Problems on Low-dimensional Topology, 2018

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This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the problem session of the conference "Intelligence of Low-dimensional Topology" held at Research Institute for Mathematical Sciences, Kyoto University in May 30 – June 1, 2018.

Contents

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1 Computational complexity of the colored Jones polynomial at a root of unity

(Oliver Dasbach)

The volume conjecture of Kashaev and Murakami & Murakami and its generalizations connect the growth of evaluations of the colored Jones polynomial $J_N\big(L; e^{2\pi\sqrt{-1}/N}\big)$ of a link L to geometric information of its link complement. For $J_2(L; q)$, *i.e.* the classical Jones polynomial, it is known by a result of Jaeger, Vertigan and Welsh [32, 61] that evaluations at all but eight points (2nd, 3rd, 4th, 6th roots of unity) are #P-hard. At those eight points polynomial time algorithms are known (see the second remark below), in particular for $J_2(L; -1)$.

Question 1.1 (O. Dasbach). *What can one say about the computational complexity of $J_N\big(L; e^{2\pi\sqrt{-1}/N}\big)$ for $N = 5$ and $N \geq 7$?*

Remark. It is known (see *e.g.* [50]) that the colored Jones polynomial $J_N(K; q)$ of a knot K can be presented by a linear sum of the Jones polynomial $J_2(K^{(n)}; q)$ of $K^{(n)}$ (for $n = 1, 2, \cdots, N-1$), where $K^{(n)}$ denotes the union of n parallel copies of K. Similarly, $J_N(L; q)$ of a link L can be presented by a linear sum of the Jones polynomial of links obtained from L by replacing each component with parallel copies of it. Further, it is known (see the remark below) that the Jones polynomial can be calculated in polynomial time at 2nd, 3rd, 4th, 6th roots of unity. Hence, when $N = 2, 3, 4, 6$, $J_N\big(L; e^{2\pi\sqrt{-1}/N}\big)$ can be calculated in polynomial time.

Remark. At a 2nd, 3rd, 4th, 6th root of unity, the Jones polynomial of a link L can be calculated in polynomial time of the number of crossings of a diagram of L in the following ways. It can be shown by a skein relation that $J_2(L; 1) = (-2)^{\#L-1}$ and $J_2(L; e^{2\pi\sqrt{-1}/3}) = 1$, where $\#L$ denotes the number of components of L. It is known that $|J_2(L; -1)|$ is equal to the order of $H_1(M_{2,L})$ if its order is finite, and 0 otherwise, where $M_{2,L}$ denotes the double branched cover of S^3 branched along L. It is known [49] that $J_2(L; \sqrt{-1}) = (-\sqrt{2})^{\#L-1}(-1)^{\mathrm{Arf}(L)}$ if $\mathrm{Arf}(L)$ exists, and 0 otherwise. It is known [43] that $J_2(L; e^{\pi\sqrt{-1}/3}) = \pm\sqrt{-1}^{\#L-1}\sqrt{-3}^{\dim H_1(M_{2,L}; \mathbb{Z}/3\mathbb{Z})}$. By using such topological interpretations, the Jones polynomial can be calculated in polynomial time at 2nd, 3rd, 4th, 6th roots of unity.

When q is not such a root of unity, it is known [32, 61] that computing $J_2(L; q)$ of an alternating link L is #P-hard.

The following remark is due to Tetsuya Ito.

Remark (T. Ito). It is known [30] that the invariants of Question 1.1 can be presented by the colored Alexander invariants, which are defined by homological representations of the braid groups generalizing the Burau representation. By using these invariants, the invariants of Question 1.1 can be calculated in polynomial time, and Question 1.1 is solved affirmatively.

2 The minimal coloring number of $\mathbb{Z}$-colorable links

(Kazuhiro Ichihara[2] and Eri Matsudo[3])

The Fox n-coloring would be one of the most well-known invariants of knots and links ($n \geq 2$ an integer). However, some of links are known to admit non-trivial Fox n-colorings for every $n \geq 2$, that is, the links with 0 determinants (see the remark at the end of this section). For such a link, we can define a $\mathbb{Z}$-coloring as follows, which is a natural generalization of the Fox n-coloring.

