NEWFORMS OF HALF-INTEGRAL WEIGHT

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This report is a summary of a joint work with Prof. Moshe Baruch on the space of newforms of half-integral weight.

Let M be odd and square-free and k be a positive integer. In a remarkable work, Niwa [8] comparing the traces of Hecke operators proved existence of Hecke isomorphism between $S_{k+1/2}(\Gamma_0(4M))$, the space of holomorphic cusp forms of weight $k+1/2$ on the congruence subgroup $\Gamma_0(4M)$ and $S_{2k}(\Gamma_0(2M))$, the space of weight $2k$ cusp forms on $\Gamma_0(2M)$. In [5, 6] Kohnen considers a certain Hecke operator on $S_{k+1/2}(\Gamma_0(4M))$ which is an analogue of Niwa's operator at level 4. This operator has two eigenvalues, one positive and one negative and the Kohnen plus space is the eigenspace of the positive eigenvalue. Kohnen considers a new space $S^{+,\mathrm{new}}_{k+1/2}(4M)$ inside his plus space and proves that this new subspace is Hecke isomorphic to $S^{\mathrm{new}}_{2k}(\Gamma_0(M))$, the space of newforms of weight $2k$ and level M, giving first instance of Atkin-Lehner type newform theory for half-integral weight forms. Ueda [11] generalises Niwa's trace computations and generalizes Kohnen's newform theory to the case M odd. Further Ueda proves existence of isomorphism between $S_{k+1/2}(\Gamma_0(8M))$ and $S_{2k}(\Gamma_0(4M))$ and with Yamana [13] defines the plus space $S^{+}_{k+1/2}(8M)$ to consist of $f=\sum_{n=1}^{\infty} a_n q^n \in S_{k+1/2}(\Gamma_0(8M))$ such that $a_n=0$ for $(-1)^k n \equiv 2,3 \pmod 4$ and gave newform theory inside this plus space. In particular they proved isomorphism between $S^{+,\mathrm{new}}_{k+1/2}(8M)$ and $S^{\mathrm{new}}_{2k}(\Gamma_0(2M))$.

In the case $M=1$, Loke and Savin [7] gave an interpretation of the Kohnen plus space in representation theory language using a 2-adic Hecke algebra of level 4. We extend the approach of Loke and Savin and study genuine Hecke algebras of the Kubota double cover of $\mathrm{SL}_2(\mathbb{Q}_p)$ modulo certain compact subgroups S and genuine central characters γ of S. This allow us to obtain certain pairs of conjugate classical operators on the space $S_{k+1/2}(\Gamma_0(2^\nu M))$ for $\nu=2,3$. We consider the common -1-eigenspace of these pairs of conjugate operators for each prime dividing the level and denote it by $S^{-}_{k+1/2}(2^\nu M)$. We prove that this common eigenspace space $S^{-}_{k+1/2}(2^\nu M)$ is hecke isomorphic to $S^{\mathrm{new}}_{2k}(\Gamma_0(2^{\nu-1}M))$ and satisfies multiplicity one in the full space $S_{k+1/2}(2^\nu M)$.

Further if $f=\sum_{n=1}^{\infty} a_n q^n$ is in the minus space at level $8M$ then $a_n=0$ for $(-1)^k n \equiv 0,1 \pmod 4$. We do not expect such Fourier coefficient condition for minus space at level $4M$.

1. Preliminaries and Notation

Let k, N denote positive integers with 4 dividing N. Let $\mathcal{G}$ be the set of all ordered pairs $(\alpha,\phi(z))$ where $\alpha=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in \mathrm{GL}_2(\mathbb{R})^+$ and $\phi(z)$ is a holomorphic function on the upper half plane $\mathbb{H}$ such that $\phi(z)^2 = t\det(\alpha)^{-1/2}(cz+d)$ with t in the unit circle S^1. For $\zeta=(\alpha,\phi(z))\in\mathcal{G}$ define the slash operator $|[\zeta]_{k+1/2}$ on functions f on $\mathbb{H}$ by $f|[\zeta]_{k+1/2}(z) = f(\alpha z)(\phi(z))^{-2k-1}$. For an even Dirichlet character χ modulo N, let

$$\Delta_0(N,\chi) := \{\alpha^* = (\alpha, j(\alpha,z)) \in \mathcal{G} \mid \alpha \in \Gamma_0(N)\} \leq \mathcal{G}.$$

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where $j(\alpha, z) = \chi(d)\varepsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{1/2}$, here $\varepsilon_d = 1$ or i according as $d \equiv 1$ or 3 (mod 4) and $\left(\frac{c}{d}\right)$ is as in Shimura's notation [10]. The space of holomorphic cusp forms $S_{k+1/2}(\Gamma_0(N), \chi)$ satisfy $f|[\alpha^*]_{k+1/2}(z) = f(z)$ for all $\alpha \in \Delta_0(N, \chi)$. We have Hecke operators $\{T_{p^2}\}_{p\nmid N}$, $\{U_{p^2}\}_{p|N}$ on $S_{k+1/2}(\Gamma_0(N), \chi)$.

Let $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ be the non-trivial central extension of $\mathrm{SL}_2(\mathbb{Q}_p)$ by $\mu_2 = \{\pm 1\}$, given by Kuobta 2-cocycle defined as below. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$, define

$$\tau(g) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0 \end{cases};$$

if $p = \infty$, set $s_p(g) = 1$ while for a finite prime p

$$s_p(g) = \begin{cases} (c, d)_p & \text{if } cd \neq 0 \text{ and } \mathrm{ord}_p(c) \text{ is odd} \\ 1 & \text{else.} \end{cases}$$

Define the 2-cocycle σ_p on $\mathrm{SL}_2(\mathbb{Q}_p)$ as follows:

$$\sigma_p(g, h) = (\tau(gh)\tau(g), \tau(gh)\tau(h))_p\, s_p(g)s_p(h)s_p(gh).$$

Then the double cover $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ is the set $\mathrm{SL}_2(\mathbb{Q}_p) \times \mu_2$ with the group law:

$$(g,\ \epsilon_1)(h,\ \epsilon_2) = (gh,\ \epsilon_1\epsilon_2\sigma_p(g, h)).$$

For any subgroup H of $\mathrm{SL}_2(\mathbb{Q}_p)$, we shall denote by $\overline{H}$ the complete inverse image of H in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. We consider the following subgroups of $\mathrm{SL}_2(\mathbb{Z}_p)$:

$$K_0(p^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) \ :\ c \in p^n\mathbb{Z}_p \right\},$$

$$K_1(p^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) \ :\ c \in p^n\mathbb{Z}_p,\ a \equiv 1 \pmod{p^n\mathbb{Z}_p} \right\}.$$

By [4] for odd primes p, $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ splits over $\mathrm{SL}_2(\mathbb{Z}_p)$ and the center M_p of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ is direct product $\{\pm I\} \times \mu_2$. However $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ does not split over $\mathrm{SL}_2(\mathbb{Z}_2)$ but instead splits over subgroup $K_1(4)$. The center M_2 of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ is a cyclic group of order 4 generated by $(-I, 1)$.

