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ON THE GROSS-KEATING INVARIANTS OF A HERMITIAN FORM OVER A NON-ARCHIMEDEAN LOCAL FIELD

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It is known that the Siegel series of a quadratic form over a non-archimedean local field is determined by the Gross-Keating invariant and it related invariants. Here, we discuss an analogue for a semi-integral hermitian form with respect to a quadratic extension of a non-archimedean local field.

1. Review of the theory of Gross-Keating invariant for a quadratic form

Here, we briefly review the theory of the Gross-Keating invariant for a quadratic form over a non-archimedean local field ([4], [5]). The symbols defined for quadratic form are distinguished by adding the subscript "quad" to avoid possible confusion. For example, $S(B)$ for quadratic form is denoted by $S(B)_{\mathrm{quad}}$.

Let F be a non-archimedean local field of characteristic 0, and $\mathfrak{o} = \mathfrak{o}_F$ its ring of integers. The order $\mathrm{ord}(x)$ of $x \in F^\times$ is normalized so that $\mathrm{ord}(\varpi) = 1$ for a prime element ϖ of F. We understand $\mathrm{ord}(0) = +\infty$.

The set of symmetric matrices $B \in \mathrm{M}_n(F)$ of size n is denoted by $\mathrm{Sym}_n(F)$. For $B \in \mathrm{Sym}_n(F)$ and $X \in \mathrm{GL}_n(F)$, we set $B[X] = {}^tXBX$. We say that $B = (b_{ij}) \in \mathrm{Sym}_n(F)$ is a half-integral symmetric matrix if

$$\begin{aligned} b_{ii} &\in \mathfrak{o}_F \qquad (1 \le i \le n), \\ 2b_{ij} &\in \mathfrak{o}_F \qquad (1 \le i \le j \le n). \end{aligned}$$

The set of all half-integral symmetric matrices of size n is denoted by $\mathcal{H}_n(\mathfrak{o})$. An element $B \in \mathcal{H}_n(\mathfrak{o})$ is non-degenerate if $\det B \neq 0$. The set of all non-degenerate elements of $\mathcal{H}_n(\mathfrak{o})$ is denoted by $\mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$.

The equivalence class of $B \in \mathcal{H}_n(\mathfrak{o})$ is denoted by $\{B\}_{\mathrm{quad}}$, i.e., $\{B\}_{\mathrm{quad}} = \{B[U] \mid U \in \mathrm{GL}_n(\mathfrak{o})\}$. We write $B \sim_{\mathrm{quad}} B'$ if $B' \in \{B\}_{\mathrm{quad}}$.

Definition 1.1. Let $B = (b_{ij}) \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$. Let $S(B)_{\mathrm{quad}}$ be the set of all non-decreasing sequences $(a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$ such that

$$\begin{aligned} &\mathrm{ord}(b_{ii}) \geq a_i && (1 \leq i \leq n), \\ &\mathrm{ord}(2b_{ij}) \geq (a_i + a_j)/2 && (1 \leq i \leq j \leq n). \end{aligned}$$

Put

$$\mathbf{S}(\{B\})_{\mathrm{quad}} = \bigcup_{B' \in \{B\}} S(B')_{\mathrm{quad}} = \bigcup_{U \in \mathrm{GL}_n(\mathfrak{o})} S(B[U])_{\mathrm{quad}}.$$

The (quadratic) Gross-Keating invariant $\mathrm{GK}(B)_{\mathrm{quad}}$ of B is the greatest element of $\mathbf{S}(\{B\})_{\mathrm{quad}}$ with respect to the lexicographic order $\succeq$ on $\mathbb{Z}_{\geq 0}^n$.

It is easy to see that $\mathbf{S}(\{B\})_{\mathrm{quad}}$ is a finite set.

A sequence of length 0 is denoted by $\emptyset$. When B is the empty matrix, we understand $\mathrm{GK}(B)_{\mathrm{quad}} = \emptyset$. By definition, the Gross-Keating invariant $\mathrm{GK}(B)_{\mathrm{quad}}$ is determined only by the equivalence class of B.

