 1–dimensional submanifold of M^4.

By Auroux–Donaldson–Katzarkov [3], the existence of a broken Lefschetz pencil is equivalent to the existence of a near-symplectic structure. Here, a *near-symplectic structure* refers to a closed 2–form $\omega \in \Omega^2(M^4)$, $d\omega = 0$, that satisfies the following: at each point $x \in M^4$, either $\omega_x^2 > 0$ (non-degenerate), or $\omega_x = 0$ and the intrinsic gradient $\nabla\omega \colon T_xM^4 \to \bigwedge^2(T_x^*M^4)$ as a linear map has rank 3. The *zero locus* of ω, i.e. the set of points $x \in M^4$ where $\omega_x = 0$, is a closed 1–dimensional submanifold of M^4 denoted by Z_ω.

Example 2.5. The 2–form $\Omega = dt \wedge dQ + *(dt \wedge dQ)$ on $\mathbb{R}^4$ with coordinates (t, x_1, x_2, x_3) is near-symplectic with Z_Ω being the t–axis. Here, $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$ is an indefinite quadratic form and $*$ is the Hodge star operator (see [18]).

It is known that every M^4 with $b_2^+(M^4) > 0$ admits a near-symplectic structure [17, 3]. We can say that near-symplectic structures are "soft".

3. Indefinite fibrations

In this section, we introduce the notion of indefinite fibrations on smooth 4–manifolds and state our main results.

Let us first recall the following singularities, which are known to be generic.

Definition 3.1. A *singularity* of a smooth map $f : M^4 \to \Sigma^2$ is a *fold* if it is locally given by

$$(t, x_1, x_2, x_3) \mapsto (t, \pm x_1^2 \pm x_2^2 \pm x_3^2).$$

A *singularity* is a *cusp* if it is locally given by

$$(t, x_1, x_2, x_3) \mapsto (t, x_1^3 + tx_1 \pm x_2^2 \pm x_3^2).$$

They are *indefinite* if both of the $+$ and $-$ signs appear in the above normal forms. Otherwise, they are *definite*.

Then, the following theorem is classically known.

Theorem 3.2 (Whitney [31], Thom [29])**.** *An arbitrary smooth map $M^4 \to \Sigma^2$ can be approximated by a map with only fold and cusp singularities. More precisely, the set of smooth maps with only fold and cusp singularities forms an open and dense subset of the mapping space $C^\infty(M^4, \Sigma^2)$ endowed with the Whitney C^∞–topology.*

In the following, a smooth map $f : M^4 \to \Sigma^2$ with fold and cusp singularities is called a *generic map*. Such a map f is called an *indefinite generic map* if its folds and cusps are all indefinite.

Remark 3.3. Generic maps are often called *Morse* 2*–functions* [11, 12].

Now, let us introduce the following class of maps, which plays a central role in this article.

Definition 3.4. A smooth map $f : M^4 \to \Sigma^2$ is an *indefinite fibration* if it is an indefinite generic map outside of a finite set of Lefschetz singularities.

Note that for an indefinite fibration, the image of its indefinite fold locus is *normally oriented*: in the direction, it corresponds to a Morse critical point of index 2, i.e. a 3–dimensional 2–handle is attached (see Fig. 1). In other words, in the direction, the topology of a fiber changes by a 2–handle attachment when crossing over the fold image from one side to the other.