For an open compact subgroup S of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ and a genuine character γ of S, let $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//S, \gamma)$ be the subalgebra of $C_c^\infty(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p))$ given by

$$\{f \in C_c^\infty(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)) : f(\tilde{k}\tilde{g}\tilde{k}') = \overline{\gamma}(\tilde{k})\overline{\gamma}(\tilde{k}')f(\tilde{g}) \text{ for } \tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p),\ \tilde{k},\ \tilde{k}' \in S\}$$

under the usual convolution which for any $f_1, f_2 \in C_c^\infty(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p))$, is defined by

$$f_1 * f_2(\tilde{h}) = \int_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)} f_1(\tilde{g})f_2(\tilde{g}^{-1}\tilde{h})d\tilde{g} = \int_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)} f_1(\tilde{h}\tilde{g})f_2(\tilde{g}^{-1})d\tilde{g},$$

where $d\tilde{g}$ is the Haar measure on $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ such that the measure of $\overline{K_0(p)}$ is one.

Loke and Savin [7] considered a genuine Hecke algebra for $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ corresponding to $\overline{K_0(4)}$ and a genuine central character. In the next section we shall study genuine Hecke algebras for $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ corresponding to $\overline{K_0(p)}$ and a given genuine character of M_p for general odd prime p. We will then consider the case of $\overline{K_0(8)}$. While doing so we will also compute the full integral weight Hecke algebra of $\mathrm{GL}_2(\mathbb{Q}_2)$ correponding to $H_0(4)$ (here

here $H_0(p^n)$ denotes the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ consisting of elements with $(2,1)$th entry in $p^n\mathbb{Z}_p$) and verify "local Shimura correspondence" in these cases.

We set up the following notation: For $s \in \mathbb{Q}_p$, $t \in \mathbb{Q}_p^\times$, consider the following elements of $\mathrm{SL}_2(\mathbb{Q}_p)$:

$$x(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},\ y(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},\ w(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix},\ h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

2. Local Hecke algebras - generators and relations

In this section, we will first consider genuine Iwahori Hecke algebras of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ for any odd prime p. We will then study the case of 2-adic Hecke algebra at the level 8 extending the work of Loke and Savin that deal with the Iwahori-type 2-adic Hecke algebra.

2.1. Iwahori Hecke Algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ modulo $\overline{K_0(p)}$, p odd.

Fix an odd prime p. Let $\tilde{\gamma}$ be a character of $K_0(p)$ such that it is trivial on $K_1(p)$. Since $\frac{K_0(p)}{K_1(p)} \cong (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ we can define $\tilde{\gamma}$ by a character of $(\mathbb{Z}/p\mathbb{Z})^\times$. Define genuine character γ on $\overline{K_0(p)}$ by defining $\gamma(A,\epsilon) = \epsilon\tilde{\gamma}(A)$ for $A \in K_0(p)$.

We say $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)},\gamma)$ "genuine " Iwahori Hecke algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ with respect to $\overline{K_0(p)}$ and central character γ. We want to describe it using generators and relations.

Lemma 2.1. *A complete set of representatives for the double cosets of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ mod $\overline{K_0(p)}$ are given by $(h(p^n),1)$, $(w(p^{-n}),1)$ where n varies over integers.*

We say that $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)},\gamma)$ is supported on $\tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ if there exists a $f \in H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p),\overline{K_0(p)},\gamma)$ such that $f(\tilde{g}) \neq 0$. We note that in general $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p),\overline{K_0(p)},\gamma)$ need not be supported on the double coset representatives $(h(p^n),1)$, $(w(p^{-n}),1)$. However we can prove the following

Proposition 2.2. *If γ is a quadratic character then $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)},\gamma)$ is supported on the double cosets of $\overline{K_0(p)}$ represented by $(h(p^n),1)$ and $(w(p^{-n}),1)$ where n varies over integers.*

For the rest of this section we shall assume γ to be quadratic.

2.2. Generators and relations.

Let $T = \{(h(t),\epsilon) : t \in \mathbb{Q}_p^\times,\ \epsilon = \pm 1\}$ and $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)}(T)$ be the normalizer of T in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. Note that $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)}(T)$ is generated by elements $(h(t),\epsilon)$, $(w(t),\epsilon)$ for $t \in \mathbb{Q}_p^\times$. We shall extend the character γ of $\overline{K_0(p)}$ to the normalizer group $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)}(T)$. We note the following relations:

$$(h(s),1)(h(t),1) = (h(st),(s,t)_p),\quad (w(s),1)(w(t),1) = (h(-st^{-1}),(s,t)_p),$$
$$(h(s),1)(w(t),1) = (w(st),(s,-t)_p),\quad (w(s),1)(h(t),1) = (w(st^{-1}),(-s,t)_p).$$

Let $\varepsilon_p = 1$ or $(-1)^{1/2}$ depending on whether $p \equiv 1$ or $3 \pmod 4$. Thus $\varepsilon_p^2 = \left(\frac{-1}{p}\right)$. For $t = p^n u \in \mathbb{Q}_p^\times$, define

$$\gamma(\overline{h(t)}) = \tilde{\gamma}(h(u)) \begin{cases} 1 & \text{if } n \text{ is even} \\ \varepsilon_p\left(\frac{u}{p}\right) & \text{if } n \text{ is odd.} \end{cases} \qquad \gamma(\overline{w(t)}) = \tilde{\gamma}(h(u)) \begin{cases} 1 & \text{if } n \text{ is even} \\ \varepsilon_p\left(\frac{-u}{p}\right) & \text{if } n \text{ is odd.} \end{cases}$$

Using the relations above one can check that γ indeed extends to a character of $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)}(T)$.

We define the following elements of $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)},\gamma)$ supported respectively on the double cosets of $(h(p^n),1)$ and $(w(p^{-n}),1)$: For $n\in\mathbb{Z}$, and $\tilde{k},\tilde{k}'\in\overline{K_0(p)}$,

$$X_{(h(p^n),1)}(\tilde{k}(h(p^n),1)\tilde{k}')=\overline{\gamma}(\tilde{k})\overline{\gamma}((h(p^n),1))\overline{\gamma}(\tilde{k}'),$$
$$X_{(w(p^{-n}),1)}(\tilde{k}(w(p^{-n}),1)\tilde{k}')=\overline{\gamma}(\tilde{k})\overline{\gamma}((w(p^{-n}),1))\overline{\gamma}(\tilde{k}').$$

Let $\mathcal{T}_n=X_{(h(p^n),1)}$ and $\mathcal{U}_n=X_{(w(p^{-n}),1)}$. Then Proposition 2.2 implies that $\mathcal{T}_n$ and $\mathcal{U}_n$ forms a $\mathbb{C}$-basis for $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)},\gamma)$.

Proposition 2.3. *We have following relations:*

(1) If $mn\geq 0$ then $\mathcal{T}_m*\mathcal{T}_n=\mathcal{T}_{m+n}$.
(2) For $n\geq 0$, $\mathcal{U}_1*\mathcal{T}_n=\mathcal{U}_{n+1}$.
(3) For $n\geq 0$, $\mathcal{T}_{-n}*\mathcal{U}_1=\mathcal{U}_{n+1}$.
(4) For $n\geq 0$, $\mathcal{U}_0*\mathcal{T}_{-n}=\mathcal{U}_{-n}$.
(5) For $n\geq 0$, $\mathcal{T}_n*\mathcal{U}_0=\mathcal{U}_{-n}$.
(6) For $n\geq 1$, $\mathcal{U}_0*\mathcal{U}_n=\overline{\gamma}(-1)\cdot\mathcal{T}_n$.
(7) For $n\geq 1$, $\mathcal{U}_n*\mathcal{U}_0=\overline{\gamma}(-1)\cdot\mathcal{T}_{-n}$.

