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On the computation of ramified Siegel series of degree 3

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1 Siegel Eisenstein series

Let $\Gamma^g = Sp(g, \mathbb{Z})$ be the symplectic group of rank g, i.e. matrix size $2g$. For an integer l,

$$\Gamma_0^g(l) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^g \,\middle|\, C \equiv 0 \bmod l \right\}, \quad \Gamma_\infty^g = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma^g \right\},$$

are the subgroups of Γ^g. Let ψ be a Dirichlet character modulo l. If $l = p$ is an odd prime, we denote $\psi = \chi_0$ the trivial character, and $\psi = \chi_p$ the quadratic character. We fix a positive integer k such that $\psi(-1) = (-1)^k$. Then we define the Siegel Eisenstein series of degree g, level l with character ψ by

$$E_{k,l,\psi}^g(Z) := \sum_{\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty^g \backslash \Gamma_0^g(l)} \psi(\det D) \det(CZ + D)^{-k}.$$

Here $Z \in \mathbb{H}_g := \{Z \in \mathrm{Sym}^g(\mathbb{C}) \mid \mathrm{Im}(Z) > 0\}$. The right hand side converges if $k > g + 1$, and $E_{k,l,\psi}^g$ is an element of $M_k(\Gamma_0^g(l), \overline{\psi})$ the space of Siegel modular forms of weight k level l with character $\overline{\psi}$. Siegel Eisenstein series is one of the most important objects among the Siegel modular forms.

Aim. *We want to have an explicit formula of the Fourier coefficients of* $E_{k,l,\psi}^g$.

This aim is quite simple and natural, however the answer is not so easy. In the full modular case ($l = 1$ case), under the various contribution of many mathematicians, finally Katsurada gave the explicit formula [Ka]. For the case of $l > 1$, known results are as follows.

- Mizuno ([Mi, 2009]) $g = 2$, l: square-free odd, ψ: primitive

- G. ([Gu, 2015]) $g=2$, $l=p$: odd prime, ψ: primitive
- Takemori ([Ta1, 2012]) $g=2$, l: any integer, ψ: primitive
- Takemori ([Ta2, 2015]) g: arbitrary, l: odd, $\psi=\prod_{p|l}\psi_p$ is primitive, $\psi_p\neq\chi_p$.

Mizuno calculated the Fourier coefficients of Siegel Eisenstein series of degree 2, by using the Maass lift of the Eisenstein series of Jacobi forms. The author calculated the Euler p-factor of the Fourier coefficients of Siegel Eisenstein series of degree 2 and level p, for an odd prime p. By the same method, Takemori [Ta1] calculated the Fourier coefficients for an any level l. Moreover in [Ta1], he found quite simple expressions, that is an important progression. Based on this result, he got the explicit formula for an arbitrary degree g ([Ta2]). In this paper, first he constructed some type of Siegel modular forms, whose Fourier coefficients are quite simple. Then he showed that such Eisenstein series indeed coincides with our $E^g_{k,l,\psi}$.

The results above are the case of primitive characters. In the case of the trivial character, one of the remarkable results is due to Böcherer([Bö]). Let $U(p)$ be the Hecke operator of level p acting on $M_k(\Gamma_0^g(p),\overline{\psi})$ so that $\sum C(N)\mathbf{e}(NZ)\mapsto\sum C(pN)\mathbf{e}(NZ)$. Then Böcherer showed

$$E^g_{k,p,\chi_0}(Z)\in\langle U(p^i)\big(E^g_{k,1}(Z)\big)\mid 0\leq i\leq g-1\rangle_{\mathbb{C}},$$

here $E^g_{k,1}$ is the Siegel Eisenstein series of level 1. Thus the Fourier coefficients of E^g_{k,p,χ_0} is reduced to that of $E^g_{k,1}$, that is already known by Katsurada. However finding the coefficients of the linear combination has an another difficulty and it is not yet known.

Similarly by looking at the action of $U(p)$ operator precisely, Dickson ([Di]) gave the explicit formula of the Fourier coefficients of E^2_{k,l,χ_0} for a square-free level l. Moreover, he also calculated all the Siege Eisenstein series, associated with each cusp. We remark that their methods works well only for the square-free level, since acting $U(p)$ for any times, we can get only the modular forms of level p, not level p^n.

The main result of this article is to give an explicit formula of the Fourier coefficients of $E^3_{k,p,\psi}$, for an odd prime p and primitive character ψ. If ψ is not quadratic, this is a part of the results of Takemori ([Ta2]), we mainly consider the case $\psi=\chi_p$.

2 ramified Siegel series

Let $E^g_{k,l,\psi}$ be the Siegel Eisenstein series of degree g, weight k, level l with character ψ. We write the Fourier expansion

$$E^g_{k,l,\psi}(Z) = \sum_{\substack{N \in \mathrm{Sym}^g(\mathbb{Z})^* \\ N \geq 0}} C(N)\mathbf{e}(NZ).$$

Here $\mathrm{Sym}^g(\mathbb{Z})^*$ denots the set of half integral matrices, and we put $\mathbf{e}(M) = \exp(2\pi\sqrt{-1}\,\mathrm{Tr}(M))$ for a square matrix M.