Let L be a link and D a regular diagram of L. A map γ from the set of the arcs of D to $\mathbb{Z}$ is called a $\mathbb{Z}$-*coloring* on D if it satisfies the condition $2\gamma(a) = \gamma(b) + \gamma(c)$ at each crossing of D with the over arc a and the under arcs b and c. A $\mathbb{Z}$-coloring which assigns the same integer to all the arcs of the diagram is called the *trivial $\mathbb{Z}$-coloring*. A link is called $\mathbb{Z}$-*colorable* if it has a diagram admitting a non-trivial $\mathbb{Z}$-coloring. (As usual, we call the integers appearing in the image of a $\mathbb{Z}$-coloring the *colors*.)

For a $\mathbb{Z}$-colorable link L, the *minimal coloring number* of a diagram D of L is defined as the minimal number of the colors for all non-trivial $\mathbb{Z}$-colorings on D, and the *minimal coloring number* of L is defined as the minimum of the minimal coloring numbers of diagrams representing the link L.

The minimal numbers of colors for knots and links admitting Fox's colorings behave interestingly, and have been studied in detail recently. On the other hand, the following was shown by the second author in [47] and by Meiqiao Zhang, Xian'an Jin and Qingying Deng in [62] independently, based on the result given in [27].

Theorem ([62], [47]) The minimal coloring number of any non-splittable $\mathbb{Z}$-colorable link is equal to 4.

We here remark that the proofs in both [47] and [62] are quite algorithmic, and so, the resultant diagrams in their proofs admitting a $\mathbb{Z}$-coloring with four colors are often very complicated.

In view of this, in [28], we have considered and studied the minimal coloring numbers of *minimal diagrams* of $\mathbb{Z}$-colorable links, that is, the diagrams representing the link with least number of crossings.

Based on the results obtained in [28], the following problems can be considered.

Problem 2.1 (K. Ichihara, E. Matsudo)**.** *Determine the minimal coloring number of minimal diagrams of $\mathbb{Z}$-colorable **torus links***.

Remark. It is known that which torus links admit non-trivial $\mathbb{Z}$-colorings. See [1] for example to compute the determinants of torus links. In [28, Theorem 1.3], we showed that, for even integer $n > 2$ and non-zero integer p, the torus link $T(pn, n)$ has a minimal diagram admitting a $\mathbb{Z}$-coloring with only four colors. We have studied several other cases, but not obtained the complete classification yet.

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Problem 2.2 (K. Ichihara, E. Matsudo). *Determine the minimal coloring number of minimal diagrams of $\mathbb{Z}$-colorable **pretzel links**.*

Remark. It is also known that which pretzel links admit non-trivial $\mathbb{Z}$-colorings. See [14] to compute their determinants. In [28], we also obtained some results for such links, but not obtained the complete classification yet.

Remark. A topological interpretation of a $\mathbb{Z}$-coloring is a homomorphism $\pi_1(S^3 - L) \to \text{Aut}(\mathbb{Z})$, where we denote by $\text{Aut}(\mathbb{Z})$ the subgroup of $\text{Map}(\mathbb{Z}, \mathbb{Z})$ generated by f_a (for $a \in \mathbb{Z}$) given by $f_a(x) = 2a - x$. This homomorphism naturally induces a homomorphism $H_1(M_{2,L}; \mathbb{Z}) \to 2\mathbb{Z}$, where $M_{2,L}$ denotes the double cover of S^3 branched along L. We note that $\mathbb{Z}$ naturally acts on the set of $\mathbb{Z}$-colorings by adding a constant to all colors of a coloring. We have a natural bijection

$$\big\{\mathbb{Z}\text{-colorings of a diagram of } L\big\}/\mathbb{Z} \; \longrightarrow \; \text{Hom}\big(H_1(M_{2,L}; \mathbb{Z}), 2\mathbb{Z}\big).$$

The determinant of L is the determinant of a presentation matrix of $H_1(M_{2,L})$ (see *e.g.* [42]). Hence, when the determinant of L is 0, the rank of $H_1(M_{2,L})$ is positive, and we have non-trivial $\mathbb{Z}$-colorings of L.