For the proof of the above proposition we need explicit decomposition of double cosets into union of single cosets which can be obtained using triangular decomposition of $K_0(p)$. As a consequence we can obtain simple convolution formulae (as in [2]) in many cases.

We consider two choices for γ as a character of $(\mathbb{Z}/p\mathbb{Z})^*$, either γ is trivial or γ is given by the Kronecker symbol $\gamma=\left(\frac{\cdot}{p}\right)$. Then we have following proposition.

Proposition 2.4. *We have following relations.*

(1) $\mathcal{U}_0^2=\begin{cases}(p-1)\mathcal{U}_0+p & \text{if } \gamma \text{ is trivial}\\ \left(\frac{-1}{p}\right)p & \text{if } \gamma=\left(\frac{\cdot}{p}\right).\end{cases}$

(2) $\mathcal{U}_1^2=\begin{cases}p & \text{if } \gamma \text{ is trivial}\\ \varepsilon_p(p-1)U_1+\left(\frac{-1}{p}\right)p & \text{if } \gamma=\left(\frac{\cdot}{p}\right).\end{cases}$

(3) If γ is trivial, then $\mathcal{T}_1*\mathcal{U}_1=p\,\mathcal{U}_0$ and $\mathcal{T}_{-1}=1/p\cdot\mathcal{U}_1*\mathcal{T}_1*\mathcal{U}_1$.

We obtain the following theorem.

Theorem 1. *The "genuine" Iwahori Hecke algebra $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)},\gamma)$ for γ trivial or $\left(\frac{\cdot}{p}\right)$ is generated as an $\mathbb{C}$-algebra by $\mathcal{U}_0$ and $\mathcal{U}_1$ with the defining relations given by above proposition.*

Corollary 2.5. *We have the following isomorphisms of Hecke algebras:*

$$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)},1)\cong H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)},\left(\frac{\cdot}{p}\right))\cong \textit{Iwahori Hecke alg. of } \mathrm{PGL}_2(\mathbb{Q}_p).$$

2.3. **Hecke Algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ modulo $\overline{K_0(8)}$.** Loke and Savin [7] described genuine local Hecke algebra of $\widetilde{\mathrm{SL}}(\mathbb{Q}_2)$ with respect to $\overline{K_0(4)}$ and central character of M_2.

Let γ be a genuine character of $\overline{K_0(4)}=K_1(4)\times M_2$ determined by its value on $(-I,1)$ such that it is trivial on $K_1(4)$. This character can be extended to the normalizer $N_{\widetilde{\mathrm{SL}}(\mathbb{Q}_2)}(T)$ by defining $\gamma((h(2^n),1))=1$ for all integers n and $\gamma((w(1),1))=(1+\gamma((-I,1)))/\sqrt{2}=:\zeta_8$, a primitive 8th root of unity. Let $H(\overline{K_0(4)},\gamma)$ be genuine Hecke algebra of $\widetilde{\mathrm{SL}}(\mathbb{Q}_2)$ with respect to $\overline{K_0^2(4)}$ and γ. Let $\mathcal{T}_n=X_{(h(2^n),1)}$, $\mathcal{U}_n=X_{(w(2^{-n}),1)}\in H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(4)},\gamma)$ (defined as in the odd prime case).

Theorem 2. *(Loke-Savin [7]) For $m, n \in \mathbb{Z}$,*

(1) If $mn \geq 0$ then $\mathcal{T}_m * \mathcal{T}_n = \mathcal{T}_{m+n}$.

(2) $\mathcal{U}_1 * \mathcal{T}_n = \mathcal{U}_{n+1}$ and $\mathcal{T}_n * \mathcal{U}_1 = \mathcal{U}_{1-n}$.

(3) $\mathcal{U}_1 * \mathcal{U}_n = \mathcal{T}_{n-1}$ and $\mathcal{U}_n * \mathcal{U}_1 = \mathcal{T}_{1-n}$.

The Hecke algebra $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(4)}, \gamma)$ is generated by $\mathcal{U}_0$ and $\mathcal{U}_1$ modulo relations $(\mathcal{U}_0 - 2\sqrt{2})(\mathcal{U}_0 + \sqrt{2}) = 0$ and $\mathcal{U}_1^2 = 1$.

We now describe the genuine Hecke algebra $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \gamma)$ for certain characters γ of $\overline{K_0(8)}$.

We define character γ of $\overline{K_0(8)}$ as follows. Since $\overline{K_0(8)}/(K_1(8) \times M_2)$ is generated by $(h(5), 1)$, define γ on $K_1(8) \times M_2$ so that it is trivial on $K_1(8)$ and takes $(-I, 1)$ to primitive 4th root of uity. So now it is enough to define γ on $(h(5), 1)$. Since $(h(5), 1)$ has order 2 there are two choices for $\gamma((h(5), 1))$. We will call the resulting two characters of $\overline{K_0(8)}$ as χ_1 and χ_2:

$$\chi_1((h(u), 1)) = \begin{cases} 1 & \text{if } u \equiv 1,\ 5 \pmod{8\mathbb{Z}_2} \\ \gamma((-I, 1)) & \text{if } u \equiv 3,\ 7 \pmod{8\mathbb{Z}_2}, \end{cases},$$

$$\chi_2((h(u), 1)) = \begin{cases} 1 & \text{if } u \equiv 1 \pmod{8\mathbb{Z}_2} \\ \gamma((-I, 1)) & \text{if } u \equiv 7 \pmod{8\mathbb{Z}_2} \\ -1 & \text{if } u \equiv 5 \pmod{8\mathbb{Z}_2} \\ -\gamma((-I, 1)) & \text{if } u \equiv 3 \pmod{8\mathbb{Z}_2}. \end{cases}$$

Note that for $\gamma = \chi_1,\ \chi_2$ and $(A, \epsilon) = (x(s), 1)(h(u), 1)(y(t), 1)(I, \epsilon\delta) \in \overline{K_0(8)}$ we have

$$(1) \qquad \gamma(A, \epsilon) = \gamma(x(s), 1) \cdot \gamma(h(u), 1) \cdot \gamma(y(t), 1) \cdot \gamma(I, \epsilon\delta) = \gamma(h(u), 1) \cdot \gamma(I, \epsilon\delta).$$

For simplicity, we put $\overline{g} := (g, 1) \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$.

We have the following proposition. The proof is a routine calculation.