To describe the Fourier coefficients $C(N)$, we define the Siegel series with character. For that we need to prepare some notations. Let

$$\mathcal{M}_g = \{(C,D) \in M_g(\mathbb{Z})^{\oplus 2} \mid C,\ D \text{ is symmetric and co-prime}\}.$$

Here (C,D) is symmetric if $C\,{}^tD = D\,{}^tC$, and (C,D) is co-prime if there exist $X, Y \in M_g(\mathbb{Z})$ such that $CX + DY = 1_g$. If we put

$$\mathcal{M}^g_g = \{(C,D) \in \mathcal{M}_g \mid \det C \neq 0\},$$

we have a bijection

$$GL_g(\mathbb{Z})\backslash\mathcal{M}^g_g \to \mathrm{Sym}^g(\mathbb{Q}) \quad (C,D) \mapsto C^{-1}D.$$

For a fixed integer l, we put

$$\mathrm{Sym}^g(\mathbb{Q})' := \{C^{-1}D \in \mathrm{Sym}^g(\mathbb{Q}) \mid (C,D) \in \mathcal{M}^g_g,\ C \equiv 0 \bmod l\}.$$

For $T = C^{-1}D \in \mathrm{Sym}^g(\mathbb{Q})$, we define $\delta(T) = |\det C|$, $\mu(T) = \det(T)\delta(T)$.

Definition. Let ψ be a Dirichlet character modulo l. For $s \in \mathbb{C}$ and $N \in \mathrm{Sym}^g(\mathbb{Q})$, we define the Siegel series with character ψ by

$$S_g(\psi, N, s) := \sum_{T \in \mathrm{Sym}^g(\mathbb{Q})' \bmod 1} \psi(\nu(T))\delta(T)^{-s}\mathbf{e}(TN).$$

The right hand side converges when $\mathrm{Re}(s) \gg 0$.

Now we consider the Fourier coefficients $C(N)$ of $E^g_{k,l,\psi}$. It suffices to treat the case of $N > 0$, since if $\mathrm{rank}\, N = r < g$, then $C(N)$ comes from the Fourier coefficients of $E^r_{k,l,\psi}(Z)$.

Proposition 2.1. *For $N > 0$, we have*

$$C(N) = \widetilde{\xi}(N,k) S_g(\psi, N, k)$$

with

$$\widetilde{\xi}(N,k) = \frac{2^{-g(g-1)/2}(-2\pi i)^{gk}}{\Gamma_g(k)} (\det N)^{k-(g+1)/2}.$$

Here we set $\Gamma_g(k) = \pi^{g(g-1)/4} \prod_{i=0}^{g-1} \Gamma(k - i/2)$.

The calculation of $\widetilde{\xi}(N,k)$ is due to Siegel [Si]. Next we show that the Siegel series has an Euler product expression. For each prime number p, let $\mathrm{Sym}^g(\mathbb{Q})_p = \bigcup_{n \geq 0} \frac{1}{p^n} \mathrm{Sym}^g(\mathbb{Z})$. If $p \mid l$ we put

$$\mathrm{Sym}^g(\mathbb{Q})'_p = \mathrm{Sym}^g(\mathbb{Q})' \cap \mathrm{Sym}^g(\mathbb{Q})_p.$$

Then $T \in \mathrm{Sym}^g(\mathbb{Q})$ has a decomposition $T = \sum_{q:\text{prime}} T_q$ with $T_q \in \mathrm{Sym}^g(\mathbb{Q})_q$, uniquely modulo $\mathrm{Sym}^g(\mathbb{Z})$. Let $l = \prod_{i=1}^r p_i^{e_i}$. Then $T \in \mathrm{Sym}^g(\mathbb{Q})'$ if and only if $T_{p_i} \in \mathrm{Sym}^g(\mathbb{Q})'_{p_i}$ for all i. If $T = \sum_{p_i \mid l} T_{p_i} + \sum_{q_i \nmid l} T_{q_i} \in \mathrm{Sym}^g(\mathbb{Q})'$, we have

$$\delta(T) = \prod_{p_i} \delta(T_{p_i}) \prod_{q_i} \delta(T_{q_i}), \quad \nu(T) \equiv \prod_{p_i \mid l} \nu(T_{p_i}) \prod_{q_i \nmid l} \delta(T_{q_i}) \bmod l.$$

Thus we have

$$S_g(\psi, N, s) = \prod_{q:\text{prime}} S_g^q(\psi, N, s), \quad \text{with}$$

$$S_g^q(\psi, N, s) = \begin{cases} \displaystyle\sum_{T \in \mathrm{Sym}^g(\mathbb{Q})_q \bmod 1} \psi(\delta(T))\, \delta(T)^{-s} \mathbf{e}(TN) & \text{if } q \nmid l, \\ \displaystyle\sum_{T \in \mathrm{Sym}^g(\mathbb{Q})'_q \bmod 1} \psi(\nu(T))\, \delta(T)^{-s} \mathbf{e}(TN) & \text{if } q \mid l. \end{cases}$$

$S_g^q(\psi, N, s)$ with $q \mid l$ is called the *ramified Siegel series*, that is the main topics of this article.