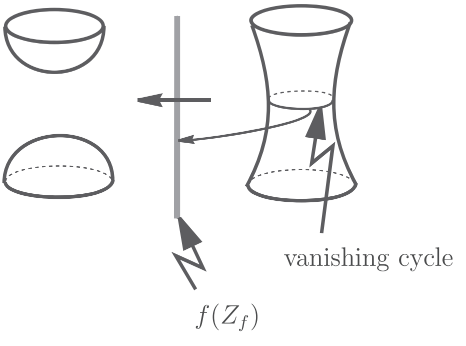


FIGURE 1. Normal orientation of fold image

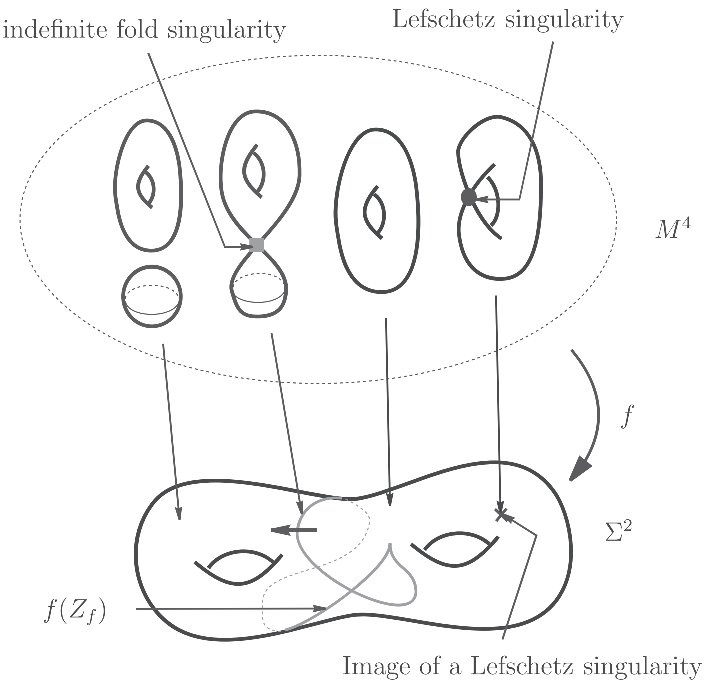


FIGURE 2. Fibers of an indefinite fibration: a schematic picture

Let us give a schematic picture of an indefinite fibration $M^4 \to \Sigma^2$. Its fibers vary as depicted in Fig. 2 when we move the corresponding points in the base surface.

Indefinite fibrations are complicated in general as shown in Fig. 2, in the sense that the image of the fold locus might have self-intersections and its position in the target surface might not be simple. Moreover, fibers might not be connected. Let us introduce the following concepts which describe "simple" indefinite fibrations.

Definition 3.5. For an indefinite fibration $f : M^4 \to S^2$ into the 2–sphere, we say that f is *directed* if $f(Z_f)$ is contained in a 2–disk D in S^2 such that the complement of a regular value $z_0 \in \operatorname{Int} D$ can be non-singularly foliated by arcs oriented from z_0 to ∂D, which intersect the image of each fold arc transversely in its normal direction. We also say that f has *embedded fold image*, if f is injective on Z_f. We say that f is *fiber-connected*, if every fiber $f^{-1}(z)$, $z \in S^2$, is connected. Similar notions are defined also for broken Lefschetz pencils.

Definition 3.6. A broken Lefschetz fibration $M^4 \to S^2$ (or a broken Lefschetz pencil) is *simplified* if it is fiber-connected, directed with embedded fold image, and it has connected fold locus.

In this article, we give a brief survey of a purely topological proof to the following theorem, originally due to Auroux–Donaldson–Katzarkov.

Theorem 3.7 (Auroux–Donaldson–Katzarkov [3]). *Let M^4 be a closed connected oriented 4–manifold.*

(1) *For a closed oriented 1–dimensional submanifold $Z \neq \emptyset$ of M^4 with $[Z] = 0$ in $H_1(M^4; \mathbb{Z})$, there exists a simplified broken Lefschetz fibration $f : M^4 \to S^2$ with $Z_f = Z$.*

(2) *Let ω be a near-symplectic form with $Z_\omega \neq \emptyset$. Then, there exists a simplified broken Lefschetz pencil g on M^4 with $Z_g = Z_\omega$.*

In particular, every closed oriented 4–manifold admits a simplified broken Lefschetz fibration by Theorem 3.7 (1). (Just consider an oriented circle embedded in a 4–disk in M^4 as Z.)

Auroux–Donaldson–Katzarkov [3] used approximately holomorphic techniques in order to show the above theorem. On the other hand, our proof is based on singularity theory; in particular, we will see that f and g as above can be derived from given generic maps by explicit algorithms.

Existence of broken Lefschetz fibrations on every 4–manifold has been proved by several authors. Baykur [4] and Gay–Kirby [12] used singularity theory; however, the results were not simplified. Akbulut–Karakurt [1] and Lekili [19] used Eliashberg's classification of over-twisted contact structures [8] and Giroux's open-book/contact-structure correspondence [14]; their results were simplified, but the construction was not explicit. Our algorithms are based on singularity theory and give explicit constructions of broken Lefschetz fibrations and pencils that are simplified.

As a digression, we have the following remark. For other dimensions, we have the following, where Z_f denotes the set of all singular points of a smooth map f.

Theorem 3.8 ([25]). *A closed orientable 3–manifold M^3 admits a smooth map $f : M^3 \to \mathbb{R}^2$ with only fold and cusp singularities such that $f|_{Z_f}$ is an injection if and only if M^3 is a graph manifold.*

Theorem 3.9 (Saeki–Yamamoto [28]). *Let M^4 be a closed oriented 4–manifold and N^3 an orientable 3–manifold. Then, for every generic map $f : M^4 \to N^3$, $f|_{Z_f}$ has at least $|\sigma(M^4)|$ triple points, where $\sigma(M^4) \in \mathbb{Z}$ is the signature of M^4.*

Note that in Theorems 3.8 and 3.9, the singular point sets Z_f are closed regular submanifolds of dimensions 1 and 2, respectively.

Therefore, not every map can be modified in such a way that the restriction to the singular point set is an embedding.

However, maps $M^4 \to S^2$ can always be homotopically modified so that it is an embedding on its singular point set.

4. Moves

In this section, we shall describe deformations of an indefinite fibration. They are based on "moves" which modify the immersed fold image locally on a small disk. Some of them are similar to Reidemeister moves in knot theory except that we do not have any over-under crossing information.