Proposition 2.6. *A complete set of representatives for the double coset $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ modulo $\overline{K}_0$ consists of $\overline{g}$ where g varies over the following elements of $SL_2(\mathbb{Q}_2)$:*
$h(2^n),\ w(2^n)$ for $n \in \mathbb{Z}$,
$h(2^n)y(4), h(2^n)y(2)$ for $n \geq 0$,
$y(4)h(2^{-n}), y(2)h(2^{-n}), w(2^{-n})y(2), y(2)w(2^{-n}), y(2)w(2^{-n})y(2)$ for $n \geq 1$,
$w(2^{-n})y(4),\ y(4)w(2^{-n}),\ y(4)w(2^{-n})y(4),\ y(2)w(2^{-n})y(4), y(4)w(2^{-n})y(2)$ for $n \geq 2$ and
$y(2)w(2^{-1})y(6)$.

We need to now compute the support of $H(//ovK_0(8), \chi_1)$ and $H(ovK_0(8), \chi_2)$. We first have the following lemma on vanishing.

Lemma 2.7. *The Hecke algebra $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_1)$ vanishes on the double cosets of $\overline{K_0(8)}$ represented by $\overline{y}(2)$, $\overline{y}(2)\overline{w}(2^{-n})$, $\overline{w}(2^{-n})\overline{y}(2)$, $\overline{y}(2)\overline{w}(2^{-n})\overline{y}(2)$, $\overline{h}(2^n)\overline{y}(2)$, $\overline{y}(2)\overline{h}(2^{-n})$, $\overline{y}(2)\overline{w}(2^{-1})\overline{y}(6)$ for $n \geq 1$ and $\overline{y}(2)\overline{w}(2^{-n})\overline{y}(4)$, $\overline{y}(4)\overline{w}(2^{-n})\overline{y}(2)$ for $n \geq 2$.*

The Hecke algebra $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_2)$ vanishes on the double cosets of $\overline{K_0(8)}$ represented by $\overline{y}(4)$, $\overline{y}(4)\overline{h}(2^{-n})$, $\overline{h}(2^n)\overline{y}(4)$ for $n \geq 1$, $\overline{y}(2)\overline{w}(2^{-1})\overline{y}(6)$, and $\overline{y}(2)\overline{w}(2^{-n})\overline{y}(4)$, $\overline{y}(4)\overline{w}(2^{-n})\overline{y}(2)$, $\overline{w}(2^{-n})\overline{y}(4)$, $\overline{y}(4)\overline{w}(2^{-n})$, $\overline{y}(4)\overline{w}(2^{-n})\overline{y}(4)$ where $n \geq 2$.

Proof. By ([3, Lemma3.1]) $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \gamma)$ is supported on $\tilde{g}$ if and only if for every $\tilde{k} \in K_{\tilde{g}} := \overline{K_0(8)} \cap \tilde{g}\overline{K_0(8)}\tilde{g}^{-1}$ we have $\gamma([\tilde{k}^{-1},\ \tilde{g}^{-1}]) = 1$. So to check the vanishing at $\tilde{g}$ we need to just find suitable $\tilde{k}$.

For example, for $A = y(2)w(2^{-n})$, take $B = \begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}$. Then $\overline{B} \in K_{\overline{A}}$ and

$$[\overline{B}^{-1}, \overline{A}^{-1}] = \left(\begin{pmatrix} 5 + 2^{2n+2} & -2 \\ 8 + 3 \cdot 2^{2n+1} & -3 \end{pmatrix}, -1 \right);$$

The above commutator is of the form $\left(\begin{pmatrix} -3 & * \\ 0 & -3 \end{pmatrix} \pmod{8\mathbb{Z}_2}, -1 \right)$ and in its triangular decomposition (as in equation 1) $\delta = 1$. Since χ_1 takes value -1, the vanishing of $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_1)$ follows on the double coset of A. The vanishing of $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_1)$, $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_2)$ at the double cosets listed in the lemma follow similarly. □

We next illustrate the case of support of $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_2)$ on $\overline{y}(2)$ before giving our proposition describing the complete support.

We note that

$$K_{\overline{y}(2)} = \left\{ \left(\begin{pmatrix} a - 2b & b \\ c + 2(a-d) - 4b & 2b + d \end{pmatrix}, \pm 1 \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0, \ \mathrm{ord}_2(b) \geq 1 \right\}.$$

has a triangular decomposition $K_{\tilde{A}} = N^{K_{\tilde{A}}} T^{K_{\tilde{A}}} \bar{N}^{K_{\tilde{A}}}$ where

$$N^{K_{\overline{y}(2)}} = \{(x(s), \pm 1) : \mathrm{ord}_2(b) \geq 1\}, \quad T^{K_{\overline{y}(2)}} = T^{\overline{K}_0}, \quad \bar{N}^{K_{\overline{y}(2)}} = \bar{N}^{\overline{K}_0}.$$

For $B = x(s)$ where $\mathrm{ord}_2(s) \geq 1$ (let $s \neq 0$) we have $B^{-1}A^{-1}BA = \begin{pmatrix} 1 + 2s + 4s^2 & 2s^2 \\ -4s & -2s + 1 \end{pmatrix}$ and $s_2(B^{-1}A^{-1}BA) = (-s, 2s-1)_2$ when $\mathrm{ord}_2(s)$ is odd, 1 else. Thus for $\mathrm{ord}_2(s) \geq 2$ we have $s_2(B^{-1}A^{-1}BA) = 1$. If $s = 2u$ with u a unit,

$$\begin{aligned} (-s, -2s+1)_2 &= (-2u, -4u+1)_2 = (-2, -4u+1)_2 (-u, -4u+1)_2 \\ &= (-2, -3)_2 = -1. \end{aligned}$$

As before, since $\mathrm{ord}_2(s) \geq 1$ the δ-factor in the triangular decomposition of $[(B, \epsilon)^{-1}, \overline{y}(2)]$ is 1 and so

$$\chi_2([(B, \epsilon)^{-1}, \overline{y}(2)]) = \begin{cases} \chi_2((h(5), -1)) = -1 \times -1 = 1 & \text{if } \mathrm{ord}_2(s) = 1 \\ 1 & \text{if } \mathrm{ord}_2(s) \geq 2. \end{cases}$$

For $B = h(u)$ and $B = y(t)$ in $K_{\overline{y}(2)}$ we check that $[(B, \epsilon)^{-1}, \overline{y}(4)] \in K_1(8) \times \{1\}$ and so χ_2 takes value 1.

We have the following proposition.

Proposition 2.8. *$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_1)$ is supported on precisely the double cosets of $\overline{K_0(8)}$ represented by*

$$\{\overline{h}(2^n), \ \overline{w}(2^{-n})\}_{n \in \mathbb{Z}} \cup \overline{y}(4) \cup \{\overline{h}(2^n)\overline{y}(4), \ \overline{y}(4)\overline{h}(2^{-n})\}_{n \geq 1} \cup$$
$$\{\overline{y}(4)\overline{w}(2^{-n}), \ \overline{w}(2^{-n})\overline{y}(4), \ \overline{y}(4)\overline{w}(2^{-n})\overline{y}(4)\}_{n \geq 2}.$$

$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_2)$ is supported on precisely the double cosets of $\overline{K_0(8)}$ represented by

$$\{\overline{h}(2^n), \ \overline{w}(2^{-n})\}_{n \in \mathbb{Z}} \ \cup \ \overline{y}(2) \ \cup$$
$$\{\overline{y}(2)\overline{w}(2^{-n}), \ \overline{w}(2^{-n})\overline{y}(2), \ \overline{y}(2)\overline{w}(2^{-n})\overline{y}(2), \ \overline{h}(2^n)\overline{y}(2), \ \overline{y}(2)\overline{h}(2^{-n})\}_{n \geq 1}.$$

2.4. **Generators and Relations.** Let γ be either χ_1 or χ_2. Following Loke and Savin [7] we extend the character to the normalizer subgroup $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)}(T)$ as before.