Remark. The usual Siegel series (without character) is defined by

$$S_g^q(N, s) = \sum_{T \in \mathrm{Sym}^g(\mathbb{Q})_q \bmod 1} \delta(T)^{-s} \mathbf{e}(TN),$$

whose explicit formula is given in [Ka]. It is known that $S_g^q(N, s) = P(q^{-s})$ with a rational function $q(X)$. Then for $q \nmid l$ we have $S_g^q(\psi, N, s) = P(\psi(q) q^{-s})$.

The remark above shows that it suffices to compute the ramified Siegel series $S_g^p(\psi, N, s)$ with $p \mid l$. It is convenient to regard that Siegel series are defined locally. Assume $p \mid l$ and $e = \mathrm{ord}_p\, l$. We set the notations over $\mathbb{Q}_p$ or $\mathbb{Z}_p$ as follows. Let

$$\mathcal{M}_g^g(\mathbb{Z}_p) = \{(C, D) \in M_g(\mathbb{Z}_p) \mid (C, D) \text{ is symmetric co-prime, } \det C \neq 0\},$$

then $GL_g(\mathbb{Z}_p)\backslash\mathcal{M}_g^g(\mathbb{Z}_p) \simeq \mathrm{Sym}^g(\mathbb{Q}_p)$. For $T = C^{-1}D \in \mathrm{Sym}^g(\mathbb{Q}_p)$, we set $\delta(T) = p^{\mathrm{ord}_p(\det C)}$, $\nu(T) = \delta(T)\det(T)$. Let

$$\mathrm{Sym}^g(\mathbb{Q}_p)' = \{C^{-1}D \mid (C, D) \in \mathcal{M}_g^g(\mathbb{Z}_p),\ C \equiv 0 \bmod p^e\}.$$

The Dirichlet character ψ is extended to the character of $\mathbb{Z}_p$ by composing the natural surjection $\mathbb{Z}_p \to \mathbb{Z}/p^e$. Finally $\mathbf{e}_p$ is defined by $\mathbf{e}_p(X) = \exp(2\pi i \varphi(\mathrm{Tr}(X)))$ for $X \in M_g(\mathbb{Z}_p)$ with the natural isomorphism

$$\varphi : \mathbb{Q}_p/\mathbb{Z}_p \simeq \bigcup_{n \geq 0} \frac{1}{p^n}\mathbb{Z}/\mathbb{Z}.$$

Then we have

$$S_g^p(N, \psi, s) = \sum_{T \in \mathrm{Sym}^g(\mathbb{Q}_p)' \bmod \mathbb{Z}_p} \psi(\nu(T))\,\delta(T)^{-s}\mathbf{e}_p(TN).$$

To compute the ramified Siegel series, we rewrite them again using the symmetric co-prime pair. Let

$$\mathcal{M}_g(p^e) = \{(C, D) \in \mathcal{M}_g^g(\mathbb{Z}_p) \mid C \equiv 0 \bmod p^e,\ \det C = p^i\ (i \geq 1)\},$$

then

$$SL_g(\mathbb{Z})\backslash\mathcal{M}_g(p^e) \to \mathrm{Sym}^g(\mathbb{Q}_p)' \quad (C, D) \mapsto C^{-1}D$$

is bijective. Since the co-prime condition is not easy to handle, we set

$$\widetilde{\mathcal{M}}_g(p^e) = \{(C, D) \mid (C, D) \text{ is symmetric, } C \equiv 0 \bmod p^e,\ \det C = p^i\ (i \geq 1)\}.$$

Lemma 2.2. *Assume that $(C, D) \in M_g^g(\mathbb{Z}_p)$ is symmetric pair and* $\det C \neq 0$. *Then there exists $M \in M_g(\mathbb{Z}_p)$ such that*

$$C = MC', \quad D = MD', \quad (C', D') \in \mathcal{M}_g^g(\mathbb{Z}_p).$$

By this lemma, we have

$$S_g^p(\psi, N, s) = \sum_{\substack{(C,D)\in SL_g(\mathbb{Z}_p)\backslash\widetilde{\mathcal{M}}_g(p^e)\\ D \bmod C}} \psi(\det D)(\det C)^{-s}\mathbf{e}_p(C^{-1}DN). \quad (2.1)$$

Indeed, if $(C, D) \in \widetilde{\mathcal{M}}_g(p^e)$ is not co-prime, then we have $C = MC'$, $D = MD'$. Since $\det C$ is p-power, we have $\det M \equiv 0 \bmod p$, thus $\det D$ is also divisible by p. Because of the term $\psi(\det D)$, the contribution of such pair (C, D) is 0.