Definition 1.2. $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$ is optimal if $\mathrm{GK}(B)_{\mathrm{quad}} \in S(B)_{\mathrm{quad}}$.

By definition, a non-degenerate half-integral symmetric matrix $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$ is equivalent to an optimal form.

For $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$, we put $D_B = (-4)^{[n/2]} \det B$. If n is even, we denote the discriminant ideal of $F(\sqrt{D_B})/F$ by $\mathfrak{D}_B$. We also put

$$\xi(B)_{\mathrm{quad}} = \begin{cases} 1 & \text{if } D_B \in F^{\times 2}, \\ -1 & \text{if } F(\sqrt{D_B})/F \text{ is unramified and } [F(\sqrt{D_B}) : F] = 2, \\ 0 & \text{if } F(\sqrt{D_B})/F \text{ is ramified.} \end{cases}$$

Here, $F^{\times 2} = \{x^2 \mid x \in F^\times\}$.

Definition 1.3. For $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$, we put

$$\Delta(B)_{\mathrm{quad}} = \begin{cases} \mathrm{ord}(D_B) & \text{if } n \text{ is odd,} \\ \mathrm{ord}(D_B) - \mathrm{ord}(\mathfrak{D}_B) + 1 - \xi(B)_{\mathrm{quad}}^2 & \text{if } n \text{ is even.} \end{cases}$$

Theorem 1.1. *Suppose that* $\underline{a} = (a_1, a_2, \ldots, a_n) = \mathrm{GK}(B)_{\mathrm{quad}}$ *for* $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$. *Then we have*

$$a_1 + a_2 + \cdots + a_n = \Delta(B)_{\mathrm{quad}}.$$

For a non-decreasing sequence $\underline{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$, we set

$$G_{\underline{a},\mathrm{quad}} = \{g = (g_{ij}) \in \mathrm{GL}_n(\mathfrak{o}) \mid \mathrm{ord}(g_{ij}) \geq (a_j - a_i)/2, \text{ if } a_i < a_j\}.$$

Theorem 1.2. *Suppose that* $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$ *is optimal and* $\mathrm{GK}(B)_{\mathrm{quad}} = \underline{a}$. *Let* $U \in \mathrm{GL}_n(\mathfrak{o})$. *Then* $B[U]$ *is optimal if and only if* $U \in G_{\underline{a},\mathrm{quad}}$.

For $B = (b_{ij})_{1\le i,j\le n} \in \mathcal{H}_n(\mathfrak{o})$ and $1 \le m \le n$, we denote the upper left $m \times m$ submatrix $(b_{ij})_{1\le i,j\le m} \in \mathcal{H}_m(\mathfrak{o})$ by $B^{(m)}$. For $\underline{a} = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n_{\ge 0}$, we put $\underline{a}^{(m)} = (a_1, a_2, \dots, a_m)$ for $m \le n$.

Theorem 1.3. *Suppose that $B \in \mathcal{H}^{\mathrm{nd}}_n(\mathfrak{o})$ is optimal and $\mathrm{GK}(B)_{\mathrm{quad}} = \underline{a}$. If $a_k < a_{k+1}$, then $B^{(k)}$ is also optimal and $\mathrm{GK}(B^{(k)})_{\mathrm{quad}} = \underline{a}^{(k)}$.*

Definition 1.4. The Clifford invariant of $B \in \mathcal{H}^{\mathrm{nd}}_n(\mathfrak{o})$ is the Hasse invariant of the Clifford algebra (resp. the even Clifford algebra) of B if n is even (resp. odd).

We denote the Clifford invariant of B by $\eta(B)$.