3 Triangulations and the 3D index of cusped 3-manifolds

(Neil Hoffman)

Let (M, T) be an cusped, orientable 3-manifold and T be an ideal triangulation of M. We say T is *1-efficient* if the only embedded normal surfaces in T with non-negative Euler characteristic are the boundary linking tori (made up solely of triangular disks). In our context, we say a triangulation is *0-efficient* if there are no embedded normal S^2 or RP^2. The concept of 0-efficient and 1-efficient triangulations was introduced by Jaco and Rubinstein [31] and they were able to show that any triangulation of an irreducible manifold can be algorithmically simplified to 0-efficient one.

The 3D-index is an invariant introduced by Dimofte, Gaiotto and Gukov [15], which associated to a 1-efficient ideal triangulation of a 3-manifold with torus boundary components. Furthermore, if two 1-efficient triangulations are associated by a 2/3, 3/2, 0/2 or 2/0 Pachner move, then the 3D-indices of both triangulations are identical by work of Garoufalidis, Hodgson, Rubinstein and Segerman [20]. This motivates the following question which appears in [19] and in discussed in Section 12 of that paper:

Question 3.1 (S. Garoufalidis, C.D. Hodgson, N.R. Hoffman, J.H. Rubinstein [19]). *Given a cusped, atoroidal 3-manifold M. Are all 1-efficient triangulations connected by 2/3, 3/2, 0/2 or 2/0 Pachner moves?*

It is worth mentioning that a more basic question is also appears to be open.

Question 3.2 (folklore). *Given a cusped, irreducible 3-manifold M. Are all 1-efficient triangulations connected by 2/3, 3/2, 0/2 or 2/0 Pachner moves?*

Henry Segerman showed that triangulations without degree 1 edges are connected in the Pachner graph [56]. The absence of degree 1 edges is a necessary condition for a triangulation to be 0-efficient.

There is also a version of the 3D index which associates to each peripheral curve a formal Laurent series. Suggesting that closed manifolds may be assigned a 3D index in this way.

Question 3.3 (N. Hoffman)**.** *Is this assignment a topological invariant? That is given two different Dehn surgery presentations of the same manifold, will the 3D index for each presentation be the same?*

A lighter version of this question was asked by Dongmin Gang [18].

Conjecture 3.4 (D. Gang [18])**.** *If the 3D index evaluated on a slope is an infinite series starting with $1 + \cdots$, if the Dehn filling corresponds to a hyperbolic 3-manifold and the 3D index is 0, 1 or ∞ (does not converge), if M is non-hyperbolic. In particular, if the Dehn filling is a lens space, then the 3D index is 0.*

We remark that Gang's conjecture is supported by a number of a computations.

Finally, let infinite collection of Dehn fillings $\{M_{\gamma_i}\}$ of a cusped manifold M such that the geometric limit of the $\{M_{\gamma_i}\}$ is M.

Question 3.5 (N. Hoffman)**.** *Is there a normalization for the 3D index such that a sequence of 3D indices computed on the $\{M_{\gamma_i}\}$ converges (term-wise) to the 3D index of M?*

4   Destabilized Heegaard surfaces of 3-manifolds

(Yeonhee Jang, Tsuyoshi Kobayashi, Makoto Ozawa and Kazuto Takao)

We briefly recall some of the standard terminology and facts. A *Heegaard surface* of a 3-manifold is an embedded closed surface which divides the manifold into two handlebodies. It is an important fact that every closed orientable 3-manifold has a Heegaard surface. For a given Heegaard surface, we can construct a new Heegaard surface of the same 3-manifold by adding a canceling pair of handles. A Heegaard surface is said to be *destabilized* if it cannot be produced by this construction.

The following is a general problem.

Problem 4.1. *Classify the Heegaard surfaces of each closed orientable* 3-*manifold.*

In particular, classifying the destabilized ones is an essential part.

