ON $p$-ADIC FAMILIES OF THE $D$-TH SAI TO-KUROKA W Lifts

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1. Introduction

Let $p$ be an odd prime, $N$ an odd positive integer not divisible by $p$ and $D$ the discriminant of an imaginary quadratic field satisfying $p | D$. We will establish $p$-adic interpolation of Fourier coefficients of the $D$-th Saito-Kurokawa lifts of primitive forms of level $N$ varying in a Coleman family. This generalizes known results for Hida families of tame level $N = 1$ to the case of Coleman families of tame level $N$. The main theorem is Theorem 6.7.

Notation and terminology. Throughout the paper, we fix an odd prime $p$, a positive integer $N$ satisfying $(N, 2p) = 1$ and a non-negative rational number $\alpha$. We assume that $Np \geq 4$ to ensure that $\Gamma_1(Np)$ is torsion-free. We denote by $\bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}}_p$ an algebraic closure of the rational number field $\mathbb{Q}$, and the $p$-adic number field $\mathbb{Q}_p$, respectively. Let $\mathbb{C}$ be the complex number field and $\mathbb{C}_p$ the $p$-adic completion of $\bar{\mathbb{Q}}_p$. We fix two embeddings $i_{\infty}: \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p: \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, and an isomorphism $\mathbb{C}_p \sim \mathbb{C}$ which commutes with $i_{\infty}$ and $i_p$. Let $va_1$ be the normalized $p$-adic additive valuation on $\mathbb{C}_p$ so that $va_1(p) = 1$. For $z \in \mathbb{C}$, we define $\sqrt{z} = z^{1/2}$ so that $\pi/2 < \arg(z^{1/2}) \leq \pi/2$ and put $z^{k/2} = (\sqrt{z})^k$ for each integer $k$. We put $e(z) := \exp(2\pi\sqrt{-1}z)$ and $e^m(z) := e(mz)$. For a Dirichlet character $\chi$, we denote by $c_\chi$ the conductor of $\chi$, $\chi_0$ the primitive character attached to $\chi$ and $G(\chi) := \sum_{i=0}^{c_\chi-1} \chi_0(i)e(i/c_\chi)$. For a non-zero integer $a$, we let $\chi_a$ denote the Kronecker symbol $\chi_a(b) := (\frac{a}{b})$ defined by [MFM, (3.1.9)].

2. Siegel cusp forms

Let $g$ be a positive integer. Note that our concern is only for $g = 1, 2$.

2.1. Definition of Siegel cusp forms

Put $1_g := \text{diag}(1, \ldots, 1), 0_g := \text{diag}(0, \ldots, 0) \in M_g(\mathbb{Z})$ and

$$J_g := \begin{pmatrix} 0_g & -1_g \\ 1_g & 0_g \end{pmatrix} \in \text{GL}_{2g}(\mathbb{Z})$$

For a commutative ring $R$ and an integer $M$,

$$\begin{align*}
\text{GSp}_g(R) &:= \{ \gamma \in \text{GL}_{2g}(R) \mid \gamma J_g \gamma = \nu(\gamma) J_g \text{ for some } \nu(g) \in R^\times \}, \\
\text{Sp}_g(R) &:= \{ \gamma \in \text{GL}_{2g}(R) \mid \gamma J_g \gamma = J_g \} = \text{Ker}(\nu), \\
\Gamma_0^2(M) &:= \{ \gamma \in \text{Sp}_g(\mathbb{Z}) \mid c_\gamma \equiv 0_g (\text{mod } M) \},
\end{align*}$$

where we denote by $c_\gamma$ the left lower $g \times g$ matrix of $\gamma$ and from now on we use the notation

$$\begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} := \gamma \in \text{GL}_{2g}(R).$$
For a subring $R$ of the real number field $\mathbb{R}$, we put

\begin{equation}
\text{GSp}_{g}^{+}(R) := \{ \gamma \in \text{GL}_{2g}(R) \mid \gamma J_{g} \gamma = \nu(\gamma) J_{g} \text{ for some } \nu(g) > 0 \}.
\end{equation}

We denote by $\mathfrak{H}_{g} := \{ Z \in \text{Sym}_{g}(\mathbb{C}) \mid \text{Im}(Z) > 0 \}$ the Siegel upper-half plane of genus $g$, where $\text{Sym}_{g}(\mathbb{C})$ is the set of symmetric $g \times g$ matrices whose entries in $\mathbb{C}$.

Let $\gamma \in \text{GSp}_{g}^{+}(\mathbb{R})$ act on $Z \in \mathfrak{H}_{g}$ by

\begin{equation}
\gamma Z := (a_{\gamma}Z + b_{\gamma})(c_{\gamma}Z + d_{\gamma})^{-1}.
\end{equation}

**Definition 2.1.** Let $\chi$ be a Dirichlet character modulo $M$. A Siegel modular form $F$ of genus $g$, weight $k$, level $M$ and character $\chi$ is a holomorphic function on $\mathfrak{H}_{g}$ satisfying

\begin{equation}
F|_{k}\gamma(Z) := \det(c_{\gamma}Z + d_{\gamma})^{-k}F(\gamma Z) = \chi^{-1}(\det(a_{\gamma}))F(Z)
\end{equation}

for any $\gamma \in \Gamma_{0}^{g}(M)$ and additional conditions of holomorphy at the cusps when $g = 1$. We denote the space of all such functions $F$ by $S_{k}^{g}(M, \chi)$. Note that a Siegel cusp form of genus $g$, weight $k$, level $M$ and character $\chi$ is a cusp form of weight $k$ and character $\chi$ for $\Gamma_{0}^{g}(M)$ in terms of [QFHO, p.78] and $S_{k}^{g}(M, \chi)$ is written as $\mathfrak{N}_{k}(\Gamma_{0}^{g}(M), \chi)$ in [QFHO, p.82].

For any $F \in S_{k}^{g}(M, \chi)$ and $Z \in \mathfrak{H}_{g}$, we write the Fourier expansion of $F$ as

\begin{equation}
F(Z) = \sum_{T \in \mathcal{L}_{>0}} a_{T}(F)e(\text{tr}TZ),
\end{equation}

where $\mathcal{L}_{>0}$ is the set of positive definite half-integral symmetric $g \times g$ matrices (see [QFHO, Theorem 2.3.12] for the Fourier expansions).

**2.2. Hecke algebras**

For a group $G$, a subgroup $\Gamma$ of $G$ and a commutative ring $R$, we put

\begin{equation}
\mathcal{H}_{R}(G, \Gamma) := R[\Gamma \backslash G / \Gamma].
\end{equation}

Put $Z_{(M)} := \bigcap_{\ell \mid M} Z_{(\ell)}$ (the intersection of the localizations $Z_{(\ell)}$ of $Z$ at $\ell Z$ for all primes $\ell \mid M$) and

\begin{equation}
\Delta_{0}^{g}(M) := \text{GSp}_{g}^{+}(\mathbb{Q}) \cap \text{GL}_{2g}(Z_{(M)}) \cap M_{2g}(Z).
\end{equation}

For a prime $\ell \nmid M$, we denote the integral Hecke algebra at $\ell$ over $\mathbb{Z}$ by

\begin{equation}
\mathcal{H}^{g}(\ell) := \mathcal{H}_{Z}((\Gamma_{0}^{g}(M) \cap \text{GL}_{2g}(Z_{(\ell)}), \chi_{0}(M))).
\end{equation}