Theorem 1.4. *Let $B, B_1 \in \mathcal{H}^{\mathrm{nd}}_n(\mathfrak{o})$. Suppose that $B \sim_{\mathrm{quad}} B_1$ and both B and B_1 are optimal. Let $\underline{a} = (a_1, a_2, \dots, a_n) = \mathrm{GK}(B)_{\mathrm{quad}} = \mathrm{GK}(B_1)_{\mathrm{quad}}$. Suppose that $a_k < a_{k+1}$ for $1 \le k < n$. Then the following assertions (1) and (2) hold.*

(1) *If k is even, then $\xi(B^{(k)})_{\mathrm{quad}} = \xi(B_1^{(k)})_{\mathrm{quad}}$.*
(2) *If k is odd, then $\eta(B^{(k)}) = \eta(B_1^{(k)})$.*

Let $\psi : F \to \mathbb{C}^\times$ be an additive character of order 0. For $B \in \mathcal{H}^{\mathrm{nd}}_n(\mathfrak{o})$, we define the Siegel series $b(B, s)$ by

$$b(B, s) = \int_{R \in \mathrm{Sym}_n(F)} \psi(\mathrm{tr}(BR))[R\mathfrak{o}^n + \mathfrak{o}^n : \mathfrak{o}^n]^{-s}\, dR.$$

This integral is convergent for $\mathrm{Re}(s) \gg 0$, and is analytically continued to the whole s-plane. Put

$$\gamma(B, X) = \begin{cases} \dfrac{(1-X)}{(1-q^{n/2}\xi(B)X)} \displaystyle\prod_{i=1}^{n/2}(1-q^{2i}X^2) & \text{if } n \text{ is even,} \\ (1-X) \displaystyle\prod_{i=1}^{(n-1)/2}(1-q^{2i}X^2) & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 1.5 (Kitaoka, Feit, Shimura)**.** *There exists a polynomial $F(B, X) \in \mathbb{Q}[X]$ such that*

$$b(B, s) = \gamma(B, q^{-s})F(B, q^{-s}).$$

Put

$$\widetilde{F}(B, X) = X^{-\mathrm{ord}(D_B)/2} F(B, q^{-(n+1)/2}X).$$

Then we have a functional equation

$$\widetilde{F}(B, X^{-1}) = \eta(B)^n \widetilde{F}(B, X).$$

Theorem 1.6. *The Siegel series $F(B, X)$ is determined by the following data:*

(1) *The Gross-Keating invariant* $\underline{a} = \mathrm{GK}(B)_{\mathrm{quad}}$.
(2) *The Kronecker invariants* $\xi(B^{(k)})_{\mathrm{quad}}$ *for* $a_k < a_{k+1}$, *with* k *even.*
(3) *The Clifford invariants* $\eta(B^{(k)})$ *for* $a_k < a_{k+1}$, *with* k *odd.*

2. The Gross-Keating invariant for hermitian forms

Let F be a non-archimedean local field. Let E/F be a *ramified* quadratic extension, and $\mathfrak{D} = \mathfrak{D}_{E/F}$ be its relative different. The trace and the norm for E/F are denoted by $\mathrm{tr}_{E/F}$ and $\mathrm{N}_{E/F}$, respectively. The non-trivial automorphism of E/F is denoted by $x \mapsto \bar{x}$. We fix a prime element ϖ_E of $\mathfrak{o}_E$ and put $\varpi = \mathrm{N}_{E/F}(\varpi_E)$. Thus ϖ is a prime element of F. We denote the discriminant ideal of E/F by $\mathbf{D} = \mathbf{D}_{E/F}$. Thus we have $\mathbf{D} = \mathrm{N}_{E/F}(\mathfrak{D})$. The order of $x \in E^\times$ is normalized so that $\mathrm{ord}(\varpi) = 1$. In particular, $\mathrm{ord}(\varpi_E) = 1/2$. Similarly, the order of an $\mathfrak{o}_E$-ideal is defined by $\mathrm{ord}(\mathfrak{p}_E^k) = k/2$.