We briefly review some of the abundant known results for the above problem. Waldhausen [60] showed the uniqueness of destabilized Heegaard surfaces of the 3-sphere, and Bonahon and Otal [10, 11] showed that of lens spaces. Hence, if there is a Heegaard surface of genus 0 or 1, then the destabilized ones of the 3-manifold are unique. On the other hand, Casson–Gordon (see [48, 54]) and Moriah–Schleimer–Sedgwick [48] gave families of 3-manifolds each of which has infinitely many destabilized Heegaard surfaces of pairwise distinct genera. The minimal genus of Casson–Gordon's Heegaard surfaces is 4, and that of Moriah–Schleimer–Sedgwick's is 3,

though we do not know whether they attain the minimums of the manifolds. We also refer the reader to [53, 58] for a theorem on the stable equivalence, to [33, 40] for theorems on the finiteness at any given genus, to [13, 36, 41] for classification algorithms, and to [25, 35, 38, 45, 55] for relevant examples.

The following is one of various open problems suggested by the above results.

Problem 4.2. *Prove or disprove the existence of a* 3-*manifold which has a Heegaard surface of genus* 2, *and infinitely many destabilized ones of pairwise distinct genera.*

5 Simplified trisections of 4-manifolds

(Osamu Saeki[4])

A *trisection* of a 4-manifold was introduced by Gay–Kirby [21], and it is expected to be a generalization of a Heegaard splitting of a 3-manifold. Roughly speaking, a trisection of a 4-manifold M is a decomposition $M = M_1 \cup M_2 \cup M_3$ such that for a fixed non-negative integer ℓ, M_i is diffeomorphic to $\natural^\ell(S^1 \times B^3)$ (4-dimensional handlebody) for $i = 1, 2, 3$, and $M_1 \cap M_2 \cap M_3$ is a closed orientable surface of genus g, and $X_k = (M_k \cap M_i) \cup (M_k \cap M_j)$ is a 3-manifold having a Heegaard splitting of genus g as the union of $M_k \cap M_i$ and $M_k \cap M_j$ for $\{k, i, j\} = \{1, 2, 3\}$. As a Heegaard splitting of a 3-manifold X can be studied by using a Morse function $X \to \mathbb{R}$, a trisection of a 4-manifold M can be studied by using a Morse 2-function $M \to \mathbb{R}^2$.

Let M be a smooth closed connected oriented 4-manifold. A smooth map $f : M \to \mathbb{R}^2$ is a *trisected Morse* 2-*function* (or a *trisection map*) if it has only fold and cusp singularities and it satisfies the following conditions (see [21, 8, 9] and Fig. 1):

(1) its image is diffeomorphic to a 2-disk, denoted by D^2,

(2) it has a single definite fold circle mapped diffeomorphically onto the boundary ∂D^2 of D^2,

(3) $D^2 \setminus \{p\}$ can be nonsingularly foliated by rays from a regular value p to ∂D^2, each intersecting the indefinite fold image always in the direction of index-2 handle attachments,

(4) three of these rays split D^2 into three sectors, where there is at most one cusp on each singular arc image in a sector,

(5) the total number of cusps in the sectors are equal,

(6) the singular arcs with cusps are situated inside.

The number g of indefinite fold arcs (possibly with a cusp) in each sector is called the *genus* of the trisection. Note that $f^{-1}(p)$ is a closed orientable surface of genus g for the regular value $p \in D^2$.

It is known that for every trisection decomposition of M, there is a trisected Morse 2-function $f : M \to \mathbb{R}^2$ which gives the given trisection decomposition.

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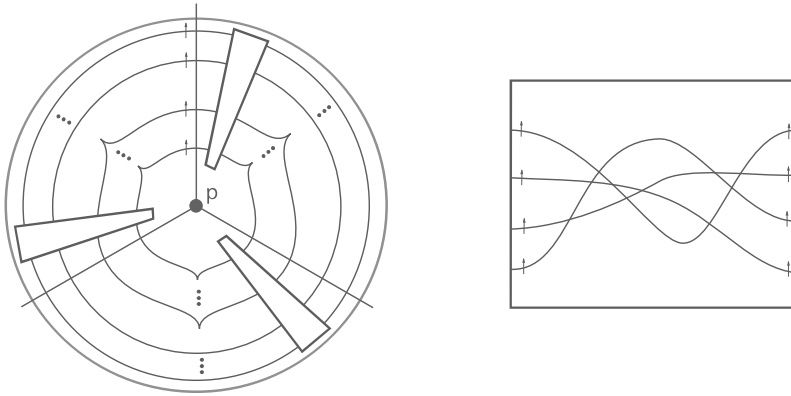Email address: `saeki@imi.kyushu-u.ac.jp`

Figure 1: The figure on the left depicts the image of a trisected Morse 2-function. In each box there is an arbitrary Cerf graphic as in the right [21, §3]. The three half lines divide the image into three parts and their inverse images correspond to 4-dimensional handlebodies giving a trisection decomposition.

