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POWERS OF THE DEDEKIND ETA FUNCTION AND HURWITZ POLYNOMIALS

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ABSTRACT. In this talk, we study the vanishing properties of Fourier coefficients of powers of the Dedekind eta function. We give a certain type of classification of this property. Further we extend the results of Atkin, Cohen, and Newman for odd powers and a list Serre presented in 1985. The topic is intimately related with Hurwitz polynomials. We also indicate possible generalization of the Lehmer conjecture. This talk contains joint work with Florian Luca, Atsushi Murase, Markus Neuhauser, Florian Rupp and Alexander Weisse.

1. INTRODUCTION

This survey is an extension of a talk given at the RIMS Workshop: Analytic and Arithmetic Theory of Automorphic Forms (15.01-19.01.2018 in Kyoto). Recent approaches and results towards the vanishing properties of the Fourier coefficients of powers of the Dedekind eta function had been presented. This contains joint work with Florian Luca, Atsushi Murase, Markus Neuhauser, Florian Rupp and Alexander Weisse [HM11, He16, HNR17, HLN18, HNR18, HN18a, HN18b, HNW18].

In his celebrated paper [Se85] Serre proved that the r-th power of the Dedekind eta function η (r even) is lacunary iff $r \in S_{\text{even}} := \{2, 4, 6, 8, 10, 14, 26\}$. For $r = 24$ Lehmer conjectured that the Fourier coefficients of the discriminant function $\Delta := \eta^{24}$ never vanish. It has always been a challenge in mathematics to understand the correspondence between multiplicative and additive structures.

In this paper we put these results and conjectures in a wider picture allowing $r \in \mathbb{C}$. We further connect the underlying structure with a family of recursively defined polynomials $P_n(x)$. The roots of these polynomials dictate the vanishing of the n-th Fourier coefficients.

Euler and Jacobi already found remarkable identities.

$$\prod_{n=1}^{\infty} (1 - X^n) = \sum_{n=-\infty}^{\infty} (-1)^n \, X^{\frac{3n^2+n}{2}}, \tag{1.1}$$

$$\prod_{n=1}^{\infty} (1 - X^n)^3 = \sum_{n=0}^{\infty} (-1)^n \, (2n+1) \, X^{\frac{n^2+n}{2}}. \tag{1.2}$$

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It is useful to reformulate these results in terms of the Dedekind eta function η, studied first by Dedekind. This makes it possible to apply the theory of modular forms, which includes the Hecke theory. Let τ be in the upper half space $\mathbb{H} := \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$ and $q := e^{2\pi i \tau}$. Dedekind introduced in 1877 the modular form of half-integral weight 1/2:

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)\,, \tag{1.3}$$

We are interested in the vanishing properties of the Fourier coefficients $a_r(n)$ defined by

$$\eta(\tau)^r := q^{\frac{r}{24}} \prod_{n=1}^{\infty} (1 - q^n)^r = q^{\frac{r}{24}} \sum_{n=0}^{\infty} a_r(n)\, q^n. \tag{1.4}$$

Note $r = 1$ and $r = 3$ are given by the examples of Euler and Jacobi. Hence the $a_1(n)$ and $a_3(n)$ vanish, if n is not represented by a given quadratic form (this can be made more precise) for each case. Such forms are denoted superlacunary [OS95]. Actually η^r $(r \in \mathbb{Z})$ is superlacunary iff $r \in S_{\text{odd}} := \{1, 3\}$ (see [OS95]). For $(-r) \in \mathbb{N}$ all coefficients $a_r(n)$ are positive integers. In particular $a_1(n) = p(n)$ are the partition numbers. Even and odd powers of η lead to modular forms of integral and half-integral weight. Hence we study them separately (see also [HNW18] introduction).

Acknowledgment

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2. Even Powers

Let r be even, then Serre [Se85] proved that $\eta(\tau)^r$ is lacunary, i.e.

$$\lim_{N \to \infty} \frac{|\{n \in \mathbb{N} \mid n \leq N,\, a_r(n) \neq 0\}|}{N} = 0, \tag{2.1}$$

if and only if $r \in S_{\text{even}} := \{2, 4, 6, 8, 10, 14, 26\}$. Lehmer conjectured that the coefficients $\tau(n)$ of the discriminant function never vanish.

$$\Delta(\tau) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)\, q^n. \tag{2.2}$$

Note that $\tau(n) := a_{24}(n-1)$ is called the Ramanujan function. Ono [On95] indicated that η^{12} has similar properties as Δ. This covers more or less the results covered in the literature, based on our knowledge. For example since η^{48} is not any more an Hecke eigenform it is not clear what to expect. Nevertheless we obtained the following recent result.