In the following, we will focus on the position of the singular point set image in the base surface of an indefinite fibration $f : M^4 \to \Sigma^2$. Without loss of generality we may assume the following:

(1) $f|_{Z_f \setminus C_f}$ is an immersion with normal crossings,
(2) $f|_{C_f \cup L_f}$ is injective and its image misses $f(Z_f \setminus C_f)$,

where C_f (resp. L_f) denotes the set of cusp singularities (resp. Lefschetz singularities) of f.

Definition 4.1. For an indefinite fibration $f : M^4 \to \Sigma^2$, the *base diagram* is the pair $(\Sigma^2, f(Z_f \cup L_f))$, where the fold image $f(Z_f \setminus C_f)$ is normally oriented and L_f is the set of Lefschetz singularities of f (see Fig. 3).

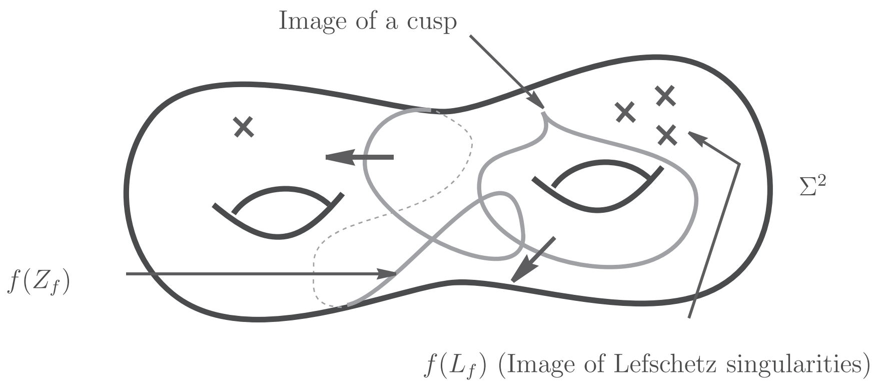


Figure 3. Example of a base diagram

First, let us recall the following existence result.

Theorem 4.2 (Williams [32], Gay–Kirby [12])**.** *If two indefinite fibrations $M^4 \to \Sigma^2$ are homotopic, then one is obtained from the other by a finite sequence of moves.*

Remark 4.3. This is an existence result. Finding explicit moves is another issue.

In the above statement, the terminology "move" is used in the following sense.

Definition 4.4. A *move* for an indefinite fibration $f : M^4 \to \Sigma^2$ is a smooth 1–parameter family $f_t : M^4 \to \Sigma^2$, $t \in [0, 1]$, of "mostly" indefinite fibrations, with $f_0 = f$, which modifies the base diagram only in a small disk neighborhood of a point in Σ^2. (Except for finitely many t's, f_t is an indefinite fibratrion.) In fact, there is a list of moves used in our paper, which will be given shortly.

A move from a local configuration of a base digram, say A, to another one, say B, is *always-realizable* if, given an indefinite fibration whose base diagram contains local configuration A, we can always find a 1–parameter family as above that realizes the relevant base diagram change; i.e. the base diagram of the terminal indefinite fibration f_1 coincides with that of f_0 except on a neighborhood of a point in Σ^2 where it has configuration B.

Remark 4.5. Not all moves are always-realizable. For example, for the Reidemeister II type move, even if we have the situation as in the left-hand side figure, we may not be able to move it to the one in the right in Fig. 4.

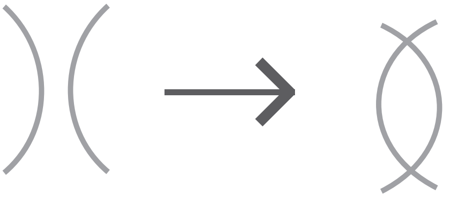


Figure 4. A Reidemeister II type move

The first list of moves is as depicted in Fig. 5. Filled arrows are always-realizable, while blank arrows are not. Cusp merge is "always-realizable" as long as the fibers over the relevant region are connected (see [20]).

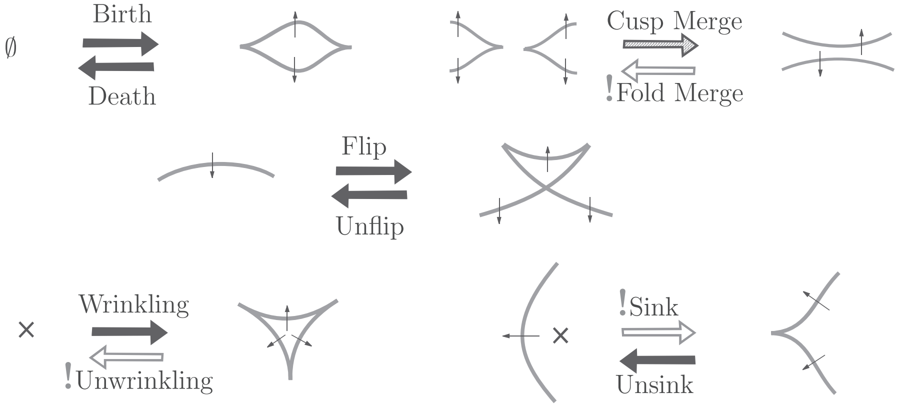


Figure 5. First list of moves

Remark 4.6. In fact, unsink and sink moves are closely related to the D_5–singularity of planar caustics, or the so called $I_{2,3}$–singularity of a map germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. A move similar to the unsink and sink moves appears as a special 1–dimensional section in a versal deformation of the D_5–singularity, see [30] and [33, Fig. 14].