As far as the author knows, the following problem is still open.

Problem 5.1 (R.I. Baykur, O. Saeki [8]). *Non-isotopic trisected Morse 2-functions may yield equivalent trisection decompositions. Describe the necessary and sufficient conditions for two trisected Morse 2-functions to give equivalent trisection decompositions.*

A trisection is said to be *simplified* if there exists an associated trisection map such that the restriction to its singular locus is an embedding [8, 9] (see Fig. 2). This is in great contrast with a general trisection map (see Fig. 1), which has Cerf boxes in between the three sectors of the image 2-disk, where folds can cross each other arbitrarily (and therefore, the images of some indefinite fold circles might wind around p multiple times).

Note that Hayano [26] has studied the condition for a given trisection decomposition to be equivalent to a simple one. He has also classified genus-2 simplified trisections.

The following manifolds are known to admit genus-3 simplified trisections [9]: S^4, connected sums of $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$ and $S^1 \times S^3$ with three summands, connected sum of either one of these manifolds with $S^2 \times S^2$, Pao's manifolds L_n, L'_n [51]. One can similarly get genus-4 examples on connected sums of lower genera trisections on these standard 4-manifolds. In addition, we have the irreducible examples on $L(p,q)$-bundles and $(S^1 \times S^2)$-bundles over S^1, which include S^2-bundles over the 2-torus T^2 and the Klein bottle Kb.

The following two problems have been posed in [9].

Problem 5.2 (R.I. Baykur, O. Saeki [9]). *Classify 4-manifolds that admit simplified genus-3 trisections. Is there any 4-manifold, other than the ones mentioned above, which admits genus-3 simplified trisections? How about genus-4?*
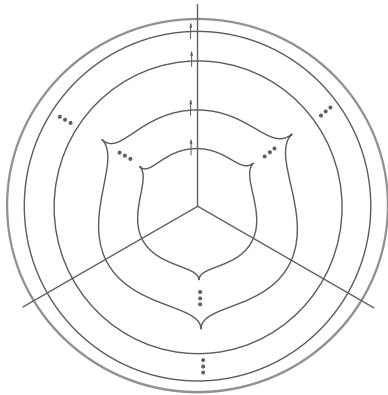
Figure 2: Singular value of a simplified trisected Morse 2-function

Question 5.3 (R.I. Baykur, O. Saeki [9]). *Is there any 4-manifold which admits a trisection, but not a simplified one of the same genus? Defining the minimal trisection genus (resp. minimal simplified trisection genus) of a 4-manifold M as the smallest genus of a trisection (resp. simplified trisection) on M, one can equivalently ask if there is a 4-manifold whose trisection genus is strictly smaller than its simplified trisection genus.*

The two genera are equal for all of the 4-manifolds with (simplified) trisections of genus $g \leq 4$ mentioned above. The answer to the analogous question for broken Lefschetz fibrations versus simplified broken Lefschetz fibrations is positive [7].

Heegaard splittings of a closed oriented 3-manifold X are closely related with a Morse function $g : X \to \mathbb{R}$. Based on this, Johnson [34] and Takao [59] used generic maps into $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ for comparing two Heegaard splittings of a given 3-manifold.

Question 5.4 (O. Saeki). *Can we use a similar idea for comparing two trisections of a given 4-manifold to get some information on the relationship between the two trisections?*

It is well known that any two Heegaard splittings of a given 3-manifold are related by a sequence of stabilizations. As an analog, it is shown [21] that any two trisections of a given 4-manifold are related by a sequence of "stabilizations". In [21], this is proved by using handle decompositions of a 4-manifold and stabilizations of Heegaard splittings of 3-manifolds.