Theorem 2.1. [HNW18]
Let r be an even positive integer. Let $r \notin S_{even}$. Let $12 \leq r \leq 132$. Then $a_r(n) \neq 0$ for $n \leq 10^8$. Let $124 \leq r \leq 550$. Then $a_r(n) \neq 0$ for $n \leq 10^7$.

The result is obtained by numerical computations. The result suggest the prediction that there exists an $n \in \mathbb{N}$ such that $a_r(n) = 0$ iff $r \in S_{\text{even}} = \{2, 4, 6, 8, 10, 14, 26\}$. This would include the case $r = 24$, known as the Lehmer's conjecture [Le47]. Hence the Lehmer conjecture would only be the tip of an iceberg. We also show in the following that the case $r = 48$, the square of the discriminant function Δ, is closely connected to a conjecture by Maeda, although in this case we are not dealing with an Hecke eigenform.

Maeda's conjecture and Δ^2.
We extended our calculations and obtained:

Theorem 2.2. *Let $a_{48}(n)$ be the Fourier coefficients of Δ^2. Let*

$$\Delta^2(\tau) = \eta^{48}(\tau) = q^2 \sum_{n=0}^{\infty} a_{48}(n)\, q^n. \tag{2.3}$$

Then for $n \leq 5 \cdot 10^9$ all coefficients are different from zero.

Maeda's conjecture [HM97], [GM12]*:* Let $S_k = S_k(\mathrm{SL}_2(\mathbb{Z}))$ be the space of modular cusp form of weight integral weight k for the full modular group. We consider the action of the Hecke operator T_m $(m > 1)$ on the finite dimensional vector space S_k. Then the characteristic polynomial is irreducible over $\mathbb{Q}$. Further the Galois group of the splitting field is the full symmetric group of the largest possible size. In particular all eigenvalues are different.

The following observation seems to be worth mentioning.

Lemma 2.3. *The Fourier coefficients of Δ^2 are non-vanishing if and only if the eigenvalues of the eigenforms of $S_{24}(\Gamma)$ are different.*

See also [DG96, KK07, HNW18]. Hence Maeda's conjecture supports the non-vanishing of all Fourier coefficients of Δ^2. The record for checking Maeda's conjecture in this case has been $n \leq 10^5$ ([GM12]). Our result implies $n \leq 5 \cdot 10^9$.

3. Odd Powers

In the odd case Serre [Se85] published a table, based on results of partly unpublished results of Atkin, Cohen and Newman

Atkin, Cohen	$r = 5$	$n = 1560, 1802, 1838, 2318, 2690, \ldots$
Atkin	$r = 7$	$n = 28017$
Newman [Ne56]	$r = 15$	$n = 53$

For these pairs (r,n) one has $a_r(n)=0$. It is not mentioned how many pairs (r,n) were studied.

In [HNR17], we showed that for $r = 9,11,13,17,19,21,23$ that $a_r(n) \neq 0$ for $n \leq 50000$. Cohen and Strömberg ([CS17] ask among other things if η^5, η^{15} and η^7 have infinitely many vanishing coefficients and also ask about their vanishing asymptotic.

In the following we report on an extended version of Serre's table in the r and n aspect. In [HNW18] we gave an extension of the conjecture of Cohen and Strömberg and asymptotics for

$$|\{n \leq N |\ a_r(n) \neq 0\}|.$$

Throughout this section, let r be an odd positive integer. We briefly introduce the concept of sources based on the Hecke theory for modular forms of half-integral weight, before we state our results.