For a matrix $X = (x_{ij}) \in \mathrm{M}_{mn}(E)$, the hermitian conjugate $X^* = (x^*_{ij}) \in \mathrm{M}_{nm}(E)$ is defined by $x^*_{ij} = \bar{x}_{ji}$. We say that $B = B^* = (b_{ij}) \in \mathrm{M}_n(E)$ is a semi-integral hermitian matrix if

$$b_{ii} \in \mathfrak{o}_F, \quad b_{ij} = \bar{b}_{ji} \in \mathfrak{D}^{-1} \quad (1 \le i, j \le n).$$

The set of all semi-integral hermitian matrices of size n is denoted by $\mathcal{H}_n(\mathfrak{o})_{E/F}$. When there is no fear of confusion, we just write $\mathcal{H}_n(\mathfrak{o})$ for $\mathcal{H}_n(\mathfrak{o})_{E/F}$. An element $B \in \mathcal{H}_n(\mathfrak{o})$ is non-degenerate if $\det B \neq 0$. The set of all non-degenerate elements of $\mathcal{H}_n(\mathfrak{o})$ is denoted by $\mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$.

Definition 2.1. For $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$, set

$$\xi_B = \xi(B) = \chi_{K/F}((-1)^{[n/2]} \det B),$$

where $\chi_{E/F} : F^\times \to \{\pm 1\}$ is the character corresponding to E/F by the local class field theory. Put $e_B = \mathrm{ord}(\det B \cdot \mathbf{D}^{[n/2]})$. One can easily see that $e_B \ge 0$ for any $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$. Put

$$\Delta(B) = \begin{cases} e_B - 1 & \text{if } n \text{ is even and } \xi_B = -1, \\ e_B & \text{otherwise.} \end{cases}$$

One can also show that $\Delta(B) \ge 0$ for any $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$.

Definition 2.2. Let $S(B)$ be the set of all non-decreasing sequences $(a_1, \dots, a_n) \in \mathbb{Z}^n_{\ge 0}$ such that

$$\begin{aligned} \mathrm{ord}(b_{ii}) &\ge a_i, & (1 \le i \le n), \\ \mathrm{ord}(b_{ij}\mathfrak{D}) &\ge (a_i + a_j)/2 & (1 \le i, j \le n). \end{aligned}$$

We also write $S(\underline{\psi})$ for $S(B)$.

Definition 2.3. Set

$$\mathbf{S}(\{B\}) = \bigcup_{B' \in \{B\}} S(B') = \bigcup_{U \in \mathrm{GL}_n(\mathfrak{o}_K)} S(B[U]).$$

The Gross-Keating invariant $\underline{a} = (a_1, a_2, \dots, a_n)$ of B is the greatest element of $\mathbf{S}(\{B\})$ with respect to the lexicographic order $\succ$ on $\mathbb{Z}^n_{\geq 0}$. The Gross-Keating invariant is denoted by $\mathrm{GK}(B)$. A sequence of length 0 is denoted by $\emptyset$. When B is the empty matrix, we understand $\mathrm{GK}(B) = \emptyset$.

By definition, the Gross-Keating invariant $\mathrm{GK}(B)$ is determined only by the equivalence class of B.

Definition 2.4. $B \in \mathcal{H}_n(\mathfrak{o})$ is optimal if $\mathrm{GK}(B) \in S(B)$.

Recall that $B \in \mathcal{H}_n(\mathfrak{o})$ is maximal if $B[U^{-1}] \in \mathcal{H}_n(\mathfrak{o})$ for some $U \in M_n(\mathfrak{o}_E)$, then $U \in \mathrm{GL}_n(\mathfrak{o}_E)$.

Proposition 2.1. *Suppose that* $B \in \mathcal{H}^{\mathrm{nd}}_n(\mathfrak{o})$. *Then* B *is maximal if and only if* $\mathrm{GK}(B) = (0, 0, \dots, 0)$.

