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An obstruction to trivializing links by n-moves

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1 Introduction

The present article is a summary of our paper [8]. We refer the reader to [8] for more details and full proofs.

Let n be a positive integer. An *n-move* on a link is a local move as illustrated in Figure 1.1. Two links are *n-move equivalent* if they are transformed into each other by a finite sequence of n-moves. Note that if n is odd then n-moves may change the number of components of a link. Since a 2-move is generated by crossing changes and vice versa, we can consider an n-move as a generalization of a crossing change. Any link can be transformed into a trivial link by a finite sequence of crossing changes. Therefore, it is natural to ask whether or not any link is n-move equivalent to a trivial link. In 1980s, Yasutaka Nakanishi proved that all links with 10 or less crossings and Montesinos links are 3-move equivalent to trivial links, and he conjectured that any link is 3-move equivalent to a trivial link (see [7, Problem 1.59 (1)]). This conjecture is called the *Montesinos-Nakanishi* 3*-move conjecture*, and have been shown to be true for several classes of links, for example, all links with 12 or less crossings, closed 4-braids and 3-bridge links [1, 9, 11].

After 20 years, in [2, 3] M. K. Dąbkowski and J. H. Przytycki introduced the *nth Burnside group* of a link as an n-move equivalent invariant, and proved that for any odd

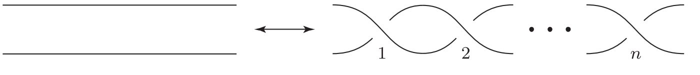

Figure 1.1: n-move

prime p there exist links which are not p-move equivalent to trivial links by using their pth Burnside groups. More precisely, they proved that the closure of the 5-braid $(\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$ and the 2-parallel of the Borromean rings are not 3-move equivalent to trivial links [2], and that the closure of the 3-braid $(\sigma_1\sigma_2)^6$ is not p-move equivalent to a trivial link for any prime number $p \geq 5$ [3]. That is, they gave counterexamples for the Montesinos-Nakanishi 3-move conjecture.

It is easy to see that the pth Burnside group is preserved by p-moves. While the pth Burnside group is a powerful invariant, it is hard to distinguish pth Burnside groups of given links in general. Hence to find a way to distinguish given Burnside groups is very important. In this article, we give an efficient way to distinguish pth Burnside groups of a given link and a trivial link (Theorem 3.1). In fact, by using Theorem 3.1, we show that there exist links, each of which is not p-move equivalent to a trivial link for any odd prime p (Theorem 3.3). Our method is naturally extended to both *virtual* and *welded* links. We prove that there exists a welded link which is not p-move equivalent to a trivial link for any odd prime p (Remark 3.5).

2 Burnside groups of links

Let L be a link in the 3-sphere S^3 and D an unoriented diagram of L. In [4, 5, 6, 13], a group $\Pi_D^{(2)}$ of D is defined as follows. Each arc of D yields a generator, and each crossing of D gives a relation $yx^{-1}yz^{-1}$, where x and z correspond to the underpasses and y corresponds to the overpass at the crossing, see Figure 2.1. The group $\Pi_D^{(2)}$ is an invariant of L. We call it the *associated core group* of L and denote it by $\Pi_L^{(2)}$.

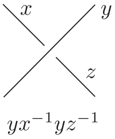

Figure 2.1: Relation of the associated core group

Remark 2.1. M. Wada [13] proved that $\Pi_L^{(2)}$ is isomorphic to the free product of the fundamental group of the double branched cover $M_L^{(2)}$ of S^3 branched along L and the infinite cyclic group $\mathbb{Z}$: $\Pi_L^{(2)} \cong \pi_1(M_L^{(2)}) * \mathbb{Z}$. Moreover, Dąbkowski and Przytycki [2, 3] pointed out that for a diagram D of L, $\pi_1(M_L^{(2)})$ is obtained from the group $\Pi_D^{(2)}$ of D by putting any fixed generator $x = 1$.

In [2, 3], for each positive integer n, Dąbkowski and Przytycki introduced n-move equivalence invariants of L by using $\Pi_L^{(2)}$ and $\pi_1(M_L^{(2)})$ as follows.