The remaining lists are as depicted in Figs. 6, 7 and 8.

Remark 4.7. These moves were first studied by Furuya in her thesis [9] as far as the author knows.

For our purpose, we also need the moves as depicted in Fig. 9 each of which is realized by a finite sequence of realizable moves. These are realized if the fibers over the regions with $(*)$ are connected. For example, exchange move is realized as in Fig. 10.

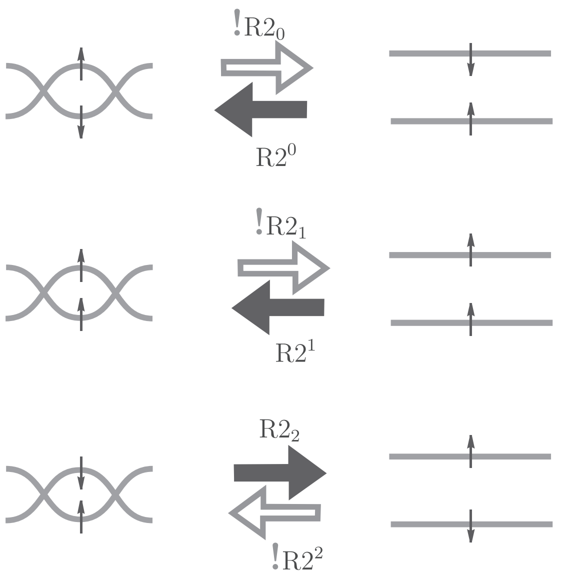


Figure 6. Reidemeister II type moves

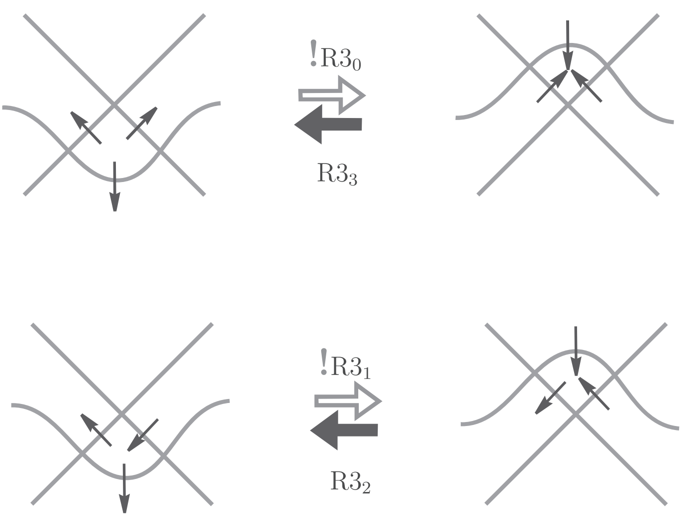


Figure 7. Reidemeister III type moves

Figure 8. Cusp-fold crossing & push/pull moves

Figure 9. Exchange move and criss-cross braiding

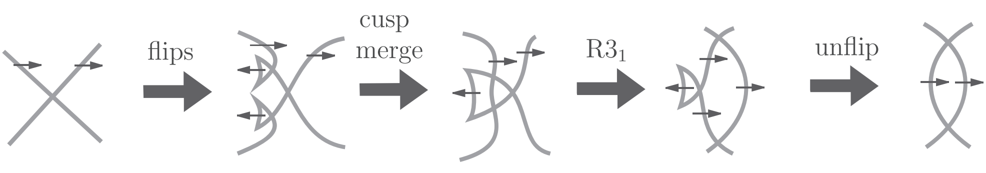


Figure 10. Base diagram moves realizing an exchange move

5. Constructions

Let us start the modification process which simplifies a given indefinite fibration. We will exclusively use always-realizable moves and the procedure will be algorithmic.

Lemma 5.1. *There exists an explicit algorithm consisting of always-realizable moves, which makes any given indefinite fibration directed.*

The above lemma can be proved by an idea as depicted in Fig. 11. We modify the fold image so that it is in a "closed braid form". This can be achieved by utilizing a sequence of always-realizable moves consisting of flip, unsink, push and Reidemeister type moves $R2^0$, $R2^1$, $R2_2$, $R3_2$, $R3_3$. For details, see [5].

Lemma 5.2. *There exists an explicit algorithm consisting of always-realizable moves, which turns any given directed indefinite fibration into a directed one with embedded fold image.*

The idea of proof is as depicted in Fig. 12. Using criss-cross braidings, we can arrange so that each "component" winds exactly once: in other words the fold image is in a closed pure braid form. Then, we can further modify it by using Reidemeister type moves so that finally it is embedded. For details, see [5].