Then $\mathcal{H}^{g}(M)$ is generated over $\mathbb{Z}$ by the following elements:

\begin{align}
T_{0}(\ell) &:= \Gamma_{0}^{g}(M)\text{diag}(1_{g}, \ell 1_{g})\Gamma_{0}^{g}(M), \\
T_{i}(\ell^{2}) &:= \Gamma_{0}^{g}(M)\text{diag}(1_{g-i}, \ell^{2}1_{g-i}, \ell^{2}1_{g-i}, \ell^{2}1_{g-i}, \ell_{1}^{2}1_{g-i}, \ell_{1}^{2}1_{g-i}, \ell_{1}^{2}1_{g-i}, \ell_{1}^{2}1_{g-i})\Gamma_{0}^{g}(M)
\end{align}

for $i = 1, 2, \ldots, g$ (see [QFHO, Theorem 3.3.23] and note $L_{p}^{n}(\ell) = \mathcal{H}_{p}^{n}(\ell) \otimes_{\mathbb{Z}} \mathbb{Q}$). We define

\begin{equation}
\mathcal{H}^{g}(M) := \otimes_{\ell}^{'} \mathcal{H}^{g}(\ell),
\end{equation}

where $\otimes_{\ell}^{'}$ is the restricted tensor product running over all primes $\ell$, i.e., $\mathcal{H}^{g}(M)$ is defined to be the $\mathbb{Z}$-algebra generated by $\mathcal{H}^{g}(\ell)$ for all primes $\ell$ over $\mathbb{Z}$. Note that $\mathcal{H}^{g}(M) = \mathcal{H}_{Z}((\Delta_{0}^{g}(M), \Gamma_{0}^{g}(M)))$ and
that $\mathcal{H}^g(M)$ is commutative ([QFHO, Theorem 3.3.7 and 3.3.12]). We let $T := \Gamma_0^g(M)\gamma \Gamma_0^g(M) = \bigcup_i \Gamma_0^g(i) \gamma_i \in \mathcal{H}^g(M)$ act on $F \in S_k^g(M, \chi)$ by
\begin{equation}
T(F)(Z) := \nu(\gamma)^{g(2k-g-1)/2} \sum_i \chi(\det(a_i)) F|_{k\gamma_i}(Z)
\end{equation}

Since $\nu(1_{2g}) = 1$ by the definition (2-1-2), the definition (2-2-7) requires that
\begin{equation}
T_g^g(\ell) \text{ acts as } \chi(\ell)^{g(k-g-1)/2} \text{ on } S_k^g(M, \chi)
\end{equation}
We refer to $F \in S_k^g(M, \chi)$ as a Hecke eigenform if $F$ is an eigenvector for any $T \in \mathcal{H}^g(M)$.

2.3. Hecke fields

For a Hecke eigenform $F \in S_k^g(M, \chi)$, we define $F \in \text{Hom}_{\mathbb{C}}(T(S_k^g(M, \chi)), \mathbb{C})$ by $\lambda_F(T)$ to be the eigenvalue of $T$ at $F$ and the Hecke field $\mathbb{Q}(F)$ of $F$ by
\begin{equation}
\mathbb{Q}(F) := \mathbb{Q}([[\lambda_F(T) \mid T \in \mathcal{H}^g(M)]])
\end{equation}
Moreover, we denote by $Z(F)$ the ring of integers of $\mathbb{Q}(F)$, $Z_{(p)}(F)$ the localization of $Z(F)$ at the prime above $p\mathbb{Z}$, $Z_p(F)$ the $p$-adic completion of $Z(F)$, and $\mathbb{Q}_p(F) := \text{Frac}(Z_p(F))$ the field of fractions. For a Dirichlet character $\psi$, we denote by $\mathbb{Q}(F, \psi)$ the field obtained by adjoining the values of $\psi$ to $\mathbb{Q}(F)$, $\mathbb{Z}(F, \psi)$ the ring of integers of $\mathbb{Q}(F, \psi)$, $\mathbb{Z}_{(p)}(F, \psi)$ the localization of $\mathbb{Z}(F, \psi)$ at the prime above $p\mathbb{Z}$, $\mathbb{Z}_p(F, \psi)$ the $p$-adic completion of $\mathbb{Z}(F, \psi)$, and $\mathbb{Q}_p(F, \psi) := \text{Frac}(Z_p(F, \psi))$ the field of fractions.

2.4. Petersson inner products

For $F, G \in S_k^g(M, \chi)$, the normalized Peterson inner product of $F$ and $G$ is defined by
\begin{equation}
\langle F, G \rangle := [Sp_g(\mathbb{Z}) : \Gamma_0^g(M)]^{-1} \int_{\Gamma_0^g(M) \backslash \mathfrak{H}_g} F(Z) \overline{G(Z)} \det(Y)^{k-(g+1)} dZ,
\end{equation}
where $Z = X + \sqrt{-1}Y = (x_{\alpha\beta}) + \sqrt{-1}(y_{\alpha\beta})$,
\begin{equation}
dZ := \prod_{1 \leq \alpha \leq \beta \leq g} dx_{\alpha\beta} y_{\alpha\beta},
\end{equation}
and $(\det Y)^{-(g+1)}dZ$ is a $Sp_g(\mathbb{R})$-invariant measure ([QFHO, Proposition 1.2.9]). By [QFHO, Theorem 2.5.3], $\langle F, G \rangle$ is absolutely convergent, independent of choice of the subgroup $\Gamma_0^g(N)$ with $F, G \in S_k^g(N, \chi)$ and a positive definite Hermitian form.

2.5. Notation and terminology for genus 1

We often omit $g$ from any notation when $g = 1$ for simplicity. We denote by $S_k^{\text{new}}(M, \varepsilon)$ the orthogonal complement of the subspace of old forms of level $M$ in $S_k(M, \varepsilon)$ with respect to the Petersson inner product. We refer to a Hecke eigenform in $S_k^{\text{new}}(M, \varepsilon)$ as a primitive form of level $M$ if $T^n(f) = a_n(f)f$ for all positive integers $n$, where $a_n(f)$ is the $n$-th Fourier coefficient of $f$. We denote by $S_k(M, \varepsilon)_{\alpha}$ the subspace of $S_k(M, \varepsilon)$ spanned by the generalized eigenspaces for eigenvalues $\lambda$ of $T_p$ with $\text{val}_p(\lambda) = \alpha$. Let $\mathbb{Z}[\varepsilon]$ be the ring generated by the values of $\varepsilon$ over $\mathbb{Z}$. For a $\mathbb{Z}[\varepsilon]$-algebra $R$ and $q := \exp(2\pi \sqrt{-1}z)$, we put
\begin{equation}
S_k(M, \varepsilon; R) := (S_k(M, \varepsilon) \cap \mathbb{Z}[\varepsilon][[q]]) \otimes_{\mathbb{Z}[\varepsilon]} R,
\end{equation}
\begin{equation}
S_k^{\text{new}}(M, \varepsilon; R) := (S_k^{\text{new}}(M, \varepsilon) \cap S_k(M, \varepsilon; \mathbb{Z}[\varepsilon])_{\alpha}) \otimes_{\mathbb{Z}[\varepsilon]} R.
For \( f \in S_k(M, \varepsilon) \) and a Dirichlet character \( \psi \), we denote by \( f \otimes \psi \in S_k(L, \varepsilon \psi^2) \) the \( \psi \)-twist of \( f \) defined by \( a_{n}(f \otimes \psi) := \psi_0(n)a_{n}(f) \) for all \( n \geq 1 \), where \( L \) is the least common multiple of \( M, c_\psi^2 \), and \( c_\psi c_\varepsilon \) ([MFM, Lemma 4.3.10.(2)]). We put \( L(s, f) := \sum_{n=1}^{\infty} a_{n}(f)n^{-s} \).