Definition 2.2 ([2, 3])**.** Suppose that $\Pi_L^{(2)} = \langle x_1, \ldots, x_m \mid R\rangle$. Then $\pi_1(M_L^{(2)}) \cong \langle x_1, \ldots, x_m \mid R, x_m\rangle$. Let W_n denote a set $\{w^n \mid w \in \langle x_1, \ldots, x_m\rangle\}$, where $\langle x_1, \ldots, x_m\rangle$ is the free group of rank m. The *unreduced nth Burnside group* $\widehat{B}_L(n)$ of L is defined as $\langle x_1, \ldots, x_m \mid R, W_n\rangle$. The *nth Burnside group* $B_L(n)$ of L is defined as $\langle x_1, \ldots, x_m \mid R, x_m, W_n\rangle$.

Proposition 2.3 ([2, 3])**.** *$\widehat{B}_L(n)$ and $B_L(n)$ are preserved by n-moves.*

We will focus on the unreduced nth Burnside group $\widehat{B}_L(n)$ from now on. Let $\widehat{B}_L^q(n)$ denote the quotient group of $\widehat{B}_L(n)$ by the qth term of the lower central series of $\widehat{B}_L(n)$ $(q = 1, 2, \ldots)$. We remark that $\widehat{B}_L(n)$ is not always finite but $\widehat{B}_L^q(n)$ is a finite group for all q, see for example [12, Chapter 2]. Then the proposition above immediately implies the following corollary.

Corollary 2.4. *$\widehat{B}_L^q(n)$ and $|\widehat{B}_L^q(n)|$ are preserved by n-moves for any q.*

Remark 2.5. Let $\mathbb{Z}_n$ denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order n. Let L be a link and D a diagram of L. A map $f : \{\text{arcs of } D\} \to \mathbb{Z}_n$ is a *Fox n-coloring* of D if f satisfies $f(x) + f(z) = 2f(y)$ for each crossing of D, where x and z correspond to the underpasses and y corresponds to the overpass at the crossing. The set of Fox n-colorings of D forms an abelian group and is an invariant of L. Moreover, it is known that the abelian group is isomorphic to $\widehat{B}_L^2(n)$ [10, Proposition 4.5].

3 Obstruction to trivializing links by p-moves

Let p be a prime number. The *Magnus $\mathbb{Z}_p$-expansion* E^p is a homomorphism from $\langle x_1, \ldots, x_m\rangle$ into the formal power series ring in non-commutative variables $X_1, \ldots, X_m$ with $\mathbb{Z}_p$ coefficients defined by $E^p(x_i) = 1 + X_i$ and $E^p(x_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots$ $(i = 1, \ldots, m)$. Then we have the following theorem.

Theorem 3.1 ([8, Theorem 4.1])**.** *Let L be a link with $\Pi_L^{(2)} \cong \langle x_1, \ldots, x_m \mid R\rangle$ and $\widehat{B}_L^2(p) \cong \mathbb{Z}_p^m$. If L is p-move equivalent to a trivial link, then for any $r \in R$,*

$$E^p(r) = 1 + \sum_{(i_1,\ldots,i_p)} c(i_1, \ldots, i_p) X_{i_1} \cdots X_{i_p} + d(p+1)$$

for some $c(i_1, \ldots, i_p) \in \mathbb{Z}_p$ such that $c(i_1, \ldots, i_p) = c(i_{\sigma(1)}, \ldots, i_{\sigma(p)})$ for any permutation σ of $\{1, \ldots, p\}$, where $(i_1, \ldots, i_p)$ runs over $\{1, \ldots, m\}^p$ and $d(k)$ denotes the terms of degree $\geq k$.

Even though 4 is not prime, we have the following theorem.

Theorem 3.2 ([8, Theorem 4.2]). *Let L be an m-component link with $\Pi_L^{(2)} \cong \langle x_1, \dots, x_m \mid R\rangle$. If L is 4-move equivalent to a trivial link, then for any $r \in R$,*

$$E^2(r) = 1 + \sum_{(i_1,i_2,i_3,i_4)} c(i_1, i_2, i_3, i_4) X_{i_1} X_{i_2} X_{i_3} X_{i_4} + d(5)$$

for some $c(i_1, i_2, i_3, i_4) \in \mathbb{Z}_2$ such that $c(i_1, i_2, i_3, i_4) = c(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}, i_{\sigma(4)})$ for any permutation σ of $\{1, 2, 3, 4\}$, where (i_1, i_2, i_3, i_4) runs over $\{1, \dots, m\}^4$.