For a non-decreasing sequence $\underline{a} = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$, we set

$$G_{\underline{a}} = \{g = (g_{ij}) \in \mathrm{GL}_n(\mathfrak{o}_E) \mid \ \mathrm{ord}(g_{ij}) \geq (a_j - a_i)/2, \text{ if } a_i < a_j\}.$$

Definition 2.5. For $\underline{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$, put

$$\mathcal{M}(\underline{a}) = \left\{ B = (b_{ij}) \in \mathcal{H}_n(\mathfrak{o}) \,\middle|\, \begin{array}{l} \mathrm{ord}(b_{ii}) \geq a_i, \\ \mathrm{ord}(\mathfrak{D} b_{ij}) \geq (a_i + a_j)/2, \end{array} \ (1 \leq i < j \leq n) \right\},$$

Note that for $B \in \mathcal{H}^{\mathrm{nd}}_n(\mathfrak{o})$, we have

$$\underline{a} \in S(B) \Longleftrightarrow B \in \mathcal{M}(\underline{a}).$$

3. Reduced forms

Let $\underline{a} = (a_1, \dots, a_n)$ be a non-decreasing sequence. We define n_s, n^*_s, and I_s for $s = 1, \dots, r$ is in the previous section. For an involution $\sigma \in \mathfrak{S}_n$, we set

$$\begin{aligned}
\mathcal{P}^0 &= \mathcal{P}^0(\sigma) = \{i \,|\, 1 \leq i \leq n, \ i = \sigma(i)\}, \\
\mathcal{P}^+ &= \mathcal{P}^+(\sigma) = \{i \,|\, 1 \leq i \leq n, \ a_i > a_{\sigma(i)}\}, \\
\mathcal{P}^{++} &= \mathcal{P}^{++}(\sigma) = \{i \,|\, 1 \leq i \leq n, \ a_i > a_{\sigma(i)} + 1\}, \\
\mathcal{P}^- &= \mathcal{P}^-(\sigma) = \{i \,|\, 1 \leq i \leq n, \ a_i < a_{\sigma(i)}\}, \\
\mathcal{P}^{--} &= \mathcal{P}^{--}(\sigma) = \{i \,|\, 1 \leq i \leq n, \ a_i + 1 < a_{\sigma(i)}\}.
\end{aligned}$$

For each block $I_1, \dots, I_r$, put

$$\mathcal{P}_s^0 = \mathcal{P}^0 \cap I_s, \quad \mathcal{P}_s^= = \mathcal{P}^= \cap I_s,$$
$$\mathcal{P}_s^+ = \mathcal{P}^+ \cap I_s, \quad \mathcal{P}_s^{++} = \mathcal{P}^{++} \cap I_s,$$
$$\mathcal{P}_s^- = \mathcal{P}^- \cap I_s, \quad \mathcal{P}_s^{--} = \mathcal{P}^{--} \cap I_s.$$

Definition 3.1. An involution $\sigma \in \mathfrak{S}_n$ is $\underline{a}$-admissible, if

$$\sum_{i=1}^{s} \sharp\mathcal{P}_i^{--} + \sum_{i=1}^{s} \sharp\mathcal{P}_i^0 \le \sum_{i=1}^{s} \sharp\mathcal{P}_i^{++} + 2$$

for $s = 1, \dots, r$.

Note that if $a_{s+1}^* > a_s^* + 1$ or $s = r$, then we have

$$\sum_{i=1}^{s} \sharp\mathcal{P}_i^{-} + \sum_{i=1}^{s} \sharp\mathcal{P}_i^0 \le \sum_{i=1}^{s} \sharp\mathcal{P}_i^{+} + 2$$

since

$$\begin{aligned}\sum_{i=1}^{s} \sharp\mathcal{P}_i^{-} - \sum_{i=1}^{s} \sharp\mathcal{P}_i^{+} =& \sharp\{i \,|\, a_i \le a_s^*,\ a_{\sigma(i)} > a_s^*\} \\ =& \sharp\{i \,|\, a_i \le a_s^*,\ a_{\sigma(i)} > a_s^* + 1\} \\ =& \sum_{i=1}^{s} \sharp\mathcal{P}_i^{--} - \sum_{i=1}^{s} \sharp\mathcal{P}_i^{++}\end{aligned}$$

in this case.