Now, we can make the fibers connected and fold locus connected.

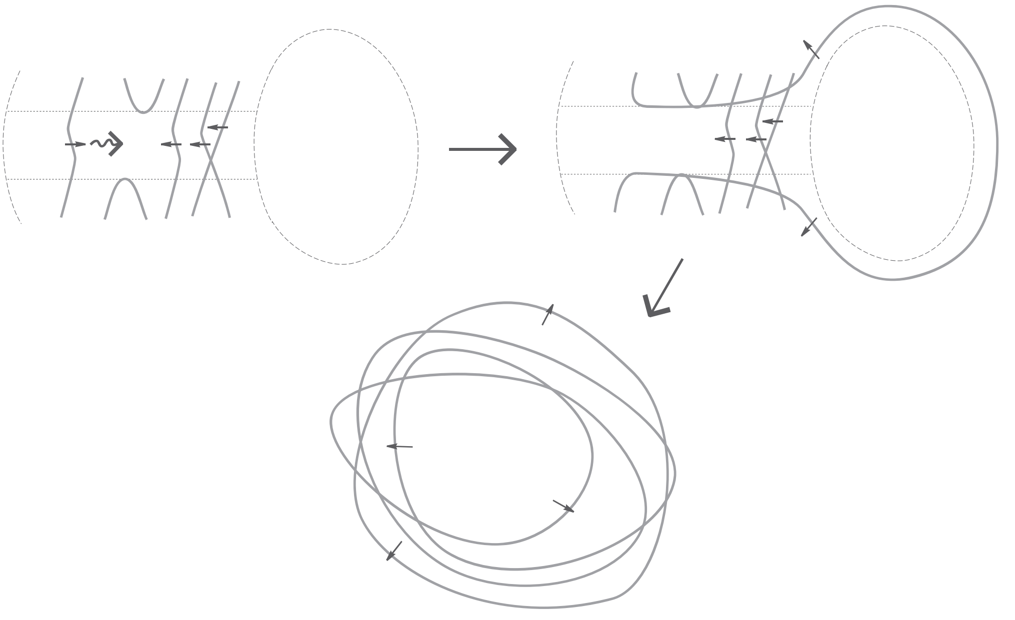


FIGURE 11. Modifying the fold image so that it is directed

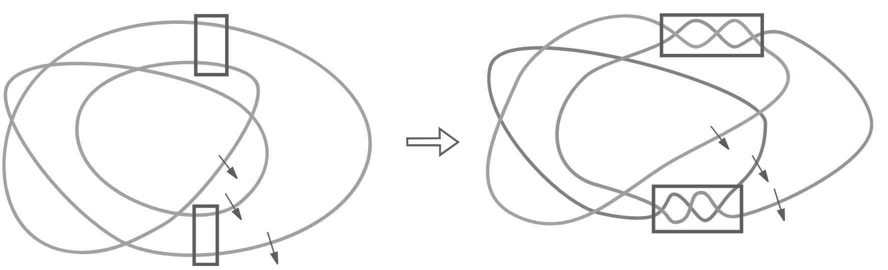


FIGURE 12. Applying criss-cross braidings so that each component winds exactly once

Lemma 5.3. *There exists an explicit algorithm consisting of always-realizable moves, which turns any given directed indefinite fibration with embedded fold image into a fiber-connected, directed one with embedded fold image such that the fold locus is connected.*

In fact, we have an explicit algorithm for such a modification as depicted in Fig. 13, which proves Lemma 5.3.

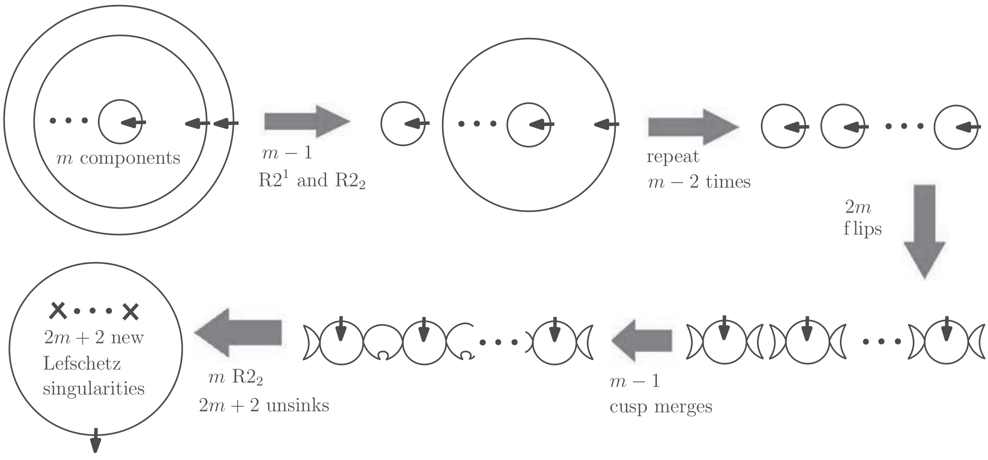


FIGURE 13. Making the fibers and the fold locus connected

Before proceeding to the modification which realizes a given 1–dimensional submanifold as the fold locus, let us recall the following result in singularity theory.