3 Calculation for degree 3 case

From now on, we assume $g = 3$, $l = p$ is an *odd* prime, and ψ is a primitive Dirichlet character modulo p. Since

$$S_g^p(\psi, N[U], s) = \psi(\det U)^{-2}S_g^p(\psi, N, s), \quad U \in GL_g(\mathbb{Z}_p),$$

we may assume N is a diagonal form. Thus we consider the case

$$N = p^m\begin{pmatrix}\alpha & & \\ & \beta p^r & \\ & & \gamma p^{r+t}\end{pmatrix}, \quad (p, \alpha\beta\gamma) = 1. \quad (3.1)$$

Let $\Lambda = SL_3(\mathbb{Z}_p)$ and $\Lambda^\gamma := \gamma^{-1}\Lambda\gamma$ for $\gamma \in GL_3(\mathbb{Q}_p)$. We put

$$\tau_{ijk} = \begin{pmatrix}p^i & & \\ & p^{i+j} & \\ & & p^{i+j+k}\end{pmatrix}.$$

Then for $(C, D) \in \widetilde{\mathcal{M}}_g^g(p)$, C is contained in $\Lambda\tau_{ijk}\Lambda$ for some i, j, k ($i \geq 1$, $j, k \geq 0$). C runs the set $\Lambda\backslash\Lambda\tau_{ijk}\Lambda$, there is a bijection

$$\Lambda\backslash\Lambda\tau_{ijk}\Lambda \simeq \Lambda \cap \Lambda^{\tau_{ijk}}\backslash\Lambda, \quad \tau_{ijk}Y \mapsto Y.$$

Let $\Xi_{ijk} := \Lambda \cap \Lambda^{\tau_{ijk}}\backslash\Lambda$. For $C = \tau_{ijk}Y$ with $Y \in \Xi_{ijk}$, if we write $D = \widetilde{D}\,{}^tY^{-1}$, then

$$(C, D) \text{ is symmetric} \iff \tau_{ijk}^{-1}\widetilde{D} \text{ is a symmetric matrix.}$$

Since

$$\mathbf{e}_p(C^{-1}DN) = \mathbf{e}_p(Y^{-1}\tau_{ijk}^{-1}\widetilde{D}^tY^{-1}N) = \mathbf{e}_p(\tau_{ijk}^{-1}\widetilde{D}\,N[Y^{-1}]),$$

as a consequence we can rewrite (2.1) to

$$S_3^p(\psi, N, s) = \sum_{i=1}^{\infty} \sum_{j,k=0}^{\infty} p^{-(3i+2j+k)s} \times \sum_{Y \in \Xi_{ijk}} \sum_{\widetilde{D} \bmod \tau_{ijk}} \psi(\det \widetilde{D})\, \mathbf{e}_p(\tau_{ijk}^{-1} \widetilde{D}\, N[Y^{-1}]). \tag{3.2}$$

In order to describe Ξ_{ijk}, we prepare some notations. Let $\mathfrak{S}_3$ be the symmetric group of degree 3, that is the Weyl group of GL_3. For $\sigma \in \mathfrak{S}_3$, let $(s_{ij})_{1 \le i,j \le 3}$ be the corresponding matrix in $O(3)$, that is

$$s_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j), \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is also denoted by σ. Note that

$$\sigma^{-1} \operatorname{diag}(a_1, a_2, a_3) \sigma = \operatorname{diag}(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}).$$

We write the elements of $\mathfrak{S}_3$ as

$$\sigma_1 = \mathrm{id},\ \sigma_2 = (2,3),\ \sigma_3 = (1,2),\ \sigma_4 = (1,2,3),\ \sigma_5 = (1,3,2),\ \sigma_6 = (1,3).$$

We set $\Xi_{ijk}^{-1} = \{Y^{-1} \mid Y \in \Xi_{ijk}\}$, since only Y^{-1} appears in (3.2).

Lemma 3.1. *The representative set Ξ_{ijk}^{-1} is given as follows.*

(1) *If $j = k = 0$, then $\Xi_{i00}^{-1} = \{1_3\}$.*

(2) *If $j \ge 1$ and $k = 0$, then $\Xi_{ij0}^{-1} = \coprod_{l=1}^{3} \mathcal{I}_l$ with*

$$\mathcal{I}_1 = \left\{ \begin{pmatrix} 1 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, u, v \in \mathbb{Z}/p^j \right\},$$

$$\mathcal{I}_2 = \left\{ \sigma_3 \begin{pmatrix} 1 & pu & v \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \,\middle|\, \begin{matrix} u \in \mathbb{Z}/p^{j-1} \\ v \in \mathbb{Z}/p^j \end{matrix} \right\},$$

$$\mathcal{I}_3 = \left\{ \sigma_5 \begin{pmatrix} 1 & pu & pv \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, u, v \in \mathbb{Z}/p^{j-1} \right\}.$$