2.6. Notation for genus 2

Let \( F \in S^2_k(M, \chi) \) be a Hecke eigenform such that \( T^2(\ell)F = \lambda_F(\ell)F \) for any prime \( \ell \) and \( T^2_1(\ell^2)F = \lambda_F(\ell^2)F \) for each prime \( \ell \mid M \), where \( T^2(\ell) \) is defined as (2-2-4) even when \( \ell \mid M \). Recall that we have always \( T^2_1(\ell^2)F = \chi(\ell^2)\ell^{2(k-3)}F \) by (2-2-8). The spinor \( L \)-function \( L(s, F, \text{spin}) \) attached to \( F \) is defined by

\[
L(s, F, \text{spin}) := \prod_{\ell|M}(1-\lambda_F(\ell)\ell^{-s})^{-1}\prod_{l(M}Q_{\ell}(\ell^{-s})^{-1}
\]

with

\[
Q_{\ell}(X) := 1 - \lambda_F(\ell)X + (\ell\lambda_F(\ell^2) + \chi(\ell^2)(\ell^2 + 1)\ell^{2k-5})X^2 - \chi(p^2)\lambda_F(\ell)\ell^{2k-3}X^3 + \chi(\ell^4)\ell^{4k-6}X^4.
\]

3. \( D \)-th Saito-Kurokawa lifts

Let \( k \geq 2 \) be an integer, \( M \) an odd positive integer and \( \chi \) a Dirichlet character modulo \( M \).

3.1. Kohnen plus spaces

Put \( \tilde{\chi} := \chi\epsilon \chi \) with \( \epsilon := \chi(-1) \). We denote the Kohnen plus space by

\[
S^+_{k-1/2}(M, \chi_\epsilon) := \{ g \in S_{2k+1}^{\text{Sh}}(4M, \overline{\chi}) | a_{n}(g) = 0 \text{ if } \chi(-1)(-1)^{k-1}n \equiv 2, 3 \pmod{4} \},
\]

where \( S_{2k+1}^{\text{Sh}}(4M, \overline{\chi}) \) is the space of cusp forms of half-integral weight \( k-1/2 \) with level \( 4M \) and a character \( \overline{\chi} \) modulo \( 4M \) in the sense of Shimura [Shi73, p. 447]. For \( g \in S^+_{k-1/2}(M, \chi) \) and each prime \( \ell \), the Hecke operator \( T^+(\ell) \) is defined by

\[
a_{n}(T^+(\ell)g) = a_{\ell^2n}(g) + \tilde{\chi}\chi_{(-1)^{k-\ell}}(\ell)a_{n}(g) + \chi(\ell^2)\ell^{2k-3}a_{n/\ell^2}(g)
\]

for any positive integer \( n \) with \( \chi(-1)(-1)^{k-1}n \equiv 0, 1 \pmod{4} \). For \( g, h \in S^+_{k-1/2}(M, \chi) \), we define the Petersson inner product by

\[
\langle g, h \rangle_{4M} := \int_{\Gamma_0(4M)\backslash \mathfrak{H}} g(z)\overline{h(z)}y^{k-5/2}dxdy (z = x + \sqrt{-1}y),
\]

\[
\{g, h\} := [\text{SL}_2(\mathbb{Z}) : \Gamma_0(4M)]^{-1}\langle g, h \rangle_{4M}.
\]

3.2. \( D \)-th Shintani lifts \( \theta_D \)

Let \( D \) be a fundamental discriminant with \( \chi(-1)(-1)^{k-1}D > 0 \) and \( (D, c_\chi) = 1 \). We define the \( D \)-th Shimura lift \( \text{Sh}_D \) by

\[
\text{Sh}_D(g) := \sum_{n \geq 1} \left( \sum_{d | n} \chi_D(\chi(d)d^{k-2}a_{n^2|D|d^2}(g)) \right) q^n
\]

(see [KT04, (3-1)]).

Remark 3.1. Although [KT04] assumes \( (D, M) = 1 \), we see that the results below do not require this assumption except for Theorem 3.3. Note that \( \text{Sh}_D = 0 \) if \( (D, c_\chi) \neq 1 \) by [KT04, (3-2)].
Assume that
\[ k \geq 3, \quad M \text{ is square-free, or } M \text{ is cubic-free and } \chi = 1. \]
Then the image of the \( D \)-th Shimura lift \( \text{Sh}_D \) is contained in the space of cusp forms ([Koh85, p.241, 1.4-9]). Now we define the \( D \)-th Shintani lift \( \theta_D : S_{2k-2}(M, \chi^2) \to S_{k-1/2}^+(M, \chi) \) as the adjoint map of \( \text{Sh}_D \) with respect to the Petersson inner products, i.e., the map satisfying
\[
[g, \theta_D(f)] = (\text{Sh}_D(g), f)
\]
for every \( g \in S_{k-1/2}^+(M, \chi) \) and \( f \in S_{2k-2}(M, \chi^2) \). Since the \( D \)-th Shimura lift \( \text{Sh}_D \) is Hecke equivariant in the sense that \( T^1(\ell) \circ \text{Sh}_D = \text{Sh}_D \circ T^+(\ell) \) for all primes \( \ell \) by [KT04, Theorem 3.1] and the Hecke operators are Hermitian operators with respect to the Petersson inner products, we have the following:

**Theorem 3.2.** Let \( k \geq 2 \) be an integer, \( M \) an odd positive integer, \( \chi \) a Dirichlet character modulo \( M \) and \( D \) a fundamental discriminant with \( \chi(-1)(-1)^{k-1}D > 0 \) and \((D, c_\chi) = 1\). Assume (3-2-2). Then the \( D \)-th Shintani lift \( \theta_D \) is a \( \mathbb{C} \)-homomorphism from \( S_{2k-2}(M, \chi^2) \) into \( S_{k-1/2}^+(M, \chi) \) and Hecke equivariant in the sense that \( T^+(\ell) \circ \theta_D = \theta_D \circ T^1(\ell) \) for all primes \( \ell \).

Suppose that \( c_\chi \| M \). Let \( \ell \) be a prime factor of \( M/c_\chi \). We put \( v_\ell := \nu c(M/c_\chi) = \nu c(M) \). Let \( \gamma_\ell \) be an element in \( \text{SL}_2(\mathbb{Z}) \) such that
\[
\gamma_\ell = \begin{cases} J_1 & \pmod{\ell^{2v_\ell}}, \\ 1_2 & \pmod{(M/\ell^{v})^2}. \end{cases}
\]
We put \( \eta_\ell := \gamma_\ell \cdot \text{diag}(\ell^{v_\ell}, 1) \) (see [MFM, (4.6.21)]). We define the eigenvalue of \( f \) for the Atkin-Lehner involution \( \eta_\ell \) by
\[
\varepsilon_\ell(f) := \chi_D^2(\ell^{v_\ell})a_{1}(f|_{2k-2}\eta_\ell).
\]
If \( v_\ell = 1 \), then we have \( a_1(f|_{2k-2}\eta_\ell) = -\chi_0^2(\ell)\ell^{-k+2}a_\ell(f) \) by [MFM, Corollary 4.6.18.(2)] and hence
\[
\varepsilon_\ell(f) = -\ell^{-k+2}a_\ell(f) \in \{\pm 1\}
\]
by [MFM, Theorem 4.6.17.(2)].