Problem 5.5 (D. Gay, R. Kirby [21]). *Find a singularity theoretical proof of the uniqueness of trisections (up to stabilization) on a given 4-manifold.*

See [21, Remark 15] for an approach to this problem.

6 Virtual embeddings between mapping class groups of surfaces

(Takuya Katayama)

Let $\Sigma_{g,p}^b$ be a compact orientable surface of genus g with p marked points and b boundary components. The *mapping class group* (or *Teichmüller modular group*) $\mathrm{Mod}(\Sigma_{g,p}^b)$ of $\Sigma_{g,p}^b$ is the group of orientation-preserving homeomorphisms of $\Sigma_{g,p}^b$ fixing the set of marked points setwise and fixing the boundary pointwise, up to isotopy relative to the marked points and the boundary. In particular, $\mathrm{Mod}(\Sigma_{0,n}^1)$ is naturally identified with the *nth braid group* B_n.

Homomorphisms between mapping class groups have been studied in many researches. A typical construction of such a homomorphism is induced by an embedding of a surface, *i.e.*, a combination of forgetting marked points, deleting boundary components, and subsurface embeddings; see [4]. Another typical construction of such a homomorphism is induced by a covering of a surface. For example, a natural double branched covering $\Sigma_{2,0}^0 \to \Sigma_{0,6}^0$ induces a surjective homomorphism $\mathrm{Mod}(\Sigma_{2,0}^0) \to \mathrm{Mod}(\Sigma_{0,6}^0)$, whose kernel is a cyclic group of order 2; see [46]. Further, natural double coverings $\Sigma_{g,0}^1 \to \Sigma_{0,2g+1}^1$ and $\Sigma_{g,0}^2 \to \Sigma_{0,2g+2}^1$ induce injective homomorphisms $B_{2g+1} \to \mathrm{Mod}(\Sigma_{g,0}^1)$ and $B_{2g+2} \to \mathrm{Mod}(\Sigma_{g,0}^2)$; see [16, Section 9.4]. The topic of homomorphisms between mapping class groups is related to the rigidity of structures on surfaces; see [5] for a survey on this topic.

Moreover, injective homomorphisms between mapping class groups have also been studied in some researches. By modifying the above mentioned homomorphisms of B_{2g+1} and B_{2g+2}, we can obtain injective homomorphisms $B_{2g+1} \to \mathrm{Mod}(\Sigma_{g+1,0}^0)$ and $B_{2g+2} \to \mathrm{Mod}(\Sigma_{g+1,0}^0)$; see [24, 37]. Further, it is known [3] that, for any $g \geq 2$, there exist $g' > g$ and an injective homomorphism $\mathrm{Mod}(\Sigma_{g,0}^0) \to \mathrm{Mod}(\Sigma_{g',0}^0)$. On the other hand, it is known [24] that, when $g \geq 3$ and $g > g'$, there are no nontrivial homomorphisms $\mathrm{Mod}(\Sigma_{g,0}^0) \to \mathrm{Mod}(\Sigma_{g',0}^0)$.

We say that a group H is *embedded* in another group G if there exists an injective homomorphism from H to G. Further, as in [6], we say that H is *virtually embedded* in G if a finite index subgroup of H is embedded in G. We note that the composition of virtual embeddings is a virtual embedding.

As an example of a virtual embedding, we can show that $\mathrm{Mod}(\Sigma_{2,0}^0)$ is virtually embedded in $\mathrm{Mod}(\Sigma_{0,6}^0)$, as follows. As mentioned above, a double covering $\Sigma_{2,0}^0 \to \Sigma_{0,6}^0$ induces a homomorphism $\mathrm{Mod}(\Sigma_{2,0}^0) \to \mathrm{Mod}(\Sigma_{0,6}^0)$ whose kernel is a cyclic group of order 2 (generated by the hyperelliptic involution ι). Since $\mathrm{Mod}(\Sigma_{2,0}^0)$ is residually finite (see [16, Section 6.4]), we can find a finite index subgroup H of $\mathrm{Mod}(\Sigma_{2,0}^0)$ such that the restriction map $H \to \mathrm{Mod}(\Sigma_{0,6}^0)$ is injective, by showing that its kernel $\{1, \iota\} \cap H$ is trivial. Hence, $\mathrm{Mod}(\Sigma_{2,0}^0)$ is virtually embedded in $\mathrm{Mod}(\Sigma_{0,6}^0)$. This construction is a topological interpretation of a virtual embedding. See also [57, Theorem 2] for a preceding result on virtual embeddings.