Lemma 3.1. *Let $\underline{a} \in \mathbb{Z}_{\ge 0}^n$ be a non-decreasing sequence and σ an $\underline{a}$-admissible involution. Then we have $\sharp\mathcal{P}^0 \le 2$. We also have $\sharp\mathcal{P}_s^{++} \le 2$ and $\sharp\mathcal{P}_s^{--} \le 2$ for $s = 1, \dots, r$.*

Proof. Put $\mathcal{Q}_s = \{i \in \mathcal{P}^{--} \,|\, a_i \le a_s^*,\ a_{\sigma(i)} > a_s^*\}$. Then we have

$$\sum_{i=1}^{s} \sharp\mathcal{P}_i^{--} - \sum_{i=1}^{s} \sharp\mathcal{P}_i^{++} = \sharp\mathcal{Q}_s$$

It follows that

$$\sharp\mathcal{Q}_s + \sum_{i=1}^{s} \sharp\mathcal{P}_i^0 \le 2$$

for $s = 1, \dots, r$. In particular, we have $\sharp\mathcal{P}^0 \le 2$. We also have $\sharp\mathcal{P}_s^{--} \le 2$, since $\mathcal{P}_s^{--} \subset \mathcal{Q}_s$. Note that if $i \in \mathcal{P}_s^{++}$, then we have $\sigma(i) \in \mathcal{Q}_{s-1}$. Hence we have $\sharp\mathcal{P}_s^{++} \le 2$. $\square$

For $B = (b_{ij}) \in \mathcal{H}_n(\mathfrak{o})$ and $1 \le i, j \le n$, we write $B_{(ij)} = \begin{pmatrix} b_{ii} & b_{ij} \\ \bar{b}_{ij} & b_{jj} \end{pmatrix}$.

Definition 3.2. $B = (b_{ij}) \in \mathcal{M}(\underline{a})$ is a reduced form of GK type $(\underline{a}, \sigma)$, if the following conditions (1), (2), (3), and (4) hold.

(1) For $i < j = \sigma(i)$, we have

$$\mathrm{GK}(B_{(ij)}) = (a_i, a_j), \qquad \xi_{B_{(ij)}} = 1.$$

(2) If $i \in \mathcal{P}^0 \cup \mathcal{P}^{--}$, then we have

$$\mathrm{ord}(b_{ii}) = a_i.$$

(3) Suppose that $i, j \in \mathcal{P}^0 \cup \mathcal{P}^{--}$ and that $i < j$. Suppose also that either $i \in \mathcal{P}^0$ or $\sigma(i) > j$. Then we have

$$\mathrm{GK}(B_{(ij)}) = (a_i, a_j), \qquad \xi_{B_{(ij)}} = -1.$$

(4) For $j \neq i, \sigma(i)$, we have

$$\mathrm{ord}(b_{ij}\mathfrak{D}) > \frac{a_i + a_j}{2}.$$

Theorem 3.1. *Suppose that $B \in \mathcal{H}_n(\mathfrak{o})$ is optimal and $\mathrm{GK}(B) = \underline{a}$. Then there exists $U \in G_{\underline{a}}$ and an $\underline{a}$-admissible involution σ such that $B[U]$ is a reduced form of GK type $(\underline{a}, \sigma)$.*

Theorem 3.2. *Suppose that $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$ and $\mathrm{GK}(B) = \underline{a} = (a_1, \ldots, a_n)$. Then we have*