**Theorem 3.3** ([KT04, (4-19, 20, 21, and 22)]). Let \( f \in S_{2k-2}^\text{new}(M, \chi^2) \) be a primitive form. Suppose that \( c_\chi \| M \) and \((D, M) = 1\). We put
\[
R_D(f) := \prod_\ell \left( 1 + \chi_D \chi(\ell^{v_\ell})\varepsilon_\ell(f) \left( \frac{1 - \chi_D \chi_0^{-1}(\ell)\ell^{-k+1}a_\ell(f)}{1 - \chi_D \chi_0(\ell)\ell^{-k+1}a_\ell(f)^c} \right) \right),
\]
where \( \prod_\ell \) is taken over all prime factors \( \ell \) of \( M/c_\chi \) and \( a_\ell(f)^c \) is the complex conjugate of \( a_\ell(f) \). Then
\[
a_{|D|}(\theta_D(f)) = R_D(f) |D|^{k-3/2}c_\chi^{2k-3}a_{2k-3-\chi^{-1}M-1}L(k + 1, f \otimes \chi_D^{-1}).
\]

**Remark 3.4.** Let the notation and the assumption be the same as the theorem above. If \( \chi^2 = 1 \) and \( M/c_\chi \) is square-free, then \( R_D(f) \in \{0, 2^{\nu(M/c_\chi)}\} \) by [Shi72, Proposition 1.3] and (3-2-6), where \( \nu(M/c_\chi) \) is the number of distinct prime factors of \( M/c_\chi \). In particular, if \( \chi = 1 \), then the following conditions are equivalent:

1. \( R_D(f) \neq 0 \).
2. \( R_D(f) = 2^{\nu(M)} \).
3. \( \chi_D(\ell) = \varepsilon_\ell(f) \) for any prime divisor \( \ell \) of \( M \).
In this case, the formula (3-2-8) is nothing but the result of Kohnen in [Koh85] and the sign of the functional equation of \( L(s, f \otimes \chi_D) \) is \((-1)^{k-1} \chi_D(-1)\), i.e., if \((-1)^{k-1} \chi_D(-1) = -1\), then \( L(k-1, f \otimes \chi_D) = 0 \).

3.3. D-th Saito-Kurokawa lifts \( SK_D \)

Let \( k \geq 2 \) be an even integer, \( M \geq 1 \) an odd integer, \( \chi \) a Dirichlet character modulo \( M \) with \( \chi(-1) = 1 \) and \( D \) a fundamental discriminant with \( D < 0 \) and \( (D, c_\chi) = 1 \). Assume (3-2-2). We then define the D-th Saito-Kurokawa lift \( SK_D \) by composing \( \theta_D \) with the Eichler-Zagier map \( EZ \) and the Maass lift \( L \) (see [Mak, Section 4 and 5] for definition of \( EZ \) and \( L \), respectively):

\[
SK_D : S_{2k-2}(M, \chi^2) \rightarrow^\theta_D S_{k-1/2}^+(M, \chi) \rightarrow^{EZ} J_{k,1}^{\text{cusp}}(M, \chi) \mapsto L^\chi S_k^2(M, \chi).
\]

Let \( f \in S_{2k-2}(M, \chi^2) \) be any element. Since \( EZ \) is an isomorphism ([Mak, Theorem 4.2]) and \( L \) is an injective homomorphism ([Mak, Theorem 5.1]), we see that

\[
SK_D(f) \neq 0 \text{ if and only if } \theta_D(f) \neq 0 \quad \text{(see [Mak, Theorem 3.6] for the non-vanishing criterion for } \theta_D(f)\text{).}
\]

For any prime \( \ell \),

\[
T^2(\ell) \circ SK_D = SK_D \circ(T^1(\ell) + \chi(\ell)(\ell^{k-2} + \ell^{k-1})),
\]

\[
(\ell T^{2k-5}(\ell^2) + \chi(\ell)^2(\ell^2 + 1)\ell^{2k-5}) \circ SK_D = SK_D \circ \left(\chi(\ell)(\ell^{k-1} + \ell^{k-2})T^1(\ell) + 2\chi(\ell)^2\ell^{2k-3}\right).
\]

Namely, we have the following:

**Theorem 3.5.** Let \( k \geq 2 \) be an even integer, \( M \geq 1 \) an odd integer, \( \chi \) a Dirichlet character modulo \( M \) with \( \chi(-1) = 1 \) and \( D \) a fundamental discriminant with \( D < 0 \) and \( (D, c_\chi) = 1 \). Assume (3-2-2). Let \( f \in S_{2k-2}^{\text{new}}(M, \chi^2) \) be a primitive form. If \( \theta_D(f) \neq 0 \), then \( SK_D(f) \in S_k^2(M, \chi) \) is a Hecke eigenform satisfying

\[
L(s, SK_D(f), \text{spin}) = L(s-k+1, \chi)L(s-k+2, \chi)L(s, f).
\]

**Remark 3.6.** It is known that the image of \( SK_D \) is characterized by the generalized Maass relation (see [Hei17]).

4. \( p \)-adic families

Let \( K \) be a complete discretely valued subfield of \( \mathbb{C}_p \). The weight space \( W \) attached to \( \mathcal{O}_K[Z_p^\times] \) is the rigid analytic variety whose \( \mathbb{C}_p \)-valued points are given by

\[
\text{Hom}_{\mathbb{C}^\times}^{\text{cont}}(Z_p^\times, \mathbb{C}_p) \cong \text{Hom}_{\mathcal{O}_K-\text{alg}}^{\text{cont}}(\mathcal{O}_K[Z_p^\times], \mathbb{C}_p).
\]

For a \( K \)-Banach algebra \( R \) and an \( R \)-valued point \( k \in W(R) \), we will use a notation \( t^k \) instead of \( k(t) \) for \( t \in Z_p^\times \). For a \( K \)-rigid analytic variety \( X \), we denote by \( A(X) \) the ring of rigid analytic functions on \( X \) and \( A^0(X) \) the subring consisting of elements that are power bounded with respect to the supremum semi-norm \( | \cdot | \) (see [BGR, Definition 6.2.1/2]). By [BGR, Proposition 6.2.3/1], we have \( A^0(X) = \{ f \in A(X) \mid |f| \leq 1 \} \). We denote by \( B_K[a, r] \) the affinoid closed disk over \( K \) of radius \( r \in |K| \) about \( a \in \mathcal{O}_K \) whose \( \mathbb{C}_p \)-valued points are given by

\[
B_K[a, r] \{x \in \mathcal{O}_K \mid |x - a|_p < r\}.
\]
4.1. Coleman families

Let \( f \in S_{w}^{\text{new}}(N, \varepsilon)_{\alpha} \) be a primitive form. Assume that the characteristic polynomial

\[
X^{2} - a_{p}(f)X + \varepsilon(p)p^{w-1} \in \mathbb{Z}(f)[X]
\]

of \( T^{1}(p) \) on the subspace spanned by \( f \) and \( f|V_{p} \) has no double roots, i.e., \( a_{p}(f)^{2} \neq \varepsilon(p)p^{w-1} \). Let \( \alpha_{p}(f) \) be the root of this polynomial satisfying \( \text{val}_{p}(\alpha_{p}(f)) = \alpha \). We refer to

\[
f^{*}(z) := f(z) - \varepsilon(p)p^{w-1}a_{p}(f)^{-1}f(pz)
\]

as the \( p \)-stabilization of \( f \). The \( p \)-stabilization \( f^{*} \) is the Hecke eigenform of level \( Np \) with the same eigenvalues as \( f \) outside \( p \) and \( T^{1}(p) \)-eigenvalue \( a_{p}(f^{*}) = \alpha_{p}(f) \).

Theorem 4.1 ([Col97]). Let \( f \in S_{w_{0}}^{\text{new}}(N, \varepsilon)_{\alpha} \) be a primitive form with \( w_{0} > \alpha + 1 \) and \( K \) a complete discretely valued subfield of \( \mathbb{C}_{p} \) containing the \( p \)-adic completion of the Hecke field \( \mathbb{Q}(f^{*}) \). Assume \( a_{p}(f)^{2} \neq \varepsilon(p)p^{w_{0}-1} \). Then there exists a positive integer \( M \) and a formal power series

\[
f = \sum_{n=1}^{\infty} a_{n}(f)q^{n} \in A^{o}(B_{K}[w_{0}, p^{-M}])([q])
\]

such that for any \( w \) in

\[
W(M) := \{ w \in \mathbb{Z} \mid w \equiv w_{0} \ (\text{mod} \ (p-1)p^{M}), w > \alpha + 1 \}
\]

except for at most one, the specialization \( f(w) \) at \( w \) given by

\[
f(w) := \sum_{n=1}^{\infty} a_{n}(f)(w)q^{n}
\]

is the \( p \)-stabilization of some primitive form in \( S_{w}^{\text{new}}(N, \varepsilon; \mathcal{O}_{K})_{\alpha} \) and \( f(w_{0}) = f^{*} \). More precisely, there exists a primitive form \( f_{w} \in S_{w}^{\text{new}}(N, \varepsilon; \mathcal{O}_{K})_{\alpha} \) satisfying the following conditions:

1. \( f(w) = f_{w}^{*} \).
2. \( f_{w_{0}} = f \).
3. \( f(w_{1}) \in S_{w_{1}}^{\text{new}}(Np, \varepsilon)_{\alpha} \) is primitive if there exists an exceptional weight \( w_{1} \in W(M) \).