By applying Theorem 3.1, we have the following theorem.

Theorem 3.3 ([8, Theorem 4.3]). *The closure of the 5-braid $(\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$ and the 2-parallel of the Borromean rings are not p-move equivalent to trivial links for any odd prime p.*

Remark 3.4. Dąbkowski and Przytycki proved Theorem 3.3 for $p = 3$ [2, Theorem 6]. In their proof, the condition that $p = 3$ is essential, and hence it seems hard to show Theorem 3.3 by using their arguments.

Proof of Theorem 3.3. Let γ be the 5-braid $(\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$ described by a diagram in Figure 3.1. We put labels x_i ($i = 1, 2, 3, 4, 5$) on initial arcs of the diagram. Progress from left to right, then the arcs are labeled by using relations of the associated core group. Thus we obtain labels Q_i of terminal arcs of γ as follows (see [2, Lemma 5]):

$$Q_i = x_1x_2^{-1}x_3x_4^{-1}x_5x_1^{-1}x_2x_3^{-1}x_4x_5^{-1}x_ix_5^{-1}x_4x_3^{-1}x_2x_1^{-1}x_5x_4^{-1}x_3x_2^{-1}x_1.$$

Let $\overline{\gamma}$ be the closure of γ. Since we have relations $Q_ix_i^{-1}$ for $\Pi_{\overline{\gamma}}^{(2)}$, $\Pi_{\overline{\gamma}}^{(2)}$ has the presentation $\langle x_1, x_2, x_3, x_4, x_5 \mid r_1, r_2, r_3, r_4, r_5\rangle$, where $r_i = Q_ix_i^{-1}$. We note that $\widehat{B}^2_{\overline{\gamma}}(p) \cong \mathbb{Z}_p^5$ for any odd prime p. On the other hand, by computing $E^p(r_1)$, then the coefficient of $X_2X_3X_4$ is 0 and that of $X_4X_2X_3$ is 2 in $E^p(r_1)$. Theorem 3.1 implies that $\overline{\gamma}$ is not p-move equivalent to a trivial link.

Let γ' be the 6-braid described by a diagram in Figure 3.2. We put labels x_i on initial arcs, y_i on terminal arcs, and Q_i on arcs of the diagram as illustrated in Figure 3.2 ($i = 1, 2, 3, 4, 5, 6$). By using relations of the associated core group, the labels Q_i are

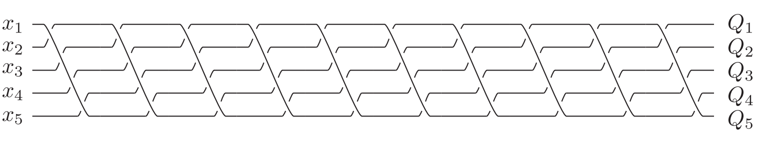

Figure 3.1: 5-braid $\gamma = (\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$

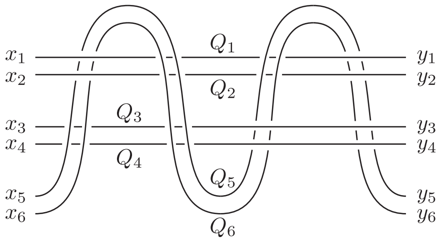

Figure 3.2: 6-braid γ' whose closure is the 2-parallel of the Borromean rings L_{2BR}

expressed as follows:

$$Q_i = \begin{cases} x_1x_2^{-1}x_5x_6^{-1}x_2x_1^{-1}x_ix_1^{-1}x_2x_6^{-1}x_5x_2^{-1}x_1 \\ \quad = y_1y_2^{-1}y_3y_4^{-1}y_5y_6^{-1}y_4y_3^{-1}y_2y_1^{-1}y_iy_1^{-1}y_2y_3^{-1}y_4y_6^{-1}y_5y_4^{-1}y_3y_2^{-1}y_1 \ (i=1,2), \\ x_6x_5^{-1}x_ix_5^{-1}x_6 \\ \quad = Q_6Q_5^{-1}y_iQ_5^{-1}Q_6 = x_1x_2^{-1}x_6x_5^{-1}x_2x_1^{-1}y_ix_1^{-1}x_2x_5^{-1}x_6x_2^{-1}x_1 \quad (i=3,4), \\ x_1x_2^{-1}x_ix_2^{-1}x_1 = y_4y_3^{-1}y_1y_2^{-1}y_3y_4^{-1}y_iy_4^{-1}y_3y_2^{-1}y_1y_3^{-1}y_4 \quad (i=5,6). \end{cases}$$