Let $f_r(\tau) := \eta(24\tau)^r$ with Fourier expansion

$$\begin{aligned} f_r(\tau) &= \sum_{D=1}^{\infty} b_r(D)\, q^D \\ \eta(\tau)^r &= \sum_{n=0}^{\infty} a_r(n)\, q^n. \end{aligned}$$

Proposition 3.1. *Let $1 \leq r < 24$ be an odd integer. Let $n_0 \in \mathbb{N}$ be given, such that $D_0 := 24n_0 + r$ satisfies $p^2 \not| D_0$ for all prime numbers $p \neq 2,3$. Let*

$$\mathcal{N}_r(n_0) := \left\{ n_0\, l^2 + r\left(l^2-1\right)/24 \mid l \in \mathbb{N},\ (l, 2\cdot 3) = 1 \right\}. \tag{3.1}$$

Let $a_r(n_0)=0$. Then $a_r(n) = 0$ for all $n \in \mathcal{N}_r(n_0)$. We call such n_0's sources.

Let further $3|r$ and $27 \not| D_0$ for the source n_0. Then $a_r(n) = 0$ is already true for all elements of

$$\left\{ n_0\, l^2 + r\left(l^2-1\right)/24 \mid l \in \mathbb{N},\ (l, 2) = 1 \right\}. \tag{3.2}$$

We refer to [HNR17] for more details. Note for $r=15$ (since $27 \not| D_0$), we obtain

$$\mathcal{N}_{15}(53) = \left\{ 53 + 429 \frac{l(l+1)}{2} \mid l \in \mathbb{N}_0 \right\}.$$

Theorem 3.2. [HNW18] *Let $r = 7,9,11$ and let $n \leq 10^{10}$. Then there exists among all possible pairs (r,n) with $a_r(n)=0$ exactly one source pair $(7, 28017)$. Let $13 \leq r \leq 27$ odd and $n \leq 10^9$. Then there is exactly one source pair $(15,53)$.*

Theorem 3.3. [HNW18] *Let r be odd and $29 \leq r < 550$ and $n \leq 10^7$. Then there exists no pair (r,n) such that $a_r(n) = 0$.*

Serre's table extented

r	Sources n_0	$\mathcal{N}_r(n_0)$	checked up to
5	$1560, 1802, \ldots$	$\{n_0 l^2 + 5 \cdot \frac{l^2-1}{24},\ (l, 2\cdot 3) = 1,\ l \in \mathbb{N}\}$	10^{10}
7	28017	$\{28017\, l^2 + 7\frac{l^2-1}{24},\ (l, 2\cdot 3) = 1,\ l \in \mathbb{N}\}$	10^{10}
9	–	$\emptyset$	10^{10}
11	–	$\emptyset$	10^{10}
13	–	$\emptyset$	10^{10}
15	53	$\{429\binom{l}{2} + 53,\ l \in \mathbb{N}\}$	10^{10}
$17 \leq r \leq 27$	–	$\emptyset$	10^9
$29 \leq r \leq 549$	–	$\emptyset$	10^8

For $r = 5$ we have the following distribution of sources.

$n \leq$	10^3	10^4	10^5	10^6	10^7	10^8	10^9	10^{10}
	0	19	70	235	579	1402	3052	6352

3.1. **Questions of Cohen and Strömberg.** Cohen and Strömberg ([CS17], Exercise 2.6) made the following conjectures: The Fourier expansion of η^5 and of η^{15} have infinitely many zero coefficients and perhaps even more than X^δ up to X for some $\delta > 0$ (perhaps any $\delta < 1/2$). The Fourier expansion of η^7 has infinitely many zero coefficients, perhaps of order $\log(X)$ up to X. We also refer to Ono ([On03], Problem 3.51).

The η^r for $r = 5, 7, 15$ are Hecke eigenforms. Further in all cases sources exist. Hence there are infinitely pairs (r, n) such that $a_r(n) = 0$.

We can answer both problems of Cohen and Strömberg in the following way. For a function $X \mapsto f(X)$ we use the Landau notation that it is $\Omega(g(X))$ if $\limsup_{X\to\infty} |f(X)/g(X)| > 0$.

Proposition 3.4. *The Fourier expansions of η^r for $r = 5, 7, 15$ have $\Omega\left(X^{1/2}\right)$ coefficients which are zero.*

More precisely the following holds.