For $n \in \mathbb{Z}$, define as before the elements $\mathcal{T}_n$ and $\mathcal{U}_n$ of $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \gamma)$ precisely supported on the $\overline{K_0(8)}$ double cosets of $(h(2^n), 1)$ and $(w(2^{-n}), 1)$ respectively. We obtain the following relations in $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \gamma)$.

Lemma 2.9. *(1) If $mn \geq 0$ then $\mathcal{T}_m * \mathcal{T}_n = \mathcal{T}_{m+n}$.*
(2) For $n \leq 0$, $\mathcal{U}_1 * \mathcal{T}_n = \mathcal{U}_{1+n}$ and for $n \geq 0$, $\mathcal{T}_n * \mathcal{U}_1 = \mathcal{U}_{1-n}$.
(3) For $n \geq 0$, $\mathcal{U}_2 * \mathcal{T}_n = \mathcal{U}_{2+n}$ and for $n \leq 0$, $\mathcal{T}_n * \mathcal{U}_2 = \mathcal{U}_{2-n}$.
(4) For $m \geq 2$, $\mathcal{U}_1 * \mathcal{U}_m = \mathcal{T}_{m-1}$ and $\mathcal{U}_m * \mathcal{U}_1 = \mathcal{T}_{1-m}$.
(5) For $m \leq 1$, $\mathcal{U}_2 * \mathcal{U}_m = \mathcal{T}_{m-2}$ and $\mathcal{U}_m * \mathcal{U}_2 = \mathcal{T}_{2-m}$.

2.5. **The algebra** $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_1)$**.** Consider the case when $\gamma = \chi_1$. Since $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_1)$ is supported on $\overline{K_0(8)}\overline{y}(4)\overline{K_0(8)}$, we define $\mathcal{V}$ to be an element of $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_1)$ that is supported precisely on $\overline{K_0(8)}\overline{y}(4)\overline{K_0(8)}$ such that $\mathcal{V}(\overline{y}(4)) = 1$.

We obtain the following relations.

Proposition 2.10. *(1) $\mathcal{V} * \mathcal{V} = 1$.*
(2) $\mathcal{U}_1 * \mathcal{V} = \mathcal{U}_1 = \mathcal{V} * \mathcal{U}_1$.
(3) $\mathcal{U}_2 * \mathcal{U}_2 = 2$.
(4) $\mathcal{U}_1 * \mathcal{U}_1 = 2 + 2\mathcal{V}$.
(5) $\mathcal{U}_2 * \mathcal{V} * \mathcal{U}_2 = \sqrt{2}\, \mathcal{V} * \mathcal{U}_2 * \mathcal{V}$.
(6) $\mathcal{U}_0 * \mathcal{U}_0 = 8 + 2\sqrt{2}\, \mathcal{U}_0 + 8\mathcal{V}$.
(7) $\mathcal{U}_0 * \mathcal{V} = \mathcal{U}_0 = \mathcal{V} * \mathcal{U}_0$ and consequently, $\frac{\mathcal{U}_0}{\sqrt{2}} * (\frac{\mathcal{U}_0}{\sqrt{2}} - 4) * (\frac{\mathcal{U}_0}{\sqrt{2}} + 2) = 0$.

Let $\widehat{\mathcal{U}_1} = \frac{1}{\sqrt{2}}\mathcal{U}_1$, $\widehat{\mathcal{U}_2} = \frac{1}{\sqrt{2}}\mathcal{U}_2$ and $\widehat{\mathcal{U}_0} = \frac{1}{2\sqrt{2}}\mathcal{U}_0$.

Theorem 3. *The Hecke algebra $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_1)$ is generated by $\widehat{\mathcal{U}_1}$, $\widehat{\mathcal{U}_2}$ and $\mathcal{V}$ modulo the relations:*
(1) $\widehat{\mathcal{U}_1}^2 = 1 + \mathcal{V}$.
(2) $\widehat{\mathcal{U}_2}^2 = 1$.
(3) $\widehat{\mathcal{U}_1}\mathcal{V} = \mathcal{V}\widehat{\mathcal{U}_1} = \widehat{\mathcal{U}_1}$.
(4) $\widehat{\mathcal{U}_2}\mathcal{V}\widehat{\mathcal{U}_2} = \mathcal{V}\widehat{\mathcal{U}_2}\mathcal{V}$.

2.6. **The algebra** $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_2)$**.** Now take $\gamma = \chi_2$. Define $\mathcal{Z}_1' \in H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_2)$ supported only on the double coset of $\overline{y}(2)\overline{w}(2^{-1})\overline{y}(2)$ such that $\mathcal{Z}_1'(\overline{y}(2)\overline{w}(2^{-1})\overline{y}(2)) = 1$. Note that

$$(\overline{y}(2), 1)(\overline{w}(2^{-1}), 1)(\overline{y}(2), 1) = (\overline{x}(1/2), 1)$$

and this normalizes $\overline{K_0(8)}$. As before we get that $\mathcal{Z}_1' * \mathcal{Z}_1' = 1$.

Define $\mathcal{V}' \in H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_2)$ supported precisely on $\overline{K_0(8)}\overline{y}(2)\overline{K_0(8)}$ such that $\mathcal{V}'(\overline{y}(2), 1) = \frac{1+\gamma((-I,1))}{\sqrt{2}}$.

Let $\widehat{\mathcal{U}_1} = \frac{1}{\sqrt{2}}\mathcal{U}_1$ and $\widehat{\mathcal{U}_2} = \frac{1}{\sqrt{2}}\mathcal{U}_2$.

Theorem 4. *The Hecke algebra $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_2)$ is generated by $\widehat{\mathcal{U}_1}$, $\widehat{\mathcal{U}_2}$ and $\mathcal{Z}_1'$ modulo the relations:*
(1) $\widehat{\mathcal{U}_1}^2 = 1$.

(2) $\widehat{\mathcal{U}_2}^2 = 1 + \mathcal{Z}_1'$.
(3) $\widehat{\mathcal{U}_2}\mathcal{Z}_1' = \widehat{\mathcal{U}_2} = \mathcal{Z}_1'\widehat{\mathcal{U}_2}$.
(4) $\widehat{\mathcal{U}_1}\mathcal{Z}_1'\widehat{\mathcal{U}_1} = \mathcal{Z}_1'\widehat{\mathcal{U}_1}\mathcal{Z}_1'$.

We further compute the full Hecke algebra of $\mathrm{GL}_2(\mathbb{Q}_2)$ corresponding to $H_0(4)$. As a corollary we obtain that

Corollary 2.11. *We have the following isomorphism of 2-dic Hecke algebras:*

$$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_1) \cong H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_2) \cong H(\mathrm{GL}_2(Q_2)//H_0(8), \chi_{\mathrm{triv}})$$

3. Translation of adelic to classical.

Let k be a natural number, M be odd and χ be an even Dirichlet character modulo $2^\nu M$. Let $\chi_0 = \chi\left(\frac{-1}{\cdot}\right)^k$.

