ON LINEAR RELATIONS BETWEEN L-VALUES AND ARITHMETIC FUNCTIONS

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1. RESULTS FROM SIEGEL

Let \( F = \mathbb{Q}(\sqrt{D}) \) be a quadratic field with discriminant \( D > 0 \), ring of integers \( \mathfrak{o} \) and different \( \mathfrak{d} \). For integer \( k \geq 2 \) and \( B \) an ideal class of \( F \), we define the Eisenstein series

\[
E_{k,B}(z) = \frac{1}{4} \zeta_{\mathfrak{d}B}(1-k) + \sum_{\xi \in \mathfrak{d}^{-1}, \xi \succ 0} \sigma_{k-1,\mathfrak{d}B}(\xi \mathfrak{d}) q^{\xi}
\]

of weight \( k \) and level \( SL_{2}(\mathfrak{o}) \), where \( z \in \mathfrak{h}^2 \) and \( q^{\xi} = \exp(2\pi \sqrt{-1} Tr(\xi z)). \)

If we let

\[
\mathcal{R}E_{k,B}(\tau) = E_{k,B}(\tau, \tau)
\]

for \( \tau \in \mathfrak{h} \), then \( \mathcal{R}E_{k,B} \in \mathbb{J}_{2k}(SL_{2}(\mathbb{Z})) \). It is easy to see that

\[
\mathcal{R}E_{k,B}(\tau) = \frac{1}{4} \zeta_{\mathfrak{d}B}(1-k) + \sum_{n=1}^{\infty} \left( \sum_{\xi \in \mathfrak{d}^{-1}} \sigma_{k-1,\mathfrak{d}B}(\xi \mathfrak{d}) \right) q^{n},
\]

where \( q^{n} = \exp(2\pi \sqrt{-1} n \tau) \). Thus by comparing the Fourier coefficients, we have the following results.

**Theorem 1** (Siegel). If \( k = 2, 4 \), then

\[
\zeta_{\mathfrak{d}}(1-k) = -\frac{B_{2k}}{k} \sum_{\xi \in \mathfrak{d}^{-1}, \xi \succ 0, \text{Tr}(\xi) = 1} \sigma_{k-1,\mathfrak{d}}(\xi \mathfrak{d}).
\]

**Corollary 1.** We have

\[
\zeta_{F}(-1) = \frac{1}{60} m^{2} < D \sum_{m \in \mathbb{Z}} \sigma_{1}(\frac{D-m^{2}}{4}) m^{2} \equiv D \mod 4
\]
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\[ \zeta_F(-3) = \frac{1}{120} \sum_{\substack{m \in \mathbb{Z} \\ m^2 \leq D \mod 4}} \sigma_5 \left( \frac{D - m^2}{4} \right). \]

Today, we want to use the same technique on the Hilbert forms of half-integral weight.

2. MAIN RESULTS

Consider the same \( F \) as in the last section. Put

\[ \omega = \begin{cases} \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \mod 4, \\ \frac{\sqrt{D}}{2} & \text{otherwise}. \end{cases} \]

Then \( \mathfrak{o} = \mathbb{Z} + \omega \mathbb{Z} \). For any two ideals \( b \) and \( c \) of \( F \) such that \( bc \subset \mathfrak{o} \), we let

\[ \Gamma [b, c] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(F) \mid a, d \in \mathfrak{o}, b \in b, c \in c \right\} \]

and put \( \Gamma = \Gamma [\mathfrak{o}^{-1}, 4\mathfrak{o}] \).

**Definition 1.** For any \( z \in \mathfrak{h}^2 \), we put

\[ \theta(z) = \sum_{\xi \in \mathfrak{o}} q^{\xi^2} \quad (z \in \mathfrak{h}^2) \]

The factor of automorphy of weight 1/2 is given by

\[ j(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)} \]

where \( \gamma \in \Gamma \).

Let \( k \geq 0 \). A Hilbert modular form of parallel weight \( k + 1/2 \) and level \( \Gamma \) is a holomorphic function \( f \) on \( \mathfrak{h}^2 \) which satisfies

\[ f(\gamma z) = j(\gamma, z)^{2k+1} f(z) \]

for any \( \gamma \in \Gamma \). The space of all such forms is denoted by \( M_{k+1/2}(\Gamma) \) and the subspace consisting cusp forms in it is denoted by \( S_{k+1/2}(\Gamma) \).