(3) *If $j=0$ and $k\geq 1$, then* $\Xi_{i0k}^{-1}=\coprod_{l=1}^{3}\mathcal{J}_l$ *with*

$$\mathcal{J}_1=\left\{\begin{pmatrix}1&0&u\\0&1&v\\0&0&1\end{pmatrix}\;\middle|\; u,v\in\mathbb{Z}/p^k\right\},$$
$$\mathcal{J}_2=\left\{\sigma_2\begin{pmatrix}1&0&u\\0&1&pv\\0&0&-1\end{pmatrix}\;\middle|\; \begin{matrix}u\in\mathbb{Z}/p^k\\ v\in\mathbb{Z}/p^{k-1}\end{matrix}\right\},$$
$$\mathcal{J}_3=\left\{\sigma_4\begin{pmatrix}1&0&pu\\0&1&pv\\0&0&1\end{pmatrix}\;\middle|\; u,v\in\mathbb{Z}/p^{k-1}\right\}.$$

(4) *If $j,k\geq 1$, then* $\Xi_{ijk}^{-1}=\coprod_{l=1}^{6}\mathcal{K}_l$ *with*

$$\mathcal{K}_1=\left\{\begin{pmatrix}1&u&w\\0&1&v\\0&0&1\end{pmatrix}\;\middle|\; \begin{matrix}u\in\mathbb{Z}/p^j\\ v\in\mathbb{Z}/p^k\\ w\in\mathbb{Z}/p^{j+k}\end{matrix}\right\},$$
$$\mathcal{K}_2=\left\{\sigma_2\begin{pmatrix}1&u&w\\0&1&pv\\0&0&-1\end{pmatrix}\;\middle|\; \begin{matrix}u\in\mathbb{Z}/p^j\\ v\in\mathbb{Z}/p^{k-1}\\ w\in\mathbb{Z}/p^{j+k}\end{matrix}\right\},$$
$$\mathcal{K}_3=\left\{\sigma_3\begin{pmatrix}1&pu&w\\0&1&v\\0&0&-1\end{pmatrix}\;\middle|\; \begin{matrix}u\in\mathbb{Z}/p^{j-1}\\ v\in\mathbb{Z}/p^k\\ w\in\mathbb{Z}/p^{j+k}\end{matrix}\right\},$$
$$\mathcal{K}_4=\left\{\sigma_4\begin{pmatrix}1&u&pw\\0&1&pv\\0&0&1\end{pmatrix}\;\middle|\; \begin{matrix}u\in\mathbb{Z}/p^j\\ v\in\mathbb{Z}/p^{k-1}\\ w\in\mathbb{Z}/p^{j+k-1}\end{matrix}\right\},$$
$$\mathcal{K}_5=\left\{\sigma_5\begin{pmatrix}1&pu&pw\\0&1&v\\0&0&1\end{pmatrix}\;\middle|\; \begin{matrix}u\in\mathbb{Z}/p^{j-1}\\ v\in\mathbb{Z}/p^k\\ w\in\mathbb{Z}/p^{j+k-1}\end{matrix}\right\},$$
$$\mathcal{K}_6=\left\{\sigma_6\begin{pmatrix}1&pu&pw\\0&1&pv\\0&0&-1\end{pmatrix}\;\middle|\; \begin{matrix}u\in\mathbb{Z}/p^{j-1}\\ v\in\mathbb{Z}/p^{k-1}\\ w\in\mathbb{Z}/p^{j+k-1}\end{matrix}\right\}.$$

Now we have a decomposition

$$S_3^p(\psi,N,s)=S(1_3)+\sum_{l=1}^{3}S(\mathcal{I}_l)+\sum_{l=1}^{3}S(\mathcal{J}_l)+\sum_{l=1}^{6}S(\mathcal{K}_l)$$

with

$$S(1_3) = \sum_{i=1}^{\infty} p^{-3is} \sum_{D \in \mathrm{Sym}^3(\mathbb{Z}/p^i)} \psi(\det D) \mathbf{e}_p(\frac{1}{p^i} DN),$$

$$S(\mathcal{I}_l) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p^{-(3i+2j)s} \sum_{Y \in \mathcal{I}_l} \sum_{\widetilde{D} \bmod \tau_{ij0}} \psi(\det \widetilde{D}) \mathbf{e}_p(\tau_{ij0}^{-1} \widetilde{D} N[Y]),$$

$$S(\mathcal{J}_l) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} p^{-(3i+k)s} \sum_{Y \in \mathcal{J}_l} \sum_{\widetilde{D} \bmod \tau_{i0k}} \psi(\det \widetilde{D}) \mathbf{e}_p(\tau_{i0k}^{-1} \widetilde{D} N[Y]),$$

$$S(\mathcal{K}_l) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} p^{-(3i+2j+k)s} \sum_{Y \in \mathcal{K}_l} \sum_{\widetilde{D} \bmod \tau_{ijk}} \psi(\det \widetilde{D}) \mathbf{e}_p(\tau_{ijk}^{-1} \widetilde{D} N[Y]).$$

3.1 Calculation of $S(1_3)$

For the calculation of $S(1_3)$, we use the following theorem.