In particular, for any positive integer \( m \) and \( w \in W(M) \), if \( w \equiv w_{0} \ (\text{mod} \ (p-1)p^{M+m}) \), then

\[
a_{n}(f_{w}) \equiv a_{n}(f) \ (\text{mod} \ p^{m}\mathcal{O}_{K})
\]

We refer to the family \( \{ f_{w} \}_{w \in W(M)} \) of primitive forms obtained in the theorem above as a Coleman family passing through \( f \) over \( K \). By the theorem above, for any positive integer \( m \) and \( w \in W(M) \), if \( w \equiv w_{0} \ (\text{mod} \ (p-1)p^{M+m}) \), then for any positive integer \( n \) with \( p \nmid n \),

\[
a_{n}(f_{w}) \equiv a_{n}(f) \ (\text{mod} \ p^{m}\mathcal{O}_{K})
\]

4.2. \( p \)-adic families of the \( D \)-th Saito-Kurokawa lifts

By Theorem 4.1 and Theorem 3.5, we immediately see that the family \( \{ F_{w} := \text{SK}_{D}(f_{w}) \}_{w} \) forms a \( p \)-adic family in the sense that the \( T \)-eigenvalue \( \lambda_{F_{w}}(T) \) gives a \( p \)-adic analytic function \( W \mapsto \lambda_{F_{w}}(T) \) from \( W(M) \) into \( K \) as follows:

Corollary 4.2. Let \( k_{0} \geq 2 \) be an even integer, \( \chi \) a Dirichlet character modulo \( N \) with \( \chi(-1) = 1 \) and \( D \) a fundamental discriminant with \( D < 0 \) and \( (D, c_{\chi}) = 1 \). Assume (3-2-2) for \( k = k_{0} \) and \( M = Np \). Let \( f \in S_{w_{0}}^{\text{new}}(N, \chi^{2})_{\alpha} \) be a primitive form with \( w_{0} := 2k_{0} - 2 > \alpha + 1 \) and \( K \) a complete discretely valued subfield of \( \mathbb{C}_{p} \) containing the \( p \)-adic completion of the Hecke field \( \mathbb{Q}(f^{*}) \). Assume
\[ a_{p}(f)^{2} \neq \chi^{2}(p)p^{w_{0}-1} \]. Let \( \{ f_{w} \}_{w \in W(M)} \) be a Coleman family passing through \( f \) over \( K \). Then for any \( w \) in
\[ W^{SK}(M) := \{ 2k-2 \in W(M) \mid k \in \mathbb{Z} \}, \]
if \( w \equiv w_{0} \pmod{(p-1)p^{M+m}} \), then for any \( T \in \mathcal{H}^{2}(Np) \),
\[ \lambda_{F_{w}^{*}}(T) \equiv \lambda_{F_{w_{0}}^{*}}(T) \pmod{p^{m}\mathcal{O}_{K}} \],
where \( F_{w}^{*} := SK_{D}(f_{w}^{*}) \). In particular, for any positive integer \( n \) with \( p \nmid n \), we have
\[ \lambda_{F_{w}^{*}}(T^{2}(n)) \equiv \lambda_{F_{w_{0}}^{*}}(T^{2}(n)) \pmod{p^{m}\mathcal{O}_{K}} \].

Remark 4.3. It is possible that \( \{ SK_{D}(f_{w}^{*}) \}_{w \in W^{SK}(M)} \) and \( \{ SK_{D}(f_{w}) \}_{w \in W^{SK}(M)} \) vanish identically. However, it does follow from \( \theta_{D}(f) \neq 0 \) that \( SK_{D}(f_{w}^{*}) \neq 0 \) and \( SK_{D}(f_{w}) \neq 0 \) for any \( w \in W^{SK}(M) \) with sufficiently large \( M \) by \( p \)-adic interpolation of Fourier coefficients.

5. Cohomological interpretation

5.1. Modular symbols and elliptic cusp forms

Let \( \Delta_{0} \) be a subsemigroup of \( M_{2}(\mathbb{Z}) \cap GL_{2}(\mathbb{Q}) \) containing \( \Gamma_{0}(M) \). Let \( \text{Div}^{0}(\mathbb{P}^{1}(\mathbb{Q})) \) be the group of divisors of degree 0 supported on the rational cusps \( \mathbb{P}^{1}(\mathbb{Q}) = \mathbb{Q} \cup \{ i\infty \} \) of the complex upper half plane \( \mathfrak{H} \). We let \( \Delta_{0} \) act on \( \mathfrak{H} \) by fractional linear transformations, i.e.,
\[ \gamma z := \begin{cases} (az+b)(cz+d)^{-1} & \text{if } \det(\gamma) > 0, \\ (a\bar{z}+b)(c\bar{z}+d)^{-1} & \text{if } \det(\gamma) < 0, \end{cases} \quad (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \in \mathfrak{H}). \]
This induces a natural action of \( \Delta_{0} \) on \( \mathfrak{H}^{*} := \mathfrak{H} \cup \mathbb{P}^{1}(\mathbb{Q}) \) and \( \mathbb{P}^{1}(\mathbb{Q}) \). Then \( \Delta_{0} \) acts on \( \text{Div}^{0}(\mathbb{P}^{1}(\mathbb{Q})) \) by linear fractional transformations. Let \( R \) be a commutative ring and \( E \) a left \( R[\Delta_{0}] \)-module. We let \( \gamma \in \Delta_{0} \) acts on \( \Phi \in \text{Hom}_{\mathbb{Z}}(\text{Div}^{0}(\mathbb{P}^{1}(\mathbb{Q})), E) \) by
\[ (\Phi|\gamma)(D) := \gamma \Phi(\gamma D). \]
Then the abstract Hecke algebra \( \mathcal{H}_{R}(\Delta_{0}, \Gamma_{0}(M)) \) acts on the group of \( E \)-valued modular symbols over \( \Gamma_{0}(M) \):
\[ \text{Symbr}_{\Gamma_{0}(M)}(E) := \text{Hom}_{\mathbb{Z}}(\text{Div}^{0}(\mathbb{P}^{1}(\mathbb{Q})), E)^{\Gamma_{0}(M)}. \]
Let \( \tilde{E} \) be the locally constant sheaf on the open modular curve \( Y := \Gamma_{0}(M) \backslash \mathfrak{H} \) attached to \( E \). Assume that
\[ \text{the orders of the torsion elements of } \Gamma_{0}(M) \text{ act invertibly on } E. \]
Then by [AS86, Proposition 4.2], there exists a Hecke equivariant canonical isomorphism
\[ H_{c}^{1}(Y, \tilde{E}) \cong \text{Symbr}_{\Gamma_{0}(M)}(E). \]
Throughout the paper, we will identify the group of compactly supported cohomology classes with the group of modular symbols under the assumption that (5-1-4). Note that (5-1-4) holds if either \( E \) is a vector space over a field of characteristic 0, \( E \) is a \( \mathbb{Z}_{p} \)-module with \( p \geq 5 \), or \( \Gamma_{0}(M) \) is torsion-free. The matrix \( \iota := \text{diag}(1, -1) \) induces the natural involution on \( \text{Symbr}_{\Gamma_{0}(M)}(E) \) and the decomposition
\[ \text{Symbr}_{\Gamma_{0}(M)}(E) = \text{Symbr}_{\Gamma_{0}(M)}^{+}(E) \oplus \text{Symbr}_{\Gamma_{0}(M)}^{-}(E) \]
if 2 acts invertibly on \( E \). Indeed, each element \( \Phi \) decomposes as \( \Phi = \Phi^{+} + \Phi^{-} \), where
\[ \Phi^{\pm} := 2^{-1}(\Phi \pm \Phi|\iota). \]
For a non-negative integer \(n\), let \(L(n; R)\) be the \(R\)-module of homogeneous polynomials in \((X, Y)\) of degree \(n\) with coefficients in \(R\). For an \(R\)-valued Dirichlet character \(\varepsilon\) modulo \(M\), we denote by \(L(n; \varepsilon; R)\) the \(R[\Gamma_0(M)]\)-module \(L(n; R)\) endowed with the following action; for \(\gamma \in \Gamma_0(M)\) and \(P \in L(n, \varepsilon; R)\),

\[
(\gamma P)(X, Y) = \varepsilon(d) P((X, Y)^t \gamma) \tag{5-1-8}
\]