Let $A_{k+1/2}(2^\nu M, \chi_0)$ be the set of adelic cuspidal automorphic forms $\Phi : \widetilde{\mathrm{SL}}_2(\mathbb{A}) \to \mathbb{C}$ satisfying certain properties as considered by Waldspurger [15]. By Gelbart-Waldspurger there is an isomorphism between $A_{k+1/2}(2^\nu M, \chi_0) \to S_{k+1/2}(\Gamma_0(2^\nu M)\chi)$, $\Phi_f \leftrightarrow f$, inducing a ring isomorphism

$$q : \mathrm{End}_{\mathbb{C}}(A_{k+1/2}(2^\nu M, \chi_0)) \to \mathrm{End}_{\mathbb{C}}(S_{k+1/2}(\Gamma_0(2^\nu M), \chi).$$

The local Hecke algebra sits inside $\mathrm{End}_{\mathbb{C}}(A_{k+1/2}(2^\nu M, \chi_0))$ as subalgebra and we will use q to translate certain elements in the local Hecke algebra that we described in the previous section to classical operators on $S_{k+1/2}(\Gamma_0(2^\nu M))$. For simplicity we will deal only with the case when χ is trivial. The quadratic character case can be treated similarly. Thus the classical operators so obtained satisfy the local Hecke algebra relations noted in the previous section.

3.1. $N = 2^\nu M$, M odd and $p \| M$.

Proposition 3.1. *Let $\mathcal{T}_1$, $\mathcal{U}_1$ and $\mathcal{U}_0 \in H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)}, 1)$ and $f \in S_{k+1/2}(\Gamma_0(2^\nu M))$. Then,*

(1) $$q(\mathcal{T}_1)(f)(z) = \left(\frac{-1}{p}\right)^k p^{-k-1/2} \sum_{s=0}^{p^2-1} f\left(\frac{z+s}{p^2}\right) = p^{(3-2k)/2} U_{p^2}(f).$$

(2) $$q(\mathcal{U}_1)(f)(z) = \overline{\varepsilon_p}\left(\frac{-1}{p}\right)\left(\frac{M/p}{p}\right) \sum_{s=0}^{p-1} f|[(\alpha_s, \phi_{\alpha_s})]_{k+1/2}(z),$$ *where*

$\alpha_s = \begin{pmatrix} p^2 n - 2^\nu Mms & m \\ 2^\nu pM(1-s) & p \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z})$ *is of determinant p^2 and m, $n \in \mathbb{Z}$ are such that $pn - (2^\nu M/p)m = 1$, and $\phi_{\alpha_s}(z) = (2^\nu M(1-s)z + 1)^{1/2}$.*

(3) $$q(\mathcal{U}_0)(f)(z) = \sum_{s=0}^{p-1} f|[(\beta_s, \phi_{\beta_s})]_{k+1/2}(z),$$ *where*

$\beta_s = \begin{pmatrix} 1 & m-s \\ 2^\nu M_1 & np - 2^\nu M_1 s \end{pmatrix} \in \Gamma_0(2^\nu M_1)$ *with $M_1 = M/p$ and m, $n \in \mathbb{Z}$ are chosen as above and $\phi_{\beta_s} = (2^\nu M_1 z + (np - 2^\nu M_1 s))^{1/2}$.*

Let us denote $q(p^{-1/2}\mathcal{U}_1)$ by $\widetilde{W}_{p^2}$ and $q(\mathcal{U}_0)$ by $\widetilde{Q}_p$.

Corollary 3.2. *On $S_{k+1/2}(\Gamma_0(2^\nu M))$ we have*

(1) *$\widetilde{W}_{p^2}$ is an involution.*

(2) $(\widetilde{Q}_p - p)(\widetilde{Q}_p + 1) = 0$.

(3) $\widetilde{Q}_p = \left(\frac{-1}{p}\right)^k p^{1-k} U_{p^2} \widetilde{W}_{p^2}$.

(4) If $f \in S_{k+1/2}(\Gamma_0(2^\nu M/p))$ *then* $\widetilde{Q}_p(f) = pf$.

We further define an operator $\widetilde{Q}'_p$ on $S_{k+1/2}(\Gamma_0(4M))$ to be the conjugate of $\widetilde{Q}_p$ by $\widetilde{W}_{p^2}$, i.e., $\widetilde{Q}'_p = \widetilde{W}_{p^2}\widetilde{Q}_p\widetilde{W}_{p^2}$. Thus $\widetilde{Q}'_p$ satisfies same quadratic as $\widetilde{Q}_p$ and we have $\widetilde{Q}'_p = \left(\frac{-1}{p}\right)^k p^{1-k}\widetilde{W}_{p^2}U_{p^2}$.

Remark 1. *We note that for a prime q such that $(q, 2M) = 1$, one can similarly obtain the usual Hecke operator T_{q^2} on $S_{k+1/2}(\Gamma_0(4M))$. In particular, if we take $\mathcal{T}_1 := X_{(h(q),1)} \in H(\overline{\mathrm{SL}_2(\mathbb{Z}_q)}, \gamma_q)$ then $q(\mathcal{T}_1) = \left(\frac{-1}{p}\right)^k p^{(3-2k)/2} T_{q^2}$.*

Moreover if p and q are distinct primes such that p^n, q^m strictly divides N then the operators $\mathcal{S} \in H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p^n)}, \gamma_p)$ and $\mathcal{T} \in H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_q)\overline{K_0(q^m)}, \gamma_q)$ in $\mathrm{End}_{\mathbb{C}}(S_{k+1/2}(\Gamma_0(N))$ commute.

In particular the operators $\widetilde{Q}_p$, $\widetilde{W}_{p^2}$ on $S_{k+1/2}(\Gamma_0(4M))$ that we defined above commute with T_{q^2} for primes q coprime to M.

3.2. $N = 2^\nu M$, M **odd and** $\nu = 2$. We consider the central character γ of M_2 such that $\gamma((-I, 1)) = -i^{2k+1}$. We make the same choice for the case below when $\nu = 3$. We translate $\mathcal{T}_1$, $\mathcal{U}_1$, $\mathcal{U}_0 \in H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(4)}, \gamma)$ to classical operators on $S_{k+1/2}(\Gamma_0(4M))$.

Proposition 3.3. *For $f \in S_{k+1/2}(\Gamma_0(4M))$,*

(1) $q(\mathcal{T}_1)(f)(z) = 2^{(3-2k)/2} U_4(f)(z)$.