Problem 6.1 (T. Katayama)**.** *Given $g \geq 2$, determine whether $\mathrm{Mod}(\Sigma_{g,0}^0)$ is virtually embedded in $\mathrm{Mod}(\Sigma_{g',p'}^{b'})$.*

Remark 1. If g was 1, we can show that it is relatively easy to make a virtual embedding of $\mathrm{Mod}(\Sigma_{1,0}^0)$ to a mapping class group, as follows. It is known, see *e.g.* [44, Proposition 4.4.2], that $\mathrm{Mod}(\Sigma_{1,0}^0)$ is $\mathrm{SL}(2;\mathbb{Z})$ and it has the rank 2 free group as a finite index subgroup. Further, we can embed the rank 2 free group into almost all mapping class groups, by considering Dehn twists along two essential simple closed curves with at least two crossings; see [29]. Hence, if g was 1, Problem 6.1 would be relatively easy.

Remark 2. When $g = 2$, we can show that $\mathrm{Mod}(\Sigma_{2,0}^0)$ is virtually embedded in $\mathrm{Mod}(\Sigma_{g',0}^0)$ for any $g' \geq 3$, as follows. As mentioned above, $\mathrm{Mod}(\Sigma_{2,0}^0)$ is virtually embedded in $\mathrm{Mod}(\Sigma_{0,6}^0)$. Further, by Remarks 2A and 2B below, $\mathrm{Mod}(\Sigma_{0,6}^0)$ is virtually embedded in $\mathrm{Mod}(\Sigma_{g',0}^0)$ for any $g' \geq 3$. Hence, $\mathrm{Mod}(\Sigma_{2,0}^0)$ is virtually embedded in $\mathrm{Mod}(\Sigma_{g',0}^0)$ for any $g' \geq 3$.

Therefore, when $g = 2$, $g' \geq 3$ and $p' + b' = 0$, Problem 6.1 is solved affirmatively. It is not known whether $\mathrm{Mod}(\Sigma_{g,0}^0)$ is virtually embedded in $\mathrm{Mod}(\Sigma_{g',p'}^{b'})$, when $g \geq 3$ or $p' + b' \geq 1$.

Remark 2A. We can show that $\mathrm{Mod}(\Sigma_{0,6}^0)$ is virtually embedded in B_5, as follows. It is known [12, Theorem 10] that the pure braid group P_5 is isomorphic to $\mathrm{PMod}(\Sigma_{0,6}^0) \times \mathbb{Z}$, where $\mathrm{PMod}(\Sigma_{0,6}^0)$ is the finite index subgroup of $\mathrm{Mod}(\Sigma_{0,6}^0)$ fixing the marked points. Since there is an embedding $\mathrm{PMod}(\Sigma_{0,6}^0) \to P_5 \to B_5$, $\mathrm{Mod}(\Sigma_{0,6}^0)$ is virtually embedded in B_5.

Remark 2B. We can show that B_5 is embedded in $\mathrm{Mod}(\Sigma_{g,0}^0)$ for any $g \geq 3$, as follows. As mentioned before, there is an embedding $B_{2g+1} \to \mathrm{Mod}(\Sigma_{g+1,0}^0)$. Hence, when $g \geq 3$, there is an embedding $B_5 \to B_{2g-1} \to \mathrm{Mod}(\Sigma_{g,0}^0)$. Therefore, B_5 is embedded in $\mathrm{Mod}(\Sigma_{g,0}^0)$ for any $g \geq 3$.

Remark 3. It is known [37, Corollary 1.7] that, when $g, g' \geq 2$, $\mathrm{Mod}(\Sigma_{g,p}^0)$ is virtually embedded in $\mathrm{Mod}(\Sigma_{g',p'}^0)$, only if $3g + p \leq 3g' + p'$ and $2g + p \leq 2g' + p'$.