For each cusp form \(f \in S_{w+2}(M, \varepsilon)\), we define the \(L(w, \varepsilon; \mathbb{C})\)-valued differential form on \(\mathfrak{H}\) by

\[
\omega_f := f(z) \left( \sum_{i=0}^{w} (-z)^i X^{w-i} Y^i \right) dz. \tag{5-1-9}
\]

The additive map

\[
\Phi_f : \text{Div}^0(\mathbb{P}^1(\mathbb{Q})) \to L(w, \varepsilon; \mathbb{C}) ; \{c_2\} - \{c_1\} \mapsto \int_{c_1}^{c_2} \omega_f \tag{5-1-10}
\]

defines a modular symbol in \(\text{Sym}_{\Gamma_0(M)}(L(w, \varepsilon; \mathbb{C}))\) and the map

\[
\Phi : S_{w+2}(M, \varepsilon) \to \text{Sym}_{\Gamma_0(M)}(L(w, \varepsilon; \mathbb{C})) ; f \mapsto \Phi_f \tag{5-1-11}
\]
is an injective \(\mathbb{C}[H^1(M)]\)-homomorphism.

### 5.2. Cohomological interpretation of \(\theta_D\)

**Proposition 5.1** ([Mak17, Proposition 3.2]). Let \(k \geq 2\) be an integer, \(M\) an odd positive integer, \(\chi\) a Dirichlet character modulo \(M\) and \(D\) a fundamental discriminant with \(\chi(-1)(-1)^{k-1}D > 0\) and \((D, c_\chi) = 1\). Assume (3-2-2). Then we can define a \(\mathbb{C}\)-homomorphism \(\Theta_D\) satisfying the commutative diagram

\[
\begin{array}{ccc}
\text{Sym}_{\Gamma_0(\mathbb{R})}(L(2k-4, \chi^2; \mathbb{C})) & \cong & \mathbb{C}[q] \\
\Phi & \cong & \Theta_D \downarrow \\
S_{2k-2}(M, \chi^2) & \cong & S_{k-1/2}^+(M, \chi^2). \\
\end{array} \tag{5-2-1}
\]

### 5.3. Algebraic lits \(\theta_D^{alg}\) and \(SK_D^{alg}\)

Let \(f \in S_{w+2}^{new}(N, \varepsilon)\) be a primitive form. Let \(V_f\) and \(V_f^*\) be the representation spaces of Galois representations \(\rho_f\) and \(\rho_{f^*}\) attached to \(f\) and \(f^*\) over \(\mathbb{Q}_p(f)\) and \(\mathbb{Q}_p(f^*)\), respectively. By the Comparison Theorem between étale and Betti induced by a fixed isomorphism \(\overline{\mathbb{Q}}_p \cong \mathbb{C}\), we have

\[
\begin{align*}
V_f \otimes_{\mathbb{Q}_p(f)} \hat{\mathbb{Q}}_p & \cong \text{Sym}_{\mathbb{C}}(f) := \bigcap_{\ell} \text{Ker} \left( (T(\ell) - a_\ell(f)) | \text{Sym}_{\Gamma_0(N)}(L(w, \varepsilon; \mathbb{C})) \right) ; \\
V_f^* \otimes_{\mathbb{Q}_p(f^*)} \hat{\mathbb{Q}}_p & \cong \text{Sym}_{\mathbb{C}}(f^*) := \bigcap_{\ell} \text{Ker} \left( (T(\ell) - a_\ell(f^*)) | \text{Sym}_{\Gamma_0(Np)}(L(w, \varepsilon; \mathbb{C})) \right) .
\end{align*} \tag{5-3-1, 5-3-2}
\]

Since the left-hand sides of both (5-3-1) and (5-3-2) are isomorphic by the Brauer-Nesbitt Theorem and the Chebotarev Density Theorem, we have

\[
\text{Sym}_{\mathbb{C}}(f) \cong \text{Sym}_{\mathbb{C}}(f^*). \tag{5-3-3}
\]

Symb(\mathbb{C})[f] \cong Symb(\mathbb{C})[f^*].
By [Kit94, Proposition 3.3], these are free of rank one over $\mathbb{C}$, and hence we may assume that \( \Phi_f \mapsto \Phi_{f^*} \) gives the isomorphism (5-3-3). By [Kit94, Proposition 3.3], the eigenmodules
\begin{align*}
(5-3-4) & \quad \text{Symb}(\mathbb{Z}_p(f))[f] := \text{Symb}(\mathbb{C})[f] \cap \text{Symb}_{\Gamma_0(N)}(L(w, \varepsilon; \mathbb{Z}_p(f))), \\
(5-3-5) & \quad \text{Symb}(\mathbb{Z}_p(f^*))[f^*] := \text{Symb}(\mathbb{C})[f^*] \cap \text{Symb}_{\Gamma_0(Np)}(L(w, \varepsilon; \mathbb{Z}_p(f^*))
\end{align*}
are free of rank one over $\mathbb{Z}_p(f)$ and $\mathbb{Z}_p(f^*)$, respectively. Let $\Phi_f$ be a generator of $\text{Symb}(\mathbb{Z}_p(f))[f]$, which is contained in
\begin{align*}
(5-3-6) & \quad \text{Symb}(\mathbb{Z}_p(f^*))[f] := \text{Symb}(\mathbb{C})[f] \cap \text{Symb}_{\Gamma_0(N)}(L(w, \varepsilon; \mathbb{Z}_p(f^*)).
\end{align*}
Then there exists $\Omega(f) \in \mathbb{C}^\times$ such that
\begin{align*}
(5-3-7) & \quad \Phi_f^\circ = \Omega(f)^{-1} \cdot \Phi_f^{-} \in \text{Symb}(\mathbb{Z}_p(f))[f].
\end{align*}
Then the isomorphism (5-3-3) implies
\begin{align*}
(5-3-8) & \quad \Omega(f)^{-1} \cdot \Phi_{f^*} \in \text{Symb}(\mathbb{Z}_p(f^*))[f^*].
\end{align*}

**Theorem 5.2.** Let $k \geq 2$ be an integer, $\chi$ a Dirichlet character modulo $N$ and $D$ a fundamental discriminant with $\chi(-1)(-1)^k D > 0$ and $(D, c_{\chi}p) = p$. Assume (3-2-2). Let $f \in S_{2k-2}^{new}(N, \chi^2)$ be a primitive form. Then
\begin{align*}
(5-3-9) & \quad (c_D(k, \chi)\Omega(f))^{-1} \theta_D(f) \in S_{k-1/2}^{+}(N, \chi; \mathbb{Z}_p(f, \chi)), \\
(5-3-10) & \quad (c_D(k, \chi)\Omega(f))^{-1} \theta_D(f^*) \in S_{k-1/2}^{+}(Np, \chi; \mathbb{Z}_p(f^*, \chi)).
\end{align*}

**Remark 5.3.** Note that the values of $\chi^2$ are contained in $\mathbb{Q}(f)$ for a primitive form $f \in S_{2k-2}^{new}(N, \chi^2)$ but the values of $\chi$ are not necessarily.