Proposition 5.4 (Ando [2])**.** *Let $f : M^4 \to \Sigma^2$ be an indefinite fibration. Then, Z_f is a closed 1–dimensional submanifold of M^4 which is canonically oriented, and the homology class $[Z_f] \in H_1(M^4;\mathbb{Z})$ represented by Z_f vanishes.*

Now, we have the following.

Theorem 5.5. *Let M^4 be a closed oriented 4–manifold and $Z \neq \emptyset$ a closed oriented 1–dimensional submanifold with $[Z] = 0$ in $H_1(M^4;\mathbb{Z})$. Then, there exists an explicit algorithm consisting of always-realizable moves, which turns any given indefinite fibration $f : M^4 \to S^2$ to a fiber-connected broken Lefschetz fibration $g : M^4 \to S^2$ with directed, embedded fold image such that $Z_g = Z$.*

The idea of proof is as depicted in Fig. 14. We start with an indefinite fibration with connected fold locus and with directed embedded fold image which can be explicitly constructed by Lemmas 5.1, 5.2 and 5.3. By using a flip, we create two cusps. Then, we merge them by using an arc that corresponds to a loop homotopic to a component of Z. Then, we repeat the procedure as in the figure until we realize the homotopy class of all the components of Z except the final one. By our homological assumption, the final component of the fold locus is homologous to the corresponding component of Z. Then we can further modify the indefinite fibration by always-realizable moves so that the final ones coincide with each other up to homotopy (for details, see [5]). Finally, recall that, as we are working in dimension 4, two closed 1–dimensional oriented submanifolds in M^4 are isotopic if and only if they are homotopic. This completes the proof of Theorem 5.5.

Then, we get the following.

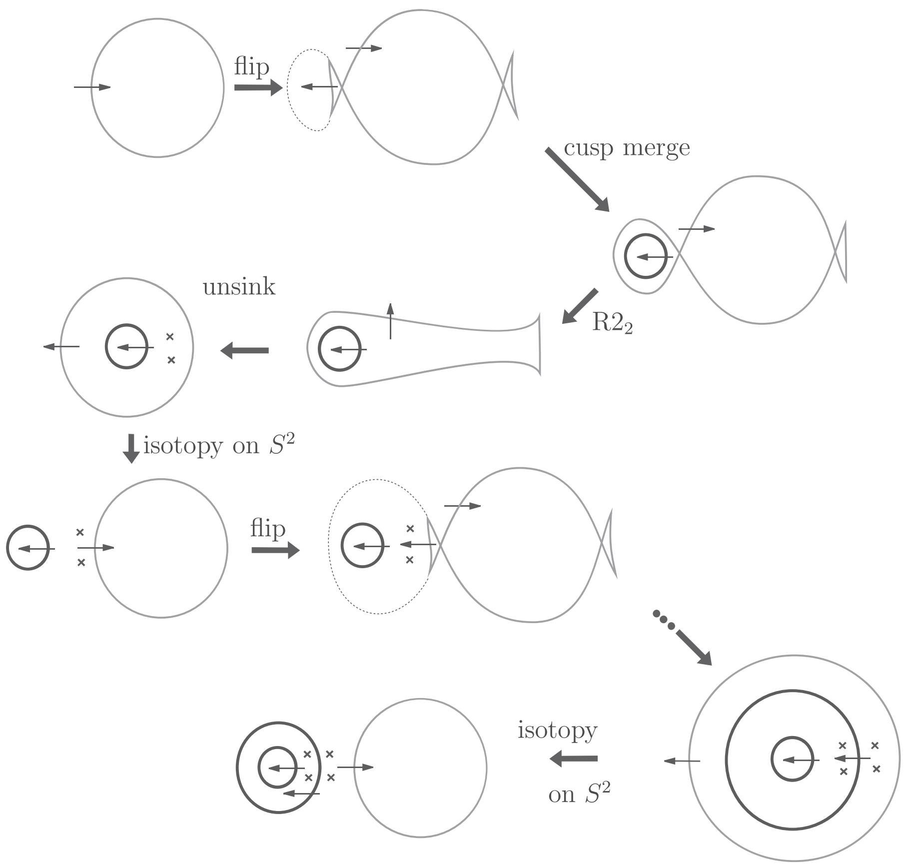


FIGURE 14. Realizing the singular locus except one component

Theorem 5.6. *Let M^4 be a closed oriented 4–manifold, and $Z \neq \emptyset$ a closed oriented 1–dimensional submanifold of M^4 with $[Z] = 0$ in $H_1(M^4; \mathbb{Z})$. Then, there exists a fiber-connected broken Lefschetz fibration $f : M^4 \to S^2$ with directed, embedded fold image such that $Z_f = Z$.*

In fact, given any generic map $M^4 \to S^2$, such f as above can be derived by an explicit algorithm. By analyzing the algorithm carefully, we see, in particular, that birth-death, wrinkling-unwrinkling and sinking moves are unnecessary for the above construction.