Now suggest that the different \( \mathfrak{d} \) has a totally positive generator \( \delta \). Put \( \Gamma' = \Gamma [\mathfrak{o}, 4\mathfrak{o}] \). A Hilbert modular form of parallel weight \( k + 1/2 \) and level \( \Gamma' \) is a function with the form

\[ f_0(z) = f(\delta^{-1} z) \]

where \( f \in M_{k+1/2}(\Gamma) \). It satisfies the automorphic condition

\[ f_0(\gamma z) = j_0(\gamma, z)^{2k+1} f_0(z) \]
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where \( \gamma \in \Gamma' \) and \( \tilde{j}_0 \) comes from \( \theta_0 \) as for \( \tilde{j} \). The space of all such forms and cusp forms are denoted by \( M_{k+1/2}(\Gamma') \) and \( S_{k+1/2}(\Gamma') \), respectively.

Note that the congruence subgroup \( \Gamma_0(4) \subset SL_2(\mathbb{Z}) \) can be embedded into \( \Gamma' \) diagonally. One can show that if \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \), we have

\[
\tilde{j}_0(\gamma, (\tau, \tau)) = (c\tau + d)\chi_{-4}(d)
\]

where \( \tau \in \mathfrak{h} \). Thus if we put

\[
\mathcal{R}f(\tau) = f_0((\tau, \tau))
\]

where \( f \in M_{k+1/2}(\Gamma) \) and \( \tau \in \mathfrak{h} \), then

\[
\mathcal{R}f \in M_{2k+1}(\Gamma_0(4), \chi_{-4}).
\]

We write \( \delta \) in the form

\[
\delta = (\alpha + \beta \omega)\sqrt{D}
\]

with \( \alpha + \beta \omega > 0 \) a unit of norm \(-1\). Then if \( f \) has the Fourier expansion

\[
f(z) = \sum_{\xi \in \mathfrak{o}} c(\xi) q^\xi \quad (z \in \mathfrak{h}^2),
\]

we can write down the Fourier expansion of \( \mathcal{R}f \) explicitly as

\[
\mathcal{R}f(\tau) = \sum_{n=0}^{\infty} \left( \sum_{(a,b) \in \mathbb{Z}^2} c(a + b\omega) \right) q^n \quad (\tau \in \mathfrak{h}).
\]

If we apply the mapping \( \mathcal{R} \) on \( \theta^2 \), we get the following result.

**Corollary 2.** For \( m > 0 \) and number field \( K \) we set

\[
r_{K,m}(x) = \# \left\{ (x_1, x_2, \ldots, x_m) \in \mathfrak{o}_K^m \mid x_1^2 + \cdots + x_m^2 = x \right\} \quad (x \in K).
\]

Then with the notations given above, we have

\[
\sum_{(a,b) \in \mathbb{Z}^2} r_{F,m}(a + b\omega) = r_{\mathbb{Q},2m}(n).
\]

From now let us consider the case for Kohnen plus space. The specific spaces were first defined by Kohnen [3] in 1980 and generalized to the case for Hilbert modular forms by Hiraga and Ikeda [2] in 2013. For any \( \xi \in F \), we denote by

\[
\xi \equiv \square \mod 4
\]

if there exists \( \lambda \in \mathfrak{o} \) such that \( \xi - \lambda^2 \in 4\mathfrak{o} \).
Definition 2. The Kohnen plus space is a subspace of $M_{k+1/2}(\Gamma)$ defined as

$$M_{k+1/2}^+(\Gamma) = \left\{ f(z) = \sum_{\xi} c(\xi) q^\xi \in M_{k+1/2}(\Gamma) \mid c(\xi) = 0 \text{ unless } (-1)^k \xi \equiv \square \mod 4 \right\}. $$

Also we put $S_{k+1/2}^+(\Gamma) = M_{k+1/2}^+(\Gamma) \cap S_{k+1/2}(\Gamma)$. Their images in $M_{k+1/2}(\Gamma')$ under the isomorphism $f \mapsto f_0$ are denoted by $M_{k+1/2}^+(\Gamma')$ and $S_{k+1/2}^+(\Gamma')$, respectively.

We also define the plus spaces contained in $M_{2k+1}(\Gamma_0(4), \chi_{-4})$ and $S_{2k+1}(\Gamma_0(4), \chi_{-4})$.

Definition 3. We put

$$M_{2k+1}^+(\Gamma_0(4), \chi_{-4}) = \left\{ h(z) = \sum_{n=0}^\infty d(n) q^n \in M_{2k+1}(\Gamma_0(4), \chi_{-4}) \mid d(n) = 0 \text{ if } \chi_{-4}(n) = (-1)^{k+1} \right\}$$

and $S_{2k+1}^+(\Gamma_0(4), \chi_{-4}) = M_{2k+1}^+(\Gamma_0(4), \chi_{-4}) \cap S_{2k+1}(\Gamma_0(4), \chi_{-4})$.