$$\sum_{i=1}^{n} a_n = \Delta(B).$$

4. The modified Gross-Keating invariant MGK(B)

Let $\underline{a} \in \mathbb{Z}_{\geq 0}$ be a sequence which is not necessarily non-decreasing and $\sigma \in \mathfrak{S}_n$ an involution. We say that $(\underline{a}, \sigma)$ is a generalized GK type is there exists a permutation $\tau \in \mathfrak{S}$ such that $(a_{\tau(1)}, a_{\tau(2)}, \ldots, a_{\tau(n)}), \tau\sigma\tau^{-1})$ is a GK type. We also say that $B \in \mathcal{H}_n(\mathfrak{o})$ is a reduced form of generalized GK type $(\underline{a}, \sigma)$ if there exists a permutation $\tau \in \mathfrak{S}_n$ such that $B[P_\sigma]$ is a reduced form of GK type $(a_{\tau(1)}, a_{\tau(2)}, \ldots, a_{\tau(n)}), \tau\sigma\tau^{-1})$, where P_σ is the permutation matrix associated with σ.

Definition 4.1. Let $(\underline{a}, \sigma)$ be a generalized GK type. Put

$$c_i = \begin{cases} a_i & \text{if } i \notin \mathcal{P}^+ \\ a_i - 1 & \text{if } i \in \mathcal{P}^+. \end{cases}$$

Definition 4.2. A GK type $(\underline{a}, \sigma)$ is well-arranged if the following condition holds.

- If $i \in \mathcal{P}_s^+$ and $j \in \mathcal{P}_s \setminus \mathcal{P}_s^+$, then we have $i < j$.

Note that if $(\underline{a}, \sigma)$ is well-arranged, then $\underline{\tilde{a}}$ is a non-decreasing sequence.

Definition 4.3. Suppose that $(\underline{a}, \sigma)$ is a well-arranged GK type. We define the subgroup $G'_{\underline{a},\sigma} \subset \mathrm{GL}_n(\mathfrak{o}_E)$ by

$$G'_{\underline{a},\sigma} = \left\{ g = (g_{ij}) \;\middle|\; \begin{array}{l} g \in \mathrm{GL}_n(\mathfrak{o}_E), \\ \mathrm{ord}(g_{ij}) \geq (\tilde{a}_j - \tilde{a}_i)/2, \text{ if } \tilde{a}_i < \tilde{a}_j, \\ \mathrm{ord}(g_{ij}) \geq 1/2, \text{ if } \tilde{a}_i = \tilde{a}_j,\, i \in \mathcal{P}^{=}, j \notin \mathcal{P}^{=} \end{array} \right\}.$$

Then $U \in G'_{\underline{a},\sigma}$ if and only if U stabilizes $\mathcal{K}_0^+, \mathcal{K}_1^+, \ldots$.

Let $I_1, \ldots, I_r$ be the blocks. Put

$$\mathcal{P}_s^{+\square} = \mathcal{P}_s^{+\square}(\sigma) = \mathcal{P}_s^{+} \setminus \mathcal{P}_s^{++} = \{ i \in \mathcal{P}_s \mid a_{\sigma(i)} = a_i - 1 \},$$
$$\mathcal{P}_s^{-\square} = \mathcal{P}_s^{-\square}(\sigma) = \mathcal{P}_s^{-} \setminus \mathcal{P}_s^{--} = \{ i \in \mathcal{P}_s \mid a_{\sigma(i)} = a_i + 1 \},$$

for $s = 1, \ldots, r$. Then we have

$$I_s = \mathcal{P}_s^{+\square} \sqcup \mathcal{P}_s^{++} \sqcup \mathcal{P}_s^{=} \sqcup \mathcal{P}_s^{0} \sqcup \mathcal{P}_s^{--} \sqcup \mathcal{P}_s^{-\square}.$$

Let $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_{\tilde{r}}$ be the block for the non-decreasing sequence $\underline{\tilde{a}}$. We also set $\tilde{n}_s = \sharp \tilde{I}_s$ and $\tilde{n}_s^* = \tilde{n}_1 + \cdots + \tilde{n}_s$ for $s = 1, \ldots, \tilde{r}$. For $s = 1, \ldots, \tilde{r}$, define $\sigma^{(\tilde{n}_s^*)} \in \mathfrak{S}_{\tilde{n}_s^*}$ by

$$\sigma^{(\tilde{n}_s^*)}(i) = \begin{cases} i & \text{if } \sigma(i) > \tilde{n}_s^*, \\ \sigma(i) & \text{otherwise.} \end{cases}$$

Then $(\underline{a}^{(\tilde{n}_s^*)}, \sigma^{(\tilde{n}_s^*)})$ is a standard GK type.