Theorem 3.2 (H. Saito [Sa, Theorem 1.3, Theorem 2.3]). *Let p be a prime number. For $N \in \mathrm{Sym}^g(\mathbb{F}_p)$ and the Dirichlet characger ψ modulo p, we define*

$$W_g^g(N, \psi) = \sum_{T \in \mathrm{Sym}^g(\mathbb{F}_p)} \psi(\det T) \mathbf{e}(\frac{1}{p} NT).$$

Then we have an explicit formula of $W_g^g(N, \psi)$.

The existence of this theorem is informed to the author by professor Hayashida in Joetsu University of education. Thanks to this theorem, we can compute $S(1_g)$ for any degree g.

For $N \in \mathrm{Sym}^g(\mathbb{Z})^*$, we put $N = p^m N'$, $N' \not\equiv 0 \bmod p$. Then

$$S(1_g) = \sum_{i=1}^{\infty} p^{-igs} \sum_{T \in \mathrm{Sym}^g(\mathbb{Z}/p^i)} \psi(\det T) \mathbf{e}\left(\frac{1}{p^{i-m}} T N'\right).$$

Decompose $T = T_1 + pT_2$ with $T_1 \in \mathrm{Sym}^g(\mathbb{F}_p)$, $T_2 = \mathrm{Sym}^g(\mathbb{Z}/p^{i-1})$, then we have

$$S(1_g) = \sum_{i=1}^{\infty} p^{-igs} \sum_{T_1 \in \mathrm{Sym}^g(\mathbb{F}_p)} \psi(\det T_1) \mathbf{e}\left(\frac{1}{p^{i-m}} T_1 N'\right)$$
$$\times \sum_{T_2 \in \mathrm{Sym}^g(\mathbb{Z}/p^{i-1})} \mathbf{e}\left(\frac{1}{p^{i-m-1}} T_2 N'\right).$$

The summation for T_2 vanishes when $i-m-1>0$, thus we may assume $i \le m+1$. If $i \le m$ then $\mathbf{e}\left(\frac{1}{p^{i-m}}T_1N'\right)=1$, thus

$$
\begin{aligned}
S(1_g) &= \sum_{i=1}^{m+1} p^{-igs+g(g+1)(i-1)/2} \sum_{T_1 \in \mathrm{Sym}^g(\mathbb{F}_p)} \psi(\det T_1)\mathbf{e}\left(\frac{1}{p^{i-m}}T_1N'\right) \\
&= \sum_{i=1}^{m} p^{-igs+g(g+1)(i-1)/2} \sum_{T_1 \in \mathrm{Sym}^2(\mathbb{F}_p)} \psi(\det T_1) \\
&\quad + p^{-(m+1)gs+g(g+1)m/2} \sum_{T_1 \in \mathrm{Sym}^g(\mathbb{F}_p)} \psi(\det T_1)\mathbf{e}\left(\frac{1}{p}T_1N'\right).
\end{aligned}
$$

The second line equals to $W_g^g(N',\psi)$. For the computation of the term $\sum \psi(\det T_1)$, it is regarded the case of $N=0$ in Theorem 3.2, or we can compute it by using the order of the orthogonal group over finite field.

Our case of degree 3 are as follows. Let $G(\psi)$ be the Gauss sum for a Dirichlet character ψ. We put

$$
\varepsilon_p = \begin{cases} 1 & p \equiv 1 \bmod 4, \\ \sqrt{-1} & p \equiv 3 \bmod 4. \end{cases}
$$

Proposition 3.3. *Let p be an odd prime and ψ a primitive Dirichlet character modulo p. For $N = p^m \operatorname{diag}(\alpha, p^r\beta, p^{r+t}\gamma)$ we have the following.*

(1) If $\psi \neq \chi_p$,

$$
S(1_3) = \begin{cases} p^{(6-3s)m-3s}\overline{\psi}(\alpha\beta\gamma)G(\chi_p)^3G(\psi)G(\psi\chi_p) & \text{if } r=t=0, \\ 0 & \text{otherwise.} \end{cases}
$$

(2) If $\psi = \chi_p$,

$$
S(1_3) = \begin{cases} -\chi_p(\alpha\beta\gamma)\varepsilon_p p^{(6-3s)m+5/2-3s} & \text{if } r=t=0 \\ 0 & \text{if } r=0,\ t>0, \\ \chi_p(-\alpha)(p-1)p^{(6-3s)m+7/2-3s} & \text{if } r>0. \end{cases}
$$

Remark. Even in the case of higher level, for example level $l=p^e$, similar arguments holds if the Dirichlet character ψ comes from the character modulo p. On the other hand if ψ is primitive in level p^e, we need the result of $W_g^g(\psi, N)$, similar to Theorem 3.2, for $\mathrm{Sym}^g(\mathbb{Z}/p^e)$. However in [Sa], Saito calculated $W_g^g(N,\psi)$ using the Bruhat decompsition of T. Thus it seems difficult to extend the result of [Sa] to the case of $\mathbb{Z}/p^e$.