It follows from [4, Corollary 1.2] that, if $6 \leq g < g' \leq 2g - 2$, $\mathrm{Mod}(\Sigma_{g,0}^0)$ is not embedded in $\mathrm{Mod}(\Sigma_{g',0}^0)$.

It is known [4, Proposition 7.1] that, if $g \geq 3$ and $g > g'$, any homomorphism $\mathrm{Mod}(\Sigma_{g,p}^b) \to \mathrm{Mod}(\Sigma_{g',p'}^{b'})$ is trivial.

Conjecture 6.2 (T. Katayama)**.** *Suppose that $g \geq 2$. If $\mathrm{Mod}(\Sigma_{g,p}^b)$ is virtually embedded in B_n for some n, then $(g, p, b) = (2, 0, 0)$.*

When $(g, p, b) = (2, 0, 0)$, we note that $\mathrm{Mod}(\Sigma_{2,0}^0)$ is virtually embedded in B_5 by Remarks 2 and 2A of Problem 6.1.

In case there is a counter-example to Conjecture 6.2, the target braid group does not admit faithful C^∞ action on $\mathbb{R}$ even virtually, and is not "virtually special" (*i.e.*, no finite index subgroup is embedded in a right-angled Artin group, see [6, Question 2]).

Problem 6.3 (T. Katayama)**.** *Find a pair (H, ϕ) of a finite index subgroup H of $\mathrm{Mod}(\Sigma_{g,p}^b)$ and a homomorphism $\phi \colon H \to \mathrm{Mod}(\Sigma_{g',p'}^{b'})$ with "small kernel".*

Remark. Some "surface manipulations" induce homomorphisms with "small kernels" (see [3], [4], [16, Chapter 3], [46] and [52]). Are there any other homomorphisms with "small kernels"?

7 Totally real immersions and embeddings of n-manifolds into $\mathbb{C}^n$

(**Naohiko Kasuya**[5])

Let M^n be a closed orientable n-manifold and $f\colon M^n \to \mathbb{C}^n$ be an immersion. A point $p \in M^n$ is said to be a *complex tangent* if $df_p(T_pM^n)$ contains a complex line. An immersion is said to be *totally real* if it has no complex tangent. It follows from Thom's transversality theorem that the set of complex tangents of a generic immersion $f\colon M^n \to \mathbb{C}^n$ is empty or forms a closed $(n-2)$-dimensional submanifold.

For totally real immersions and embeddings, the following theorems are known. These are called the *h*-principle for totally real immersions and embeddings.

Theorem (Gromov [22], Lees [39]). An n-manifold M^n admits a totally real immersion into $\mathbb{C}^n$ if and only if the complexified tangent bundle $\mathbb{C}TM^n$ is trivial.

Theorem (Gromov [23], Forstnerič [17]). Let M^n be a closed orientable n-manifold with $n \geq 3$. Then, M^n admits a totally real embedding into $\mathbb{C}^n$ if and only if it admits a totally real immersion into $\mathbb{C}^n$ which is regularly homotopic to an embedding.

In the following, we consider the case where $n = 3$. It follows from the *h*-principle that any closed orientable 3-manifold admits a totally real embedding into $\mathbb{C}^3$. However, there are few explicit examples of totally real embeddings. The following example is due to Ahern and Rudin.

Example (Ahern-Rudin [2]). Let $P(z,w) = \bar{z}\bar{w}(|w|^2 + i|z|^2)$. We consider the 3-sphere as the unit sphere $S^3 = \big\{(z,w) \mid |z|^2 + |w|^2 = 1\big\} \subset \mathbb{C}^2$. Then, the embedding $f\colon S^3 \to \mathbb{C}^3$ defined by

$$f(z,w) = \big(z, w, P(z,w)\big)$$

is a totally real embedding.

Inspired by their example, Forstnerič constructed some other examples of totally real embeddings of quotients of the 3-sphere, including $\mathbb{R}P^3$. But we want more examples.

Problem 7.1 (N. Kasuya). *Construct interesting examples of totally real embeddings of 3-manifolds into $\mathbb{C}^3$ (without using the h-principle). For example, can a Haefliger knot be explicitly realized as a totally real submanifold of $\mathbb{C}^3$?*

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