We fix, once and for all, the complex period $\Omega(f)$ as (5-3-7) and define
\begin{align*}
(5-3-11) & \quad \theta_D^{alg}(f) := \Omega(f)^{-1} \theta_D(f), \\
(5-3-12) & \quad \theta_D^{alg}(f^*) := \Omega(f)^{-1} \theta_D(f^*), \\
(5-3-13) & \quad SK_D^{alg}(f) := L(\text{EZ}(\theta_D^{alg}(f))) = \Omega(f)^{-1} SK_D(f), \\
(5-3-14) & \quad SK_D^{alg}(f^*) := L(\text{EZ}(\theta_D^{alg}(f^*))) = \Omega(f)^{-1} SK_D(f^*).
\end{align*}
Note that $c_D(k, \chi)^{-1} SK_D^{alg}(f) \in \mathbb{Z}_p(f, \chi)[[q]]$ and $c_D(k, \chi)^{-1} SK_D^{alg}(f^*) \in \mathbb{Z}_p(f^*, \chi)[[q]]$ by (3-3-3).

For a Dirichlet character $\psi$ and $j \in [1, k-1] \cap \mathbb{Z}$ with $\psi(-1)(-1)^{j-1} = -1$, we have
\begin{align*}
(5-3-15) & \quad L^{alg}(j, f \otimes \psi) := \frac{G(\psi^{-1})\Gamma(j)\Lambda(j, f \otimes \psi)}{(-2\pi\sqrt{-1})^j \Omega(f)} \in \mathbb{Z}_p(f, \psi)
\end{align*}
by [Kit94, Lemma 4.1]. If $c_{\chi} \parallel N$ and $(D, N) = 1$, then
\begin{align*}
(5-3-16) & \quad a_{|D|}(\theta_D^{alg}(f)) = (-1)^{k-1} c_D(k, \chi)(Dc_{\chi})^{k-2} R_D(f) L^{alg}(k-1, f \otimes \chi_D^{-1}).
\end{align*}
By inner product formula obtained in [Mak], we have the following:

**Corollary 5.4.** Let $k \geq 2$ be an even integer, $M \geq 1$ a square-free odd integer, $\chi$ a Dirichlet character modulo $M$ with $\chi^2 = 1$ and $\chi(-1) = 1$ and $D$ a fundamental discriminant with $D < 0$ and $(D, M) = 1$. Let $f \in S_{2k-2}^{new}(M, 1)$ be a primitive form. Then
\begin{align*}
(5-3-17) & \quad \frac{\|SK_D^{alg}(f)\|^2}{\|f\|^2} = C_D(k, M, \chi)L^{alg}(k, f) L^{alg}(k-1, f \otimes \chi_DX^{-1}).
\end{align*}
where \( \| SK_D^{alg}(f) \|^2 := \langle SK_D^{alg}(f), SK_D^{alg}(f) \rangle, \| f \|^2 := \langle f, f \rangle \) and
\[
C_D(k, M, \chi) := \frac{(-2\sqrt{-1})^{2k-1}R_D(f)|D|^{k-3/2}M^2c_{\chi}^{2k-3}\text{res}_{s=1}L(s, \chi)}{2^{3}3G(\chi_D\chi)}
\]
In particular, if \( \chi = 1 \), then
\[
\frac{\| SK_D^{alg}(f) \|^2}{\| f \|^2} \in \mathbb{Q}(f)
\]
Remark 5.5. Since \( \text{res}_{s=1}L(s, \chi) \in \overline{\mathbb{Q}} \) if and only if \( \chi = 1 \) by [MFM, Theorem 3.3.4], we see that
\[
\frac{\| SK_D^{alg}(f) \|^2}{\| f \|^2} \in \overline{\mathbb{Q}} \text{ if and only if } \chi = 1.
\]

6. \( p \)-adic interpolation of Fourier coefficients

In this section, we first present \( p \)-adic interpolation of \( \{a_n(\theta_D(f^*_w))\}_{w \in W^{SK}(M)} \). Using this result, we establish \( p \)-adic interpolation of \( \{a_T(\theta_D(f^*_w))\}_{w \in W^{SK}(\lambda l)} \) for \( T \in \mathcal{L}_{>0} \) following Guerzhoy’s method in [Gue00] which is discussed in [Kaw] as well.

6.1. \( p \)-adic interpolation of \( \theta_D^{alg}(f^*) \)

**Theorem 6.1** ([Mak17, Theorem 5.7]). Let \( k_0 \geq 2 \) be an integer, \( \chi \) a Dirichlet character with \( c_{\chi} | N \) and \( D \) a fundamental discriminant with \( \chi(-1)(-1)^{k_0-1}D > 0 \) and \( (D, c_{\chi}p) = p \). Assume (3-2-2) for \( k = k_0 \) and \( M = Np \). Let \( f \in S_{w_0}^{new}(N, \chi)^{2} \) be a primitive form with \( w_0 := 2k_0 - 2 > \alpha + 1 \) and \( K \) a complete discretely valued subfield of \( \mathbb{C}_p \) containing the \( p \)-adic completion of the field obtained by adjoining \( c_D(k_0, \chi) \) to \( \mathbb{Q}(f^*, \chi) \). Assume \( a_{p^0}(f)^2 \neq \chi^2(p)p^{wo^0-1} \) and \( \{f_w\}_{w \in W(M)} \) be a Coleman family passing through \( f \) over \( K \). Then for sufficiently large \( M \), we can define \( \theta_D(f) \in A(K_{w_0}, \mathbb{Q}^{-M})[\mathbb{Q}] \) such that for any \( w \in W^{SK}(M) \), there exists \( e_w \in K^x \) independent of \( D \) satisfying
\[
(6-1-1) \quad \theta_D(f)(w) = e_w \theta_D^{alg}(f^*_w)
\]
and \( e_{w_0} = 1 \).

6.2. \( p \)-adic interpolation of \( \theta_D^{alg}(f) \)

**Theorem 6.2**. Let \( k_0 \geq 2 \) be an integer, \( \chi \) a Dirichlet character with \( c_{\chi} | N \) and \( D_0 \) a fundamental discriminant with \( \chi(-1)(-1)^{k_0-1}D_0 > 0 \) and \( (D_0, c_{\chi}p) = p \). Assume (3-2-2) for \( k = k_0 \) and \( M = Np \). Let \( f \in S_{w_0}^{new}(N, \chi)^{2} \) be a primitive form with \( w_0 := 2k_0 - 2 > \alpha + 1 \) and \( K \) a complete discretely valued subfield of \( \mathbb{C}_p \) containing the \( p \)-adic completion of the field obtained by adjoining \( c_{D_0,k_0}(\chi) \) to \( \mathbb{Q}(f^*, \chi) \). Assume \( a_{p^0}(f)^2 \neq \chi^2(p)p^{wo^0-1} \) and \( \{f_w\}_{w \in W(M)} \) be a Coleman family passing through \( f \) over \( K \). Let \( D \) be a fundamental discriminant with \( \chi(-1)(-1)^{k_0-1}D > 0 \) and \( (D, c_{\chi}) = 1 \). When both \( p | D \) and \( \chi_{D_0D/p^{2}}(p) = -1 \) holds, we further assume \( \chi^2 = 1, a_{|D|}(\theta_D(f)) \neq 0 \), and the following condition:
\[
(6-2-1) \quad \bigcap_{\ell \mid N} \text{Ker} \left( (T^+(\ell) - a_{\ell}(f)) \mid S_{k-1/2}(N, \chi) \right) \cong \mathbb{C},
\]
where \( \ell \mid N \) runs over all primes \( \ell \mid N \). Then for sufficiently large \( M \), we can define \( a_{|D|}(\theta_{D_0}(F^*)) \in A(B_K[w_0, p^{-M}]) \) such that for any \( w = 2k - 2 \in W^{SK}(M) \), there exists \( e_w \in K^x \) satisfying
\[
(6-2-2) \quad a_{|D|}(\theta_{D_0}(F^*))(w) = e_w \left( 1 - \chi_Dx_0^{-1}(p)p^{k-2}a_{p}(f^*_w)^{-1} \right) a_{|D|} \left( \theta_{D_0}^{alg}(f^*_w) \right)
\]
and $e_{w_0} = 1$. In particular, for any positive integer $m$ and $w \in W(M)$, if $w \equiv w_0 \pmod{(p-1)p^M+m}$ and $m > \text{val}_p \left( (1-\chi_D \chi_0^{-1}(p)p^{k_0-2}a_p(f^{*-1})a_{|D|}(\theta_{D_0}(f^{*}))) \right)$, then