The proof of Theorem 5.6 starts with an arbitrary generic map $M^4 \to S^2$ which may not be indefinite. In order to get an indefinite fibration, we need to eliminate definite folds.

Theorem 5.7 ([26, 27])**.** *An arbitrary smooth map $M^4 \to S^2$ is homotopic to an indefinite generic map without definite folds.*

In other words, we can eliminate definite folds by homotopy. As is shown in [27], such an elimination procedure can be explicitly realized algorithmically.

We can now apply our algorithm to homotope the resulting indefinite fibration to a broken Lefschetz fibration with connected fibers and directed, embedded fold image whose fold locus realizes Z. This completes the proof of Theorem 5.6, which is nothing but Theorem 3.7 (1).

As corollaries, we have the following.

Corollary 5.8 (Baykur [4]). *Every closed oriented 4–manifold admits a broken Lefschetz fibration over S^2.*

Corollary 5.9. *There exists an explicit algorithm consisting of always-realizable moves, which turns any generic map $M^4 \to S^2$ to a simplified broken Lefschetz fibration.*

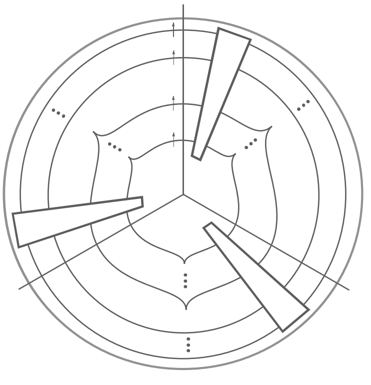
This is the first purely topological and explicit construction of broken Lefschetz fibrations on arbitrary 4–manifolds with embedded fold images.

For the proof of Theorem 3.7 (2) concerning broken Lefschetz pencils, the reader is referred to [5].

6. Trisections

In this section, we apply our techniques concerning explicit moves for indefinite fibrations to the simplification of trisections.

Gay–Kirby [13] showed that every connected closed orientable 4–manifold M^4 admits a generic map into $\mathbb{R}^2$ whose image is as depicted in Fig. 15. Such a map is called a *trisected Morse 2–function*.



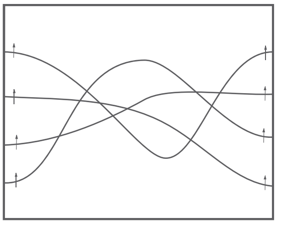


Figure 15. **Left**: Image of a generic map corresponding to a trisection: the outermost circle is the image of definite folds, and the other curves are the images of indefinite fold and cusps. In each box there is an arbitrary Cerf graphic. The three half lines divide the image into three parts and their inverse images give the trisection decomposition. **Right**: An example of a Cerf graphic.

Then, the three half lines divide the image into three parts and their inverse images are all diffeomorphic to $\natural(S^1 \times D^3)$. Every 4–manifold admits such a *trisection*. (For a precise definition of a trisection decomposition, the reader is referred to [13].)

Our explicit algorithm for simplifying indefinite fibrations leads to the following.

Theorem 6.1. *Every closed connected orientable 4–manifold M^4 admits a simplified trisection, i.e. a trisection with trivial Cerf graphics* (*see Fig.* 16).

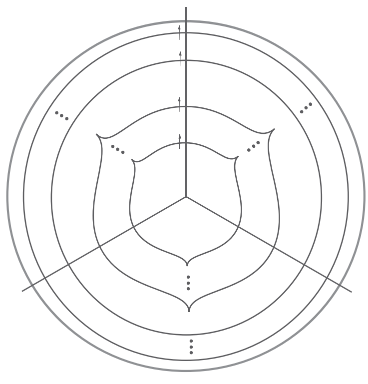


Figure 16. Image of a simplified trisection

A generic map whose image is as depicted in Fig. 16 is called a *simplified trisected Morse* 2*–function*.

The proof of Theorem 6.1 goes as follows (for details, see [5]). We start with a fiber-connected directed broken Lefschetz fibration over S^2 with connected fold locus and with embedded fold image, i.e. a simplified broken Lefschetz fibration. We then modify the map over a small disk consisting of regular values in order to get a map into $\mathbb{R}^2$ whose image is bounded by the embedded image of a definite fold circle. Then, we use our always-realizable moves to modify the resulting map so that we get a simplified trisected Morse 2–function.

In fact, we can go in the reverse direction as well; i.e. given a simplified trisected Morse 2–function, we can get a simplified broken Lefschetz fibration by an explicit algorithm.

By analyzing the above procedures carefully, we can get some formulae which compare the data of the original map with that of the resulting map. In the following, a *simplified* (g, k)*–trisection* is a simplified trisected Morse 2–function whose singular value set is as depicted in Fig. 17. The *genus* of a trisection refers to that of the central regular fiber, which coincides with g. For a simplified broken Lefschetz fibration, its *genus* refers to that of a higher genus regular fiber (see Fig. 18).