(1) *For $r = 5$ there are more than $\frac{1}{119}\sqrt{X}$ coefficients zero if $X \geq 31615\,10466$.*
(2) *For $r = 7$ there are more than $\frac{1}{508}\sqrt{X}$ coefficients zero if $X \geq 10^{10}$.*
(3) *For $r = 15$ there are more than $\frac{1}{15}\sqrt{X}$ coefficients zero if $X \geq 96157$.*

Remark. The numerical data up to $n = 10^{10}$ for $r = 5$ seems to suggest that there is even a δ larger than $1/2$ such that $\Omega\left(X^\delta\right)$ coefficients are zero.

3.2. **Question of Ono.** Ono ([On03], Problem 3.51. Ono asked the opposite question in terms of Cohen and Strömberg. He inquired about the amount of non-vanishing coefficients for r odd and $r \geq 5$.

4. Roots of Polynomials and the Dedekind eta function

We introduce polynomials $P_n(x)$. The roots of these polynomial dictate the vanishing properties of the n-th Fourier coefficients of the attached power's of the Dedekind eta function. Gian-Carlo Rota (1985) said already:

"The one contribution of mine that I hope will be remembered has consisted in pointing out that all sorts of problems of combinatorics can be viewed as problems of the location of the zeros of certain polynomials...".

We start with the definition

$$\sum_{n=0}^{\infty} P_n(z)\, q^n = \prod_{n\geq 1} (1-q^n)^{-z} \quad (z \in \mathbb{C}). \tag{4.1}$$

Hence $P_0(x) = 1$ and $P_1(x) = x$. Let $P_n(x) = \frac{x}{n!}\widetilde{P}_n(x)$. Then $\widetilde{P}_n(x) \in \mathbb{Z}$, a normalized polynomial of degree $n-1$ with strictly positive coefficients. We also observe that $P_n(x)$ is integer-valued. In addition we recall the useful and well-known identity

$$\prod_{n\geq 1}(1-q^n) = \exp\left(-\sum_{n=1}^{\infty} \sigma(n)\,\frac{q^n}{n}\right). \tag{4.2}$$

Here $\sigma(n) := \sum_{d|n} d$. This essentially says that the logarithmic derivative of the Dedekind eta function is equal to the holomorphic Eisenstein series of degree 2.

Definition. *Let $g(n)$ be an arithmetic function. Let $P_0^g(x) := 1$. Then we define the polynomials $P_n^g(x)$ by:*

$$P_n^g(x) = \frac{x}{n}\left(\sum_{k=1}^{n} g(k) P_{n-k}^g(X)\right), \qquad n \geq 1. \tag{4.3}$$

Then $P_n^{\sigma}(x) = P_n(x)$. Since $\sigma(n)$ is a complicated function, one may use other arithmetic functions $g(n)$ to interpolate $P_n(x)$ by $P_n^g(x)$.

The first ten polynomials $P_n(x)$ appeared the first time in the work of Newman [Ne55] and Serre [Se85] (in a different notation). Let for example $n = 6$, then

$$P_5(x) = x(x+3)(x+6)\,R(x),$$

where $R(x)$ is irreducible over $\mathcal{Q}$. This implies that only the 5-th Fourier coefficient for $\eta^r, (r \in \mathbb{Z})$ is vanishing iff $r = 3$ or $r = 6$. It was already known by Newman that for $n < 5$ all roots of $P_n(x)$ are integral, but not for $5 \leq n \leq 10$.

4.1. **Root Distribution.** The following result [HNR18] displays the distribution of the roots for $n \leq 50$. We record the amount of roots which are integral, irrational and in $\mathbb{C} \setminus \mathbb{R}$ upto $n \leq 50$. For $n = 10$ the first time non-real roots appear. Since $P_n(x) \in \mathbb{R}$, with z also the complex conjugate of z is a root.