(2) $q(\mathcal{U}_1)(f)(z) = \overline{\zeta_8}\left(\frac{2}{M}\right)\left(\frac{-1}{M}\right)^{k+3/2} f|[W_4, \phi_W(z)]_{k+1/2}(z)$ *where* $W_4 = \begin{pmatrix} 4n & m \\ 4M & 8 \end{pmatrix}$ *with $a, b \in \mathbb{Z}$ such that $8n - mM = 1$ and $\phi_W(z) = (2Mz+4)^{1/2}$.*

(3) $q(\mathcal{U}_0)(f)(z) = \overline{\zeta_8}\left(\frac{-1}{M}\right)^{k+3/2} \sum_{s=0}^{3} f|[A_s, \phi_{A_s}(z)]_{k+1/2}(z)$ *where* $A_s = \begin{pmatrix} n & -ns+m \\ M & -Ms+4 \end{pmatrix}$ *with $m, n \in \mathbb{Z}$ such that $4n - mM = 1$ and $\phi_W(z) = (Mz + 4 - Ms)^{1/2}$.*

Define $\widetilde{Q}_2 := q(\mathcal{U}_0)/\sqrt{2} = q(\mathcal{T}_1)q(\mathcal{U}_1)/\sqrt{2}$ and $\widetilde{W}_4 := q(\mathcal{U}_1)$ and $\widetilde{Q}'_2$ to be the conjugate of $\widetilde{Q}_2$ by $\widetilde{W}_4$. The Kohnen's plus space at level $4M$ is the 2-eigenspace of $\widetilde{Q}'_2$. Note that $\widetilde{Q}_2$ and $\widetilde{Q}'_2$ are self-adjoint with respect to the Petersson inner product. The operators $\widetilde{Q}'_p$ and $\widetilde{Q}_p$ are p-adic analogue of Kohnen's operator $\widetilde{Q}'_2$ and it's conjugate $\widetilde{Q}_2$.

3.3. $N = 2^\nu M$, M **odd and** $\nu = 3$.

Proposition 3.4. *Let $\mathcal{T}_1$, $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{V} \in H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_1)$ and $f \in S_{k+1/2}(\Gamma_0(8M))$.*

(1) $q(\mathcal{T}_1)(f)(z) = 2^{-(2k+1)/2}\sum_{s=0}^{3} f((z+s)/4) = 2^{(3-2k)/2}U_4(f)(z)$.

(2) $q(\mathcal{U}_1)(f)(z) = \overline{\zeta_8}\left(\frac{-1}{M}\right)^{k+3/2}\left(\frac{2}{M}\right) 2f|[W_4, \phi_{W_4}(z)]_{k+1/2}(z)$ *where* $W_4 = \begin{pmatrix} 4n & m \\ 4M & 8 \end{pmatrix}$ *with $n, m \in \mathbb{Z}$ such that $8n - mM = 1$ and $\phi_{W_4}(z) = (2Mz + 4)^{1/2}$.*

(3) $q(\mathcal{U}_2)(f)(z) = \overline{\zeta_8}\left(\frac{-1}{M}\right)^{k+3/2}\sum_{s=0}^{1} f|[W_8, \phi_{W_8}(z)]_{k+1/2}(z)$ *where* $W_8 = \begin{pmatrix} 16n - 8mMs & m \\ 16M - 128Ms & 16 \end{pmatrix}$ *with* $n, m \in \mathbb{Z}$ *such that* $16n - mM = 1$ *and* $\phi_{W_8}(z) = ((4M - 32Ms)z + 4)^{1/2}$.

(4) $q(\mathcal{V})(f)(z) = f|[\begin{pmatrix} 1 & 0 \\ 4M & 1 \end{pmatrix}, (4Mz+1)^{1/2}]_{k+1/2}(z)$.

Define operators $\widetilde{W_8} := q(\widehat{\mathcal{U}_2})$ and $\widetilde{V_4} := q(\mathcal{V})$ on $S_{k+1/2}(\Gamma_0(8M))$ where $\widehat{\mathcal{U}_2}$, $\mathcal{V}$ are elements in $H(\widetilde{\mathrm{SL}_2}(\mathbb{Q}_p)//\overline{K_0(8)}, \chi_1)$. Note that both $\widetilde{V_4}$ and $\widetilde{W_8}$ are involution. Define $\widetilde{V_4'}$ to be the conjugate of $\widetilde{V_4}$ by $\widetilde{W_8}$.

Corollary 3.5. *$S_{k+1/2}(\Gamma_0(4M))$ is contained in the $+1$ eigenspace of $\widetilde{V_4}$ and $q(\mathcal{U}_1^2) = 4$ on $S_{k+1/2}(\Gamma_0(4M))$.*

Proposition 3.6. *The operators $\widetilde{Q_p}$, $\widetilde{Q_p'}$, $\widetilde{W_{p^2}}$, $\widetilde{W_8}$ and $\widetilde{V_4}$, $\widetilde{V_4'}$ are self-adjoint with respect to the Petersson inner product.*

Indeed note that the Petersson inner product is essentially the L^2- inner product of corresponding automorphic forms and checking self-adjointness reduces to checking equality of certain double cosets.

3.4. **Comparison with Kohnen's projection map.** Kohnen [6, Page 37] and later Ueda-Yamana [13] define function $P_8(f) = f|[\xi + \xi^{-1}]_{k+1/2}$ where $\xi = (\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{\pi i/4})$. We have the following observation.

Proposition 3.7. $\frac{1}{\sqrt{2}}\left(\frac{2}{2k+1}\right) P_8 = \widetilde{V_4}\widetilde{W_8}\widetilde{V_4} = \widetilde{W_8}\widetilde{V_4}\widetilde{W_8} = \widetilde{V_4'}$.

Extending Kohnen's definition, Ueda-Yamana [13] define the plus space $S^+_{k+1/2}(8M)$ to consist of $f = \sum_{n=1}^{\infty} a_n q^n \in S_{k+1/2}(\Gamma_0(8M))$ such that $a_n = 0$ for $(-1)^k n \equiv 2, 3 \pmod 4$.

Corollary 3.8. *$S^+_{k+1/2}(8M)$ is the $+1$ eigenspace of $\widetilde{V_4'}$. The -1 eigenspace of $\widetilde{V_4'}$ consists of f such that $a_n = 0$ for $(-1)^k n \equiv 0, 1 \pmod 4$.*

Proof. From [6, equation(2)], $P_8(f) = \sqrt{2}\left(\frac{2}{2k+1}\right)\left(\sum_n^{(1)} a_n q^n - \sum_n^{(2)} a_n q^n\right)$ where $\sum_n^{(1)}$ resp. $\sum_n^{(2)}$ runs over n with $(-1)^k n \equiv 0, 1 \pmod 4$ resp. $(-1)^k n \equiv 2, 3 \pmod 4$. The result now follows using above proposition. $\square$

4. Main result

Theorem 5. *Let M be odd, square-free. Let $S^-_{k+1/2}(4M) \subseteq S_{k+1/2}(4M)$ be the common -1-eigenspace of operators $\widetilde{Q_p}$, $\widetilde{Q_p'}$ for all $p \mid 2M$. Further let $S^-_{k+1/2}(8M) \subseteq S_{k+1/2}(8M)$ be the common -1-eigenspace of operators $\widetilde{Q_p}$, $\widetilde{Q_p'}$ for all $p \mid M$ and $\widetilde{V_4}$, $\widetilde{V_4'}$.*

(1) The space $S^-_{k+1/2}(2^\nu M)$ has a basis of eigenforms under T_{q^2}, $(q, 2M) = 1$ and U_{p^2}, $p \mid 2M$ for $\nu = 2, 3$.