For $k > 0$ being odd, the space $S_{2k+1}^+(\Gamma_0(4), \chi_{-4})$ was defined by Kojima [4] in 1982 and shown to be isomorphic to $M_{2k+2}(\Gamma^2(\mathcal{O}))$, the space of Hermitian modular forms of weight $2k+2$ and degree 2. The plus spaces of odd weights can be described exactly in the sense taking certain linear combinations of the normalized Hecke eigenforms of the whole space as a basis.

Theorem 2. Let $k \geq 0$. We have

$$\mathcal{R}(M_{k+1/2}^+(\Gamma)) \subset M_{2k+1}^+(\Gamma_0(4), \chi_{-4})$$

and

$$\mathcal{R}(S_{k+1/2}^+(\Gamma)) \subset S_{2k+1}^+(\Gamma_0(4), \chi_{-4}).$$

This theorem can be proved in an elementary way, but can also be proved in a representation theoretical way, which reflects the nature of both plus spaces of half-integral and integral weights more.

As an application, we apply $\mathcal{R}$ on the Eisenstein series in $M_{k+1/2}^+(\Gamma)$, which was introduced in [5]. Let $\chi$ be a character of the ideal class group $Cl(F)$ of $F$. The Eisenstein series in $M_{k+1/2}^+(\Gamma)$ with respect to
\( \chi \) is given by

\[
E_{k+1/2, \chi}(z) = L_F(1 - 2k, \overline{\chi}^2) + \sum_{(-1)^k \xi \equiv \square \mod 4 \atop \xi > 0} \mathcal{H}_k(\xi, \chi) q^\xi 
\]

where

\[
\mathcal{H}_k(\xi, \chi) = \chi(D_{-1^k \xi}) L_F(1 - k, \chi_{(-1)^k \xi} \chi) \\
\times \sum_{\mu_F(a) \chi_{(-1)^k \xi}(a) N_{F/Q}(a) = k-1 \sigma_{F,2k-1, \chi^2}(f_{(-1)^k \xi} a^{-1})}
\]

Here \( D_x \) and \( \chi_x \) are the relative discriminant and the quadratic character associated to \( F(\sqrt{x})/F \), respectively, and \( f_x \) is the integral ideal such that \( f_x^2 D_x = (x) \). Further more, \( \mu_F \) is the Möbius function with respect to \( F \) and

\[
\sigma_{F,m, \chi'}(b) = \sum_{r \mid b} N_{F/Q}(r)^m \chi'(r).
\]

The Eisenstein series given above is a generalization of the one given by Cohen in [1], whose Fourier coefficients are called generalized Hurwitz class numbers.

The two Eisenstein series in \( M_{2k+1}(\Gamma_0(4), \chi_{-4}) \) are given by

\[
E_{2k+1, \chi_{-4}}(\tau) = 1 + \frac{2}{L(-2k, \chi_{-4})} \sum_{n=1}^\infty \sigma_{2k, \chi_{-4}}(n) q^n \\
\]

and

\[
F_{2k+1, \chi_{-4}}(\tau) = \frac{(-1)^k 2}{L(-2k, \chi_{-4})} \sum_{n=1}^\infty \sigma_{2k, \chi_{-4}}'(n) q^n 
\]

where

\[
\sigma_{2k, \chi_{-4}}'(n) = \sum_{r \mid n} r^{2k} \chi_{-4}(n/r).
\]

The series \( F_{2k+1, \chi_{-4}} \) is the normalized image of \( E_{2k+1, \chi_{-4}} \) under the Fricke involution. By comparison of the constant term at the cusps of \( \Gamma_0(4) \), we have the following result.

**Theorem 3.** For \( k \geq 0 \), we have

\[
\mathcal{R}G_{k+1/2, \chi} - L_F(1 - 2k, \overline{\chi}^2)(E_{2k+1, \chi_{-4}} + (-1)^k F_{2k+1, \chi_{-4}}) \in S_{2k+1}(\Gamma_0(4), \chi_{-4}).
\]

In particular, since \( S_3(\Gamma_0(4), \chi_{-4}) = 0 \), we have

\[
\sum_{(a,b) \in \mathbb{Z}^2 \atop a\beta - b\alpha = n} \mathcal{H}_1(a + b\omega, \chi) = -4L_F(1 - 2k, \overline{\chi}^2)(\sigma_{2, \chi_{-4}}(n) - \sigma_{2, \chi_{-4}}'(n)).
\]
For example, let \( F = \mathbb{Q}(\sqrt{5}) \) and \( \delta = \omega \sqrt{5} \) where \( \omega = (1 + \sqrt{5})/2 \).