Since the closure of γ' is the 2-parallel of the Borromean rings L_{2BR}, $\Pi^{(2)}_{L_{2BR}}$ has the presentation $\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid r_1, r_2, r_3, r_4, r_5, r_6 \rangle$, where

$$r_i = \begin{cases} (x_1x_2^{-1}x_5x_6^{-1}x_2x_1^{-1}x_ix_1^{-1}x_2x_6^{-1}x_5x_2^{-1}x_1)^{-1} \\ \quad \times x_1x_2^{-1}x_3x_4^{-1}x_5x_6^{-1}x_4x_3^{-1}x_2x_1^{-1}x_ix_1^{-1}x_2x_3^{-1}x_4x_6^{-1}x_5x_4^{-1}x_3x_2^{-1}x_1 \ (i=1,2), \\ (x_6x_5^{-1}x_ix_5^{-1}x_6)^{-1}x_1x_2^{-1}x_6x_5^{-1}x_2x_1^{-1}x_ix_1^{-1}x_2x_5^{-1}x_6x_2^{-1}x_1 \quad (i=3,4), \\ (x_1x_2^{-1}x_ix_2^{-1}x_1)^{-1}x_4x_3^{-1}x_1x_2^{-1}x_3x_4^{-1}x_ix_4^{-1}x_3x_2^{-1}x_1x_3^{-1}x_4 \quad (i=5,6). \end{cases}$$

We note that $\widehat{B}^2_{L_{2BR}}(p) \cong \mathbb{Z}_p^6$ for any odd prime p. On the other hand, by computing $E^p(r_6)$, then the coefficient of $X_2X_4X_6$ is 1 and that of $X_4X_6X_2$ is 0 in $E^p(r_6)$. Theorem 3.1 implies that L_{2BR} is not p-move equivalent to a trivial link. □

Remark 3.5. For a welded link L, we can similarly define the *associated core group* $\Pi^{(2)}_L$ and the *unreduced nth Burnside group* $\widehat{B}_L(n)$ of L. We note that Theorems 3.1 and 3.2 hold for welded links. Hence, we can show that there exists a welded link which is not p-move equivalent to a trivial link for any odd prime p as follows. Let b be the welded 4-braid described by a virtual diagram in Figure 3.3. We put labels x_i and Q_i $(i = 1, 2, 3, 4)$ on initial and terminal arcs of the diagram, respectively. By using relations of the associated core group, the labels Q_i are expressed as follows:

$$Q_i = \begin{cases} x_4x_1^{-1}x_2x_4^{-1}x_1x_2^{-1}x_3x_2^{-1}x_1x_4^{-1}x_2x_1^{-1}x_4 & \text{if } i=3, \\ x_i & \text{otherwise.} \end{cases}$$

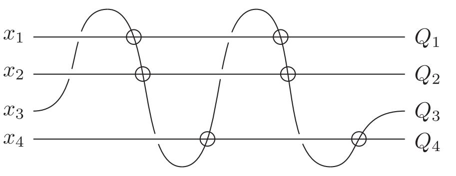

Figure 3.3: Welded 4-braid b

Let $\overline{b}$ be the closure of b, then $\Pi^{(2)}_{\overline{b}} \cong \langle x_1, x_2, x_3, x_4 \mid Q_3 x_3^{-1} \rangle$. We note that $\widehat{B}^2_{\overline{b}}(p) \cong \mathbb{Z}^4_p$ for any odd prime p. On the other hand, by computing $E^p(Q_3 x_3^{-1})$, we have that the coefficient of $X_4X_2X_3$ is 1 and that of $X_4X_3X_2$ is 0 in $E^p(Q_3x_3^{-1})$. Therefore, we have that $\overline{b}$ is not p-move equivalent to a trivial link by Theorem 3.1.

Remark 3.6. All of the three links $\overline{\gamma}, L_{2BR}$ and $\overline{b}$ above are not 4-move equivalent to trivial links by Theorem 3.2 because terms of degree 3 survive in $E^2(r)$ for some relation r of $\Pi^{(2)}_L$ ($L = \overline{\gamma}, L_{2BR}, \overline{b}$).

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