(2) $S^-_{k+1/2}(2^\nu M) \xrightarrow{\sim} S^{\mathrm{new}}_{2k}(2^{\nu-1}M)$. (Hecke isomorphism)

(3) $S^-_{k+1/2}(2^\nu M)$ has multiplicity-one in the full space $S_{k+1/2}(2^\nu M)$

(4) If $f = \sum_{n=1}^{\infty} a_n q^n \in S^-_{k+1/2}(8M)$ then $a_n = 0$ for $(-1)^k n \equiv 0, 1 \pmod 4$

Remark 2. *In the simplest case $M = 1$, we note the following observation:*

(1) $S^+_{k+1/2}(8) = S^+(4) \oplus \widetilde{W}_8 A^+(4) \oplus \widetilde{W}_8 S^-(4)$ *where* $A^+(4) = \widetilde{W}_4 S^+(4)$.

(2) Let $S^{\min}_{k+1/2}(8)$ *is the subspace of* $S_{k+1/2}(\Gamma_0(8))$ *consisting of* $f = \sum_{n=1}^{\infty} a_n q^n$ *such that* $a_n = 0$ *for* $(-1)^k n \equiv 0, 1 \pmod 4$. *Then given a newform* F *of weight* $2k$ *and level dividing* 4, *there exists a unique Shimura-equivalent form in* $S_{k+1/2}(8, F) \cap S^{\min}_{k+1/2}(8)$.

Remark 3. *We note that the decomposition of the space* $S_{k+1/2}(\Gamma_0(8M))$ *is completely analogous to that of* $S_{2k}(\Gamma_0(4M))$ *when we look at it through the local Hecke algebra. We illustrate this in the case* $M = 1$.

$$\begin{aligned} S_{2k}(\Gamma_0(4)) &= (S_{2k}(\Gamma_0(1)) \oplus q(\mathcal{U}_1) S_{2k}(\Gamma_0(1)) \oplus q(\mathcal{U}_2) S_{2k}(\Gamma_0(1))) \\ &\quad \oplus (S^{\text{new}}_{2k}(\Gamma_0(2)) \oplus q(\mathcal{U}_2) S^{\text{new}}_{2k}(\Gamma_0(2))) \oplus S^{\text{new}}_{2k}(\Gamma_0(4)). \end{aligned}$$

In the above $\mathcal{U}_1$, $\mathcal{U}_2$ *are elements in the Hecke algebra of* $PGL_2(\mathbb{Q}_2)$ *corresponding to the subgroup* $H_0(4)$ *coming from Weyl elements* $\left(\begin{smallmatrix} 0 & -1 \\ 2 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right)$ *respectively* [2]. *Also we can obtain the following relations from that paper,* $q(\mathcal{U}_2) q(\mathcal{U}_1) S_{2k}(\Gamma_0(1)) = q(\mathcal{U}_1) S_{2k}(\Gamma_0(1))$.

Now let's look at the space $S_{k+1/2}(\Gamma_0(8M))$. *We have*

$$\begin{aligned} S_{k+1/2}(\Gamma_0(8)) &= (A^+(4) \oplus q(\mathcal{U}_1) A^+(4) \oplus q(\mathcal{U}_2) A^+(4)) \\ &\quad \oplus (S^-(4) \oplus q(\mathcal{U}_2) S^-(4)) \oplus S^-(8). \end{aligned}$$

Here $\mathcal{U}_1$, $\mathcal{U}_2$ *are elements in the Hecke algebra* $H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p) // \overline{K_0(8)}, \chi_1)$ *coming from the Weyl elements* $w(2^{-1})$, $w(2^{-2})$ *respectively. Recall from* [3] *that* $A^+(4) = \widetilde{W}_4 S^+(4) = q(\mathcal{U}_1) S^+(4)$. *As in the integral weight we have* $q(\mathcal{U}_2) q(\mathcal{U}_1) A^+(4) = q(\mathcal{U}_2) S^+(4) = S^+(4) = q(\mathcal{U}_1) A^+(4)$.

Example 1. The space $S_{3/2}(\Gamma_0(152))$ is eight dimensional and there are four primitive Hecke eigenforms of weight 2 and level dividing 76, namely F_{19} of level 19, G_{38}, H_{38} of level 38 and P_{76} of level 76. We have

$$S_{3/2}(\Gamma_0(152)) = S_{3/2}(152, F_{19}) \oplus S_{3/2}(152, G_{38}) \oplus S_{3/2}(152, H_{38}) \oplus S_{3/2}(152, K_{76}).$$

We compute the Shimura decomposition [9]. As we would expect from the above remark, $S_{3/2}(152, F_{19})$ is 3-dimensional space and is spanned by

$$\begin{aligned} f1 &= q + q^5 - 2q^6 - q^9 - q^{17} + 2q^{25} + 2q^{30} + 2q^{42} - 3q^{45} + O(q^{50}) \\ f2 &= q^4 - 2q^{11} - 2q^{16} + 2q^{19} + q^{20} - 2q^{24} + 3q^{28} + 2q^{35} - q^{36} + O(q^{40}) \\ f3 &= q^7 - q^{11} - 2q^{16} + q^{19} + 2q^{28} + q^{35} - 2q^{39} - q^{43} + 2q^{44} - q^{47} + O(q^{50}), \end{aligned}$$

$S_{3/2}(152, G_{38})$ is 2-dimensional space and is spanned by

$$\begin{aligned} g1 &= q - 2q^5 + q^6 + 2q^9 - q^{17} - q^{25} - 3q^{26} - 4q^{30} + 3q^{38} + 5q^{42} + O(q^{50}) \\ g2 &= q^4 + q^7 - q^{16} - 2q^{20} - 3q^{23} + q^{24} - q^{28} + 2q^{36} + q^{39} + 2q^{47} + O(q^{50}), \end{aligned}$$

$S_{3/2}(152, H_{38})$ is 2-dimensional space and is spanned by

$$\begin{aligned} h1 &= q^2 + 2q^{10} - 3q^{13} - q^{14} - 2q^{18} - q^{21} + 2q^{22} + q^{29} + O(q^{30}) \\ h2 &= q^3 - q^8 + q^{12} - q^{19} - q^{27} - q^{32} - 2q^{40} + q^{48} + O(q^{50}) \end{aligned}$$

and $S_{3/2}(152, K_{76})$ is 1-dimensional space and is spanned by

$$k1 = q^2 - q^{10} - q^{14} + q^{18} + 2q^{21} - q^{22} - 2q^{29} - 2q^{33} - q^{34} + 2q^{37} + q^{38} - 2q^{41} + O(q^{50}).$$

The Kohnen plus space $S^+_{3/2}(152)$ is four dimensional and is spanned by $\{f_2, f3, g2, h2\}$. We further note that $S_{3/2}(76, F_{19})$ is two dimensional and spanned by $\{f1 + f3, f2 - f3\}$

and $S^-_{3/2}(76)$ is two dimensional spanned by $\{g1 - g2, h1 - h2\}$. The minus space at level 152, $S^-_{3/2}(152)$ is one dimesnional spanned by k_1 and is Shimura equivalent to K_{76}. Note that k_1 satisfies the Fourier coefficient condition as noted in the theorem.

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