3.2 Contribution for the other terms

For the remaining terms, we have the following.

Lemma 3.4. *For $l \geq 2$, we have*

$$S(\mathcal{I}_l) = S(\mathcal{K}_l) = 0.$$

By this lemma, it suffices to consider $S(\mathcal{I}_1)$, $S(\mathcal{J}_l)$ $(1 \leq l \leq 3)$ and $S(\mathcal{K}_1)$. To calculate theses terms, we use Theorem 3.2 and the following lemma.

Lemma 3.5. (1) *If $(\lambda, p) = 1$,*

$$\sum_{x \in \mathbb{Z}/p^n} \mathbf{e}\left(\frac{\lambda x^2}{p^n}\right) = \begin{cases} p^{n/2} & n \text{ is even} \\ \varepsilon_p\, \chi_p(\lambda) p^{n/2} & n \text{ is odd.} \end{cases}$$

(2) *Let $i \geq 1$ and $(\lambda, p) = 1$. If ψ is a primitive Dirichlet character modulo p, then*

$$\sum_{a \in \mathbb{Z}/p^i} \psi(a) \mathbf{e}\left(\frac{\lambda a}{p^{i-m}}\right) = \begin{cases} p^m \overline{\psi}(\lambda) G(\psi) & i = m+1 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand if $\psi = \chi_0$,

$$\sum_{a \in \mathbb{Z}/p^i} \chi_0(a) \mathbf{e}\left(\frac{\lambda a}{p^{i-m}}\right) = \begin{cases} (p-1)p^{i-1} & i \leq m \\ -p^m & i = m+1 \\ 0 & i \geq m+2. \end{cases}$$

Now we explain the calculation of $S(\mathcal{K}_1)$. Since $\tau_{ijk}^{-1} \widetilde{D}$ is symmetric, it is of the form

$$\tau_{ijk}^{-1} \widetilde{D} = p^{-i} \begin{pmatrix} a & d & f \\ * & p^{-j} b & p^{-j} e \\ * & * & p^{-(j+k)} c \end{pmatrix},$$

here $*$ means that it is a symmetric matrix. Then $\det \widetilde{D} \equiv abc \bmod p$ and $a, \ldots, f$ run

$$a, d, f \in \mathbb{Z}/p^i, \quad b, e \in \mathbb{Z}/p^{i+j}, \quad c \in \mathbb{Z}/p^{i+j+k}.$$

Thus we have

$$S(\mathcal{K}_1) = \sum_{i,j,k} \sum_{a,\ldots,f} \sum_{u,v,w} p^{-(3i+2j+k)s} \psi(a)\psi(b)\psi(c)$$

$$\times\, \mathbf{e}\left(\frac{1}{p^{i-m}} \begin{pmatrix} \alpha & u\alpha & w\alpha \\ * & u^2\alpha + p^r \beta & \alpha u w + p^r \beta v \\ * & * & w^2 \alpha + v^2 p^r \beta + p^{r+t} \gamma \end{pmatrix} \begin{pmatrix} a & d & f \\ * & p^{-j} b & p^{-j} e \\ * & * & p^{-(j+k)} c \end{pmatrix}\right)$$

with $i, j, k \geq 1$, $u \in \mathbb{Z}/p^j$, $v \in \mathbb{Z}/p^k$ and $w \in \mathbb{Z}/p^{j+k}$.

By Lemma 3.5 (2), the summation for a remains only when $i = m+1$. Then the summation for b or d vanish if u or w are co-prime to p, respectively. Thus we change $u \mapsto pu$ and $w \mapsto pw$, and we have

$$S(\mathcal{K}_1) = \overline{\psi}(\alpha)G(\psi)p^{-3(m+1)s+3m+2}\sum_{j,k}\sum_{b,e,c}\sum_{u,v,w} p^{-(2j+k)s}\psi(b)\psi(c)$$
$$\times \mathbf{e}\left(\frac{1}{p^{j+1}}\begin{pmatrix} p^2u^2\alpha + p^r\beta & \alpha p^2uw + p^r\beta v \\ * & w^2\alpha + v^2p^r\beta + p^{r+t}\gamma \end{pmatrix}\begin{pmatrix} b & e \\ * & p^{-k}c \end{pmatrix}\right)$$

with

$$j, k \in \mathbb{Z}_{\geq 1}, \quad \begin{cases} b, e \in \mathbb{Z}/p^{j+m+1} \\ c \in \mathbb{Z}/p^{j+k+m+1} \end{cases}, \quad \begin{cases} u \in \mathbb{Z}/p^{j-1} \\ v \in \mathbb{Z}/p^k \\ w \in \mathbb{Z}/p^{j+k-1}. \end{cases}$$