Suppose that $B \in \mathcal{H}_n(\mathfrak{o})$ is a reduced form of standard GK type $(\underline{a}, \sigma)$. Then $B^{(\tilde{n}_s^*)}$ is a reduce form of GK type $(\underline{a}^{(\tilde{n}_s^*)}, \sigma^{(\tilde{n}_s^*)})$.

Theorem 4.1. *Suppose that $B, B' \in \mathcal{H}_n(\mathfrak{o})$ are mutually equivalent reduced form of GK type $(\underline{a}, \sigma)$ and $(\underline{a}, \sigma')$, respectively. We assume both σ and σ' are standard $\underline{a}$-admissible involutions. Then we have $G'_{\underline{a},\sigma} = G'_{\underline{a},\sigma'}$. Moreover, if $B' = B[U]$ with $U \in \mathrm{GL}_n(\mathfrak{o}_E)$, then we have $U \in G'_{\underline{a},\sigma}$.*

***Corollary* 1.** The sequence $\underline{c} = (c_1, c_2, \ldots, c_n)$ depends only on the equivalence class of B.

We call $\underline{c}$ the modified Gross-Keating invariant of B. It is denoted by $\mathrm{MGK}(B)$.

Theorem 4.2. *Suppose that $B \in \mathcal{H}_n(\mathfrak{o})$ is a reduced form of GK type $(\underline{a}, \sigma)$. Then we have $\mathrm{GK}(B) = \underline{a}$. In particular, B is optimal.*

Theorem 4.3. *Suppose that $B \in \mathcal{H}_n(\mathfrak{o})$ is a reduced form of GK type $(\underline{a}, \sigma)$.*

(1) *If $\tilde{n}_s^*$ is even, then $\xi(B^{\tilde{n}_s^*})$ depends only on the equivalence class of B.*
(2) *If $\tilde{n}_s^*$ is odd and if $c_{s+1} \geq c_s + \mathrm{ord}(\mathbf{D})$, then $\xi(B^{\tilde{n}_s^*})$ depends only on the equivalence class of B.*

5. A conjecture on the Siegel series

Let $\psi : F \to \mathbb{C}^\times$ be an additive character of order 0.
For $B \in \mathcal{H}_n^{\mathrm{nd}}(\mathfrak{o})$ we define the Siegel series $b(B, s)$ by

$$b(B,s) = \int_{R \in \mathrm{Her}_n(E)} \psi(\mathrm{tr}(BR))[R\mathfrak{o}_E^n + \mathfrak{o}_E^n : \mathfrak{o}_E^n]^{-s/2}\, dR.$$

This integral is convergent for $\mathrm{Re}(s) \gg 0$, and is analytically continued to the whole s-plane. Put

$$\gamma_{E/F}(X) = \prod_{i=0}^{[(n-1)/2]} (1 - q^{2i}X).$$

Then there exists a unique polynomial $F(B, X)$ in X such that

$$b(B,s) = F(B, q^{-s})\gamma_{E/F}(q^{-s})$$

We then define a Laurent polynomial $\widetilde{F}(B, X)$ by

$$\widetilde{F}(B,X) = X^{e_B} F(B, q^{-n}X^{-2}).$$

It is known that the following functional equation holds.

$$\widetilde{F}(B, X^{-1}) = \xi(B)^{n-1}\widetilde{F}(B,X).$$

Conjecture 5.1. The Laurent polynomial $\tilde{F}(B, X)$ obtained from the Siegel series for B is determined by $\mathrm{GK}(B)$, $\mathrm{MGK}(B)$, and $\{\xi(B^{\tilde{n}_s^*})\}_{\tilde{n}_s^* \text{ is even}}$.

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