n	$\mathbb{Z}$	$\mathbb{R} \setminus \mathbb{Z}$	$\mathbb{C} \setminus \mathbb{R}$	n	$\mathbb{Z}$	$\mathbb{R} \setminus \mathbb{Z}$	$\mathbb{C} \setminus \mathbb{R}$
1	1	0	0	26	4	16	6
2	2	0	0	27	5	22	0
3	3	0	0	28	4	20	4
4	4	0	0	29	5	22	2
5	3	2	0	30	3	19	8
6	3	3	0	31	6	21	4
7	4	3	0	32	6	24	2
8	4	4	0	33	4	29	0
9	6	3	0	34	6	18	10
10	2	6	2	35	4	21	10
11	5	6	0	36	3	23	10
12	3	7	2	37	5	30	2
13	5	8	0	38	4	28	6
14	5	7	2	39	7	28	4
15	3	10	2	40	3	29	8
16	3	13	0	41	6	31	4
17	6	11	0	42	6	30	6
18	5	11	2	43	6	33	4
19	7	12	0	44	6	34	4
20	4	12	4	45	3	28	14
21	3	14	4	46	4	30	12
22	3	15	4	47	6	37	4
23	5	18	0	48	6	36	6
24	5	15	4	49	6	33	10
25	4	17	4	50	4	38	8

4.2. **Stable Polynomials.** We discovered that the polynomials $P_n(x)$ ($n \leq 700$) are stable [HNR18, HNW18]. More precise all the roots of $\widetilde{P}_(x)$ for $n \leq 700$ have the property that the real part is negative. By abuse of notation we also call $P_n(x)$

stable. Stable polynomials are also denoted Hurwitz polynomials. This property in general would imply that the real parts of the roots of $P_n(x)$ are bounded from above by $3n(n-1)/2$, since the real parts would have the same sign (for $n \leq 2$). This observation makes it also possible to study the roots of the polynomials with methods from the theory of dynamic systems and automatic control theory, where the stability of the underlying characteristic polynomial implies the stability of the system. Let $Q(x) \in \mathbb{R}[x]$. Then $Q(x)$ stability implies that all coefficients of

$$P(x) = \sum_{k=0}^{n} a_k \, x^k \qquad (a_n \neq 0)$$

are positive. The converse is not true.

It is remarkable that already 150 years ago, Maxwell ([Ma68, Ga05]) asked for a criterion to check the stability without calculating the roots. This has been given by Routh and Hurwitz independently. We state the Routh-Hurwitz criterion [Hu95]. A polynomial is stable if and only if the following matrix

$$H = \begin{pmatrix} a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\ a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\ 0 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ 0 & a_n & a_{n-2} & a_{n-4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has positive leading principal minors. Let $H = (h_{i,j})$. Then the lth leading principal minor is given by $\Delta_l := \det(H_l)$, where

$$H_l = (h_{i,j})_{i,j=1}^{l} .$$

It would be interesting to know if any roots (besides 0) are on the imaginary axes [HNR18].

Using algebraic methods and some analytic number theory gives

Theorem 4.1. [HLN18] *Let $P_n(x)$ be given. Let ξ be a root of unity and let $P_n(\xi) = 0$. Then $\xi = -1$.*

This implies for example that $P_n(i) \neq 0$.

4.3. **Simple roots.** It is not clear if the roots of $P_n(x)$ are simple, although our numerical calculations suggest that this true. Nevertheless we have the following results. Our first result on the derivatives of $P_n(x)$ is the following.

Theorem 4.2. [HN18a] *Let $n = p^m$, where p is a prime and $m \in \mathbb{N}$. Then*

$$P_n'(x_0) \in \mathbb{Q} \setminus \mathbb{Z} \tag{4.4}$$

for any integer x_0.

This implies that for n equal to powers of primes, every integral root is simple.

4.4. **Further examples for $g(n)$.**
Let us fix some notation.

$$P_n^g(X) = \frac{X}{n!} \sum_{k=0}^{n-1} A_k^n(g)\, X^k.$$

Let $\varphi_1(n) = n$ and $\varphi_2(n) = n^2$. In the following we study the properties of the associated polynomials $P_n^{\varphi_1}(X)$ and $P_n^{\varphi_2}(X)$. Their properties are related to $P_n(X) = P_n^{\sigma}(X)$, since $\varphi_1(n) < \sigma(n) < \varphi_2(n)$. for $n > 1$. We obtain $A_{n-1}^n(\varphi_1) = A_{n-1}^n = A_{n-1}^n(\varphi) = 1$. Let further $0 \leq k \leq n-2$, then

$$A_k^n(\varphi_1) < A_k^n < A_k^n(\varphi_2). \tag{4.5}$$

Theorem 4.3. *Let $\varphi_1(n) = n$. Then the coefficients of $P_n^{\varphi_1}(X)$ are given by*

$$A_k^n(\varphi_1) = \frac{n!}{(k+1)!} \binom{n-1}{k}. \tag{4.6}$$

Although the binomial coefficients are twisted by $1/(k+1)!$ they keep the log-concavity.