\begin{align*}
\text{val}_p \left( e_w \left( 1-\chi_D \chi_0^{-1}(p)p^{k-2}a_p(f^{*-1})a_{|D|}(\theta_{D_0}(f^{*})) \right) \right) &= \text{val}_p \left( (1-\chi_D \chi_0^{-1}(p)p^{k_0-2}a_p(f^{*-1})a_{|D|}(\theta_{D_0}(f^{*}))) \right).
\end{align*}

**Remark 6.3.** Let the notation be the same as above.

(1) By (6-2-3), we see that $\theta_D(f_w) \neq 0$ if $\theta_D(f) \neq 0$.

(2) If $N$ is square-free, then the condition (6-2-1) holds by [Koh82, Theorem 2 ii)].

Combining the theorem above with Theorem 3.3 gives the following:

**Corollary 6.4.** Let $k_0 \geq 2$ be an integer and $\chi$ a Dirichlet character with $c_\chi \parallel N$. Assume (3-2-2) for $k = k_0$ and $M = Np$. Let $f \in S_{w_0}^{new}(N, \chi^2)_0$ be a primitive form with $w_0 := 2k_0 - 2 > \alpha + 1$ and $K$ a complete discretely valued subfield of $\mathbb{C}_p$ containing the $p$-adic completion of the field obtained by adjoining $c_{D_0}(k_0, \chi)$ to $\mathbb{Q}(f^{*}, \chi)$. Assume $a_p(f)^2 = \chi^2(p)p^{w_0-1}$. Let $\{f_w\}_{w \in W(M)}$ be a Coleman family passing through $f$ over $K$. Then for sufficiently large $M$, we can define $SK_{D}^{alg}(f) \in A(\mathcal{B}_K[w_0, p^{-M}])[q]_2$ such that for any $w \in W^{SK}(M)$, there exists $e_w \in K^\times$ satisfying

\begin{align*}
SK_{D}^{alg}(f)(w) = e_w SK_{D}^{alg}(f_w)
\end{align*}

and $e_{w_0} = 1$.

**Remark 6.5.**

\begin{align*}
\frac{G(\chi_D \chi)L(k-1,f_w \otimes \chi_{D_0}\chi^{-1})}{G(\chi_{D}\chi)L(k-1,f_w \otimes \chi_D\chi^{-1})} = \frac{L^{alg}(k-1,f_w \otimes \chi_{D_0}\chi^{-1})}{L^{alg}(k-1,f_w \otimes \chi_{D}\chi^{-1})} \in \mathbb{Q}(f_w, \chi)
\end{align*}

6.4. $p$-adic interpolation of $SK_{D}^{alg}(f^{*})$

**Theorem 6.6.** Let $k_0 \geq 2$ be an even integer, $\chi$ a Dirichlet character modulo $N$ and $D_0 < 0$ a fundamental discriminant with $(D_0, c_\chi) = 1$ and $D \equiv 1 \pmod{4}$. Let $f \in S_{w_0}^{new}(N, \chi^2)_0$ be a primitive form with $w_0 := 2k_0 - 2 > \alpha + 1$ and $K$ a complete discretely valued subfield of $\mathbb{C}_p$ containing the $p$-adic completion of the field obtained by adjoining $c_{D_0}(k_0, \chi)$ to $\mathbb{Q}(f^{*}, \chi)$. Assume $a_p(f)^2 = \chi^2(p)p^{w_0-1}$. Let $\{f_w\}_{w \in W(M)}$ be a Coleman family passing through $f$ over $K$. Then for sufficiently large $M$, we can define $SK_{D}^{alg}(f) \in A(\mathcal{B}_K[w_0, p^{-M}])[q]_2$ such that for any $w \in W^{SK}(M)$, there exists $e_w \in K^\times$ satisfying

\begin{align*}
SK_{D}^{alg}(f)(w) = e_w SK_{D}^{alg}(f_w)
\end{align*}

and $e_{w_0} = 1$.

**Theorem 6.7.** Let $k_0 \geq 2$ be an even integer, $\chi$ a Dirichlet character modulo $N$ and $D_0 < 0$ a fundamental discriminant with $(D_0, c_\chi) = 1$ and $D \equiv 1 \pmod{4}$. Let $f \in S_{w_0}^{new}(N, \chi^2)_0$ be a primitive form with $w_0 := 2k_0 - 2 > \alpha + 1$ and $K$ a complete discretely valued subfield of $\mathbb{C}_p$ containing the $p$-adic completion of the field obtained by adjoining $c_{D_0}(k_0, \chi)$ to $\mathbb{Q}(f^{*}, \chi)$. Assume $a_p(f)^2 = \chi^2(p)p^{w_0-1}$. Let $\{f_w\}_{w \in W(M)}$ be a Coleman family passing through $f$ over $K$. Then for sufficiently large $M$, we can define $SK_{D}^{alg}(f) \in A(\mathcal{B}_K[w_0, p^{-M}])[q]_2$ such that for any $w \in W^{SK}(M)$, there exists $e_w \in K^\times$ satisfying

\begin{align*}
SK_{D}^{alg}(f)(w) = e_w SK_{D}^{alg}(f_w)
\end{align*}

and $e_{w_0} = 1$. When both $p \mid D$ and $\chi_{D_0}\chi(p)p^{w_0-1}$ is a fundamental discriminant with $(D, c_\chi) = 1$ and $D \equiv 1 \pmod{4}$. When both $p \mid D$ and $\chi_{D_0}\chi(p)p^{w_0-1}$ is a fundamental discriminant with $(D, c_\chi) = 1$ and $D \equiv 1 \pmod{4}$. When both $p \mid D$ and $\chi_{D_0}\chi(p)p^{w_0-1}$ is a fundamental discriminant with $(D, c_\chi) = 1$ and $D \equiv 1 \pmod{4}$. When both $p \mid D$ and $\chi_{D_0}\chi(p)p^{w_0-1}$ is a fundamental discriminant with $(D, c_\chi) = 1$ and $D \equiv 1 \pmod{4}$.
\[ \chi^2 = 1, \quad a_{D_0}(\theta_{D_0}(f)) \neq 0, \text{ and (6-2-1).} \] Then for sufficiently large \( M \), we can define an element \( a_T(\text{SK}_{D_0}(f^*)) \in A(B_K[w_0, p^{-M}]) \) such that for any \( w = 2k - 2 \in W^{SK}(M) \), there exists \( e_w \in K^\times \) satisfying

\[
(6-4-1) \quad a_T(\text{SK}_{D_0}(f^*)) (w) = e_w \left( 1 - \chi_D \chi_0^{-1}(p)p^{k-2}a_p(f_w^*)^{-1} \right) a_T(\text{SK}^{\text{alg}}_{D_0}(f_w))
\]

and \( e_{w_0} = 1 \).

References


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