Then, we have the following.

Theorem 6.2. (1) *Suppose a closed connected oriented* 4*–manifold M^4 admits a genus–g simplified broken Lefschetz fibration $f : M^4 \to S^2$, with $k \geq 0$ Lefschetz singularities and $\ell \in \{0, 1\}$ components of Z_f. Then, there exists a simplified (g', k')–trisection of M^4 with $g' = 2g + k - \ell + 2$ and $k' = 2g - \ell$.*

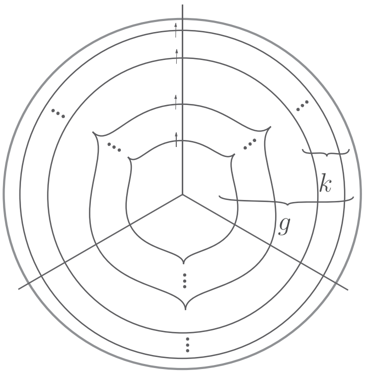


FIGURE 17. Simplified (g,k)–trisection

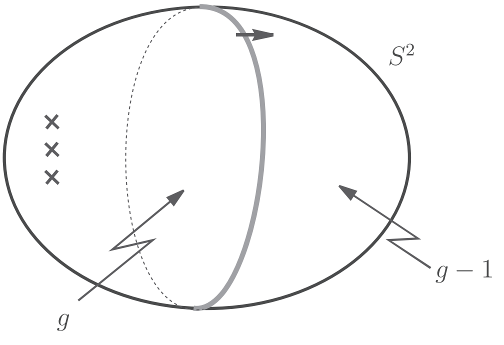


FIGURE 18. Simplified broken Lefschetz fibration with genus g

(2) *Conversely, suppose M^4 admits a simplified (g',k')–trisection. Then, there exists a genus–g simplified broken Lefschetz fibration $f: M^4 \to S^2$ with k Lefschetz singularities and one Z_f component, where $g = g'+3$ and $k = 5g'-3k'+8$.*

In fact, given a simplified broken Lefschetz fibration (or a simplified trisection), we can explicitly construct an associated simplified trisection (resp. simplified broken Lefschetz fibration).

Due to our explicit construction, we can estimate the trisection genera for explicit 4–manifolds, where the *trisection genus* of a 4–manifold is the minimal genus over all its trisections.

Example 6.3. We have the following examples of simplified trisections (see [5]).

(1) S^4 admits a simplified $(0,0)$–trisection.
(2) $\mathbb{C}P^2$ admits a simplified $(1,0)$–trisection.
(3) $S^1 \times S^3$ admits a simplified $(1,1)$–trisection.
(4) $(\pm\mathbb{C}P^2)\sharp(\pm\mathbb{C}P^2)$ and $S^2 \times S^2$ admit simplified $(2,0)$–trisections.

(5) $\mathbb{C}P^2 \sharp (S^1 \times S^3)$ admits a simplified $(2,1)$–trisection.
(6) $(S^1 \times S^3) \sharp (S^1 \times S^3)$ admits a simplified $(2,2)$–trisection.

Furthermore, we have that there exists an infinite family L_n, L'_n, $n \geq 2$, of 4–manifolds admitting simplified $(3,1)$–trisections, where L_n and L'_n are Pao's manifolds which are rational homology 4–spheres with effective torus actions [24].

We also have the following. For details, see [5].

Proposition 6.4. (1) *There exists an exotic simplified* $(20,4)$*–trisection in the homeomorphism class of* $\mathbb{C}P^2 \sharp 7\overline{\mathbb{C}P^2}$*. In other words, there exists a smooth* 4*–manifold* X *homeomorphic to but not diffeomorphic to* $\mathbb{C}P^2 \sharp 7\overline{\mathbb{C}P^2}$ *such that both* X *and* $\mathbb{C}P^2 \sharp 7\overline{\mathbb{C}P^2}$ *admit simplified* $(20,4)$*–trisections.*

(2) *There exists an infinite family of exotic simplified* $(34,8)$*–trisections in the homeomorphism class of* $\mathbb{C}P^2 \sharp 9\overline{\mathbb{C}P^2}$*.*

Problem 6.5. The following problems remain open as far as the author knows.

(1) Which 4–manifolds admit simplified $(3,k)$–trisections for some k? Is there any 4–manifold, other than the ones mentioned above (and their connected sums), which admits a simplified $(3,k)$–trisection for some k? How about simplified $(4,k)$–trisections?
(2) Is there any 4–manifold which admits a trisection, but not a simplified one of the same "genus"?
(3) Find the smallest genus g for which there are infinitely many non-diffeomorphic 4–manifolds in a homeomorphism class admitting simplified (g,k)–trisections for some fixed k.

It should be noted that Hayano [16] classifies 4–manifolds admitting simplified $(2,k)$–trisections for some k (see also [22]).

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