Corollary 4.4. *The sequence of the coefficients of $P_n^{\varphi_1}(X)$*

$$\{A_k^n(\varphi_1)\}_{k=0}^{n-1}$$

is strongly log-concave and hence unimodal.

Theorem 4.5. *Let $\varphi_2(n) = n^2$. Then the coefficients of $P_n^{\varphi_2}(X)$ are given by*

$$A_k^n(\varphi_2) = \frac{n!}{(k+1)!} \binom{n+k}{2k+1}. \tag{4.7}$$

Corollary 4.6. *The sequence of the coefficients of $P_n^{\varphi_2}(X)$*

$$\{A_k^n(\varphi_2)\}_{k=0}^{n-1}$$

is strongly log-concave and hence unimodal.

5. Discrete dynamic systems approach

Work in progress with Markus Neuhauser. Numerical calculations [HNR18] indicate that the root distribution of P_n^g for $g(n) = n, \sigma(n), n^2$ is quite complicated. Although these polynomials are stable for $n \leq 700$ a general proof seems to be far of reach. It is obvious that real roots are negative, but the general pattern is not understood yet. Even it is not clear yet that for infinitely many n non-real roots appear. Note that for example for $n = 10$ the first time non-real roots appear, but for $n = 33$ all roots are real. Hence we believe it is worth studying the polynomials $P_n^g(x)$ also via methods from discrete dynamic systems.

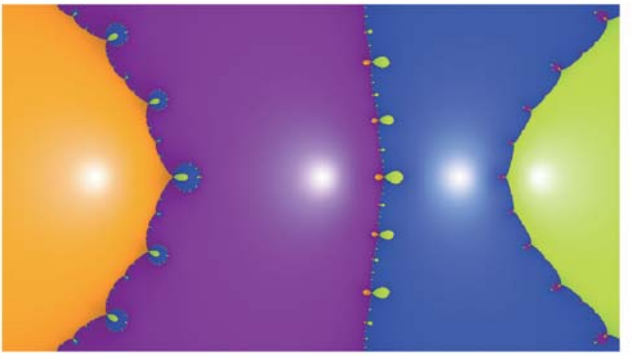

FIGURE 1. Basins of attraction associated to the Newton fractal of the polynomial $\widetilde{P}_5^{\varphi_1}(X)$.

6. Characterization of powers of the Dedekind eta function as Borcherds products

The Dedekind eta function is the simplest Borcherds product in the sense that the divisor is trivial on is characterizes units, and for the full modular group powers of the Dedekind eta function.

Theorem 6.1. *Let f be a modular form for* $\mathrm{SL}_2(\mathbb{Z})$ *of half-integral or integral weight k and possible multiplier system. Let f satisfy the functional equation*

$$f(p\tau)\prod_{a=1}^{p} f\left(\frac{\tau+a}{p}\right) = f(\tau)^{p+1} \tag{6.1}$$

for one prime number p then f is proportional to η^{2k}.

See [He16] for more results and proofs.

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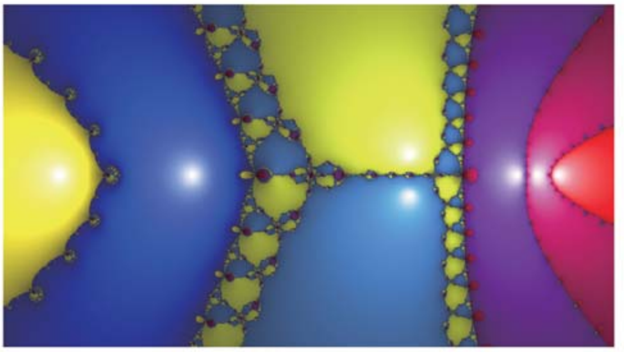

FIGURE 2. Basins of attraction associated to the Newton fractal of the polynomial $\widetilde{P_{10}^{\sigma}}(X)$. Note that only the first seven zeros are visible in the picture.

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