Next the summation for u and w are given by

$$\begin{aligned} U &= \sum_{u \in \mathbb{Z}/p^{j-1}} \sum_{w \in \mathbb{Z}/p^{j+k-1}} \mathbf{e}\left(\frac{\alpha}{p^{j-1}}(u^2b + 2uwe + w^2p^{-k}c)\right) \\ &= \sum_{u \in \mathbb{Z}/p^{j-1}} \sum_{w \in \mathbb{Z}/p^{j+k-1}} \mathbf{e}\left(\frac{\alpha b(u + b^{-1}ew)^2}{p^{j-1}} + \frac{\alpha(c - p^kb^{-1}e^2)w^2}{p^{j+k-1}}\right). \end{aligned}$$

By lemma 3.5 (1), U depends weather j and k are even or odd. The result is

$$U = \begin{cases} \varepsilon_p^2\chi_p(bc)p^{j+k/2-1} & j,\ k \text{ are even} \\ \varepsilon_p\chi_p(\alpha b)p^{j+k/2-1} & j \text{ is even, } k \text{ is odd} \\ p^{j+k/2-1} & j \text{ is odd, } k \text{ is even} \\ \varepsilon_p\chi_p(\alpha c)p^{j+k/2-1} & j,\ k \text{ are odd.} \end{cases} \tag{3.3}$$

We decompose $S(\mathcal{K}_1) = \sum_{\mu,\nu=0}^{1} S_{\mu\nu}$, so that in $S_{\mu\nu}$, j and k run satisfying $(j,k) \equiv (\mu,\nu) \bmod 2$.

Continuing to calculate in a similar way, we get the final results. Note that if ψ is not quadratic character, $S_{\mu\nu}$ remains only when $(\mu,\nu) \equiv (r,t) \bmod 2$ and $S_{\mu\nu}$ becomes a single term. On the other hand if $\psi = \chi_p$, more complicated term appears, because the character χ_p in (3.3) cancels with the original Dirichlet character $\psi = \chi_p$, and the trivial character χ_0 appears in the summation. Thus Lemma 3.5 (2) shows that the summation becomes complicated.

3.3 final results

We state the results of the ramified Siegel series for $\psi = \chi_p$ case.

Theorem 3.6. *Let p be an odd prime number and $N = p^m \operatorname{diag}(\alpha, p^r\beta, p^{r+t}\gamma)$.*

(1) *If r is even,*

$$\begin{aligned}
S_3^p(\chi_p, N, s) &= \chi_p(-\alpha)\varepsilon_p(p-1)p^{(3-s)m-1/2-s} \\
&\times \left\{ (p^2-1)\sum_{i=1}^{m+1} p^{(3-2s)i} \sum_{j=0}^{r/2-1} p^{(5-2s)j} + (1-p^{(5/2-s)r}) \sum_{i=1}^{m+1} p^{(3-2s)i} \right. \\
&\left. + p^{(3-2s)(m+1)+1} \left(\sum_{j=1}^{r-1} p^{(4-2s)j} + (p-1)\sum_{j=1}^{r/2-1} p^{(8-4s)j} \sum_{k=0}^{r/2-1-j} p^{(5-2s)k} \right) \right\} \\
&+ \chi_p(-\gamma)\chi_p(-\alpha\beta)^{t+1}\varepsilon_p p^{(3-s)m+(5/2-s)r+(2-s)t-1/2-s} \\
&\times \left\{ (p-1)\sum_{i=1}^{m+r/2} p^{(3-2s)i} - p^{(3-2s)(m+r/2+1)} \right\}.
\end{aligned}$$

(2) *If r is odd*

$$\begin{aligned}
S_3^p(\chi_p, N, s) &= \chi_p(-\alpha)\varepsilon_p(p-1)p^{(3-s)m-1/2-s} \\
&\times \left\{ (p^2-1)\sum_{i=1}^{m+1} p^{(3-2s)i} \sum_{j=0}^{r/2-1} p^{(5-2s)j} + (1+\chi_p(\alpha\beta)p^{(5/2-s)r+1/2}) \sum_{i=1}^{m+1} p^{(3-2s)i} \right. \\
&\left. + p^{(3-2s)(m+1)+1} \left(\sum_{j=1}^{r-1} p^{(4-2s)j} + (p-1)\sum_{j=1}^{(r-1)/2} p^{(8-4s)j} \sum_{k=0}^{(r-1)/2-j} p^{(5-2s)k} \right) \right\} \\
&+ \chi_p(-\beta)\varepsilon_p p^{(3-s)m+(5/2-s)r-1-s} \\
&\times \left\{ (p-1)\sum_{k=0}^{t-1} \left(\chi_p(\alpha\beta)p^{2-s}\right)^k - \left(\chi_p(\alpha\beta)p^{2-s}\right)^t \right\} \\
&\times \left\{ (p-1)\sum_{i=1}^{m+(r+1)/2} p^{(3-2s)i} + \chi_p(\alpha\beta)p^{(3-2s)(m+r/2+1)+1/2} \right\}.
\end{aligned}$$

We note that the final formula contains the term coinsides with the degree 2 ramified Siegel seires $S_2^p(\chi_p, \widetilde{N}, s)$ with $\widetilde{N} = p^m \operatorname{diag}(\alpha, p^r\beta)$ (cf. [Ta1, Proposition 3.1]).

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