Then by applying the theorem on \( E_{3/2,1} \) and \( E_{5/2,1} \) we may get

\[
L_F(0, \chi_{-2-\omega}) = -\frac{2}{15} (\sigma_{2,\chi-4}(2) - \sigma'_{2,\chi-4}(2)),
\]

\[
2L_F(0, \chi_{-3}) + 2L_F(0, \chi_{-3+\omega}) = -\frac{2}{15} (\sigma_{2,\chi-4}(3) - \sigma'_{2,\chi-4}(3)),
\]

\[
2L_F(0, \chi_{-4}) = -\frac{2}{15} (\sigma_{2,\chi-4}(4) - \sigma'_{2,\chi-4}(4)),
\]

\[
2L_F(0, \chi_{-3}) + 2L_F(0, \chi_{-6-\omega}) = -\frac{2}{15} (\sigma_{2,\chi-4}(6) - \sigma'_{2,\chi-4}(6)),
\]

\[
2\zeta_F(-1) = \frac{1}{75} (\sigma_{4,\chi-4}(1) + \sigma'_{4,\chi-4}(1) + 3s(1)),
\]

\[
4\zeta_F(-1) + 2L_F(-1, \chi_{5+\omega}) = \frac{1}{75} (\sigma_{4,\chi-4}(5) + \sigma'_{4,\chi-4}(5) + 3s(5)),
\]

and so on. Here

\[
s(n) = \frac{1}{4} \sum_{a^2 + b^2 = n} (a + b\sqrt{-1})^4.
\]

3. REPRESENTATION THEORETIC VIEW OF THE THEOREM

Let \( \psi_0 = \prod_{0,v \leq \infty} \psi_{0,v} \) be the additive character of \( \mathbb{A}_F/F \) such that for \( v | \infty \) it satisfies

\[
\psi_{0,v}(x) = \exp(2\pi \sqrt{-1}(-1)^k x) \quad (x \in \mathbb{R}).
\]

Put \( \psi = \psi_0(\delta^{-1}) \). We denote the metaplectic double covering of \( SL_2 \) by \( Mp_2 \), which is with respect to the Kubota 2-cocycle. There exists an irreducible representation \( \Omega_{\psi} \) of \( \prod_{v | 2} Mp_2(\mathfrak{o}_v) \) associated to \( \psi \). This representation is a subquotient of the restricted Weil representation associated to \( \psi \) and is of 4-dimension. Note that a modular form of weight \( k + 1/2 \) can be lifted to an automorphic form on \( SL_2(F) \backslash Mp_2(\mathbb{A}_F) \). Hiraga and Ikeda [?] showed the following theorem

**Theorem 4** (Hiraga, Ikeda). *Let \( f_0 \in M_{k+1/2}(\Gamma') \) and \( W_4f_0 \) be its image under the fricke involution with respect to \( z \mapsto -\frac{1}{4z} \). A sufficient necessary condition for \( f_0 \) to be in the plus space is*

\[
< \rho(\gamma)W_4f_0 \mid \gamma \in \prod_{v | 2} Mp_2(\mathfrak{o}_v) > \cong \Omega_{\psi}.
\]
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Regardless of the choice of $F$, $\text{SL}_2(\mathbb{Z}_2)$ can be embedded into $\prod_{v|2} \text{Mp}_2(\mathfrak{o}_v)$ and the restriction of $\Omega_\psi$ to $\text{SL}_2(\mathbb{Z}_2)$ remains the same. One can show

$$\left. \Omega_\psi \right|_{\text{SL}_2(\mathbb{Z}_2)} = \pi_k \oplus \sigma_k$$

where $\pi_k$ and $\sigma_k$ are irreducible representations of dimension 3 and 1, respectively. They only depend on the parity of $k$. The following theorem was shown by Kojima [4] (for odd $k$, but the proof can be easily extended to general positive integer $k$).

**Theorem 5** (Kojima). Let $h \in M_{2k+1}(\Gamma_0(4), \chi_{-4})$. A sufficient necessary condition for $h$ to be in the plus space is

$$< \rho(\gamma)W_4 h | \gamma \in \text{SL}_2(\mathbb{Z}_2) > \cong \pi_k.$$  

By Theorem 4, it is easy to see that if $f \in M_{k+1/2}^+(\Gamma)$ then $W_4 \mathcal{R} f$ generates a irreducible representation of $\text{SL}_2(\mathbb{Z}_2)$ equivalent to $\pi_k$. Thus we get $\mathcal{R} f \in M_{2k+1}^+(\Gamma_0(4), \chi_{-4})$.

**References**


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