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Linear continuous operators acting on the space of entire functions of a given order

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Abstract

We consider the relationship between linear continuous operators acting on the space of entire functions of one variable of a given order and linear differential operators of infinite order satisfying certain growth conditions for the coefficients. We found that these two classes of operators are equivalent.

§ 1. Introduction

Let p and c be positive numbers. We denote by $A_{p,c}$ the set of all entire functions f of one variable z satisfying

$$||f||_c := \sup_{z\in\mathbb{C}} |f(z)| \exp(-c|z|^p) < \infty.$$

This set becomes a Banach space with the norm $||\cdot||_c$. If $c > c' > 0$, the natural inclusion map $A_{p,c} \hookrightarrow A_{p,c'}$ is compact. Hence we can consider the inductive limit of the family $\{A_{p,c}\}_{c>0}$ and denote it by A_p:

$$A_p := \varinjlim A_{p,c}.$$

This becomes a DFS space.

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Definition 1.1. ([1], Definition 2.3., [2]) Let p be a positive number. The set $\mathcal{D}_{p,0}$ consists of differential operators of infinite order of the form

$$P(z,\partial_z)=\sum_{n=0}^{\infty}a_n(z)\partial_z^n \tag{1.1}$$

satisfying:

(1) The coefficients $a_n(z)$ $(n=0,1,2,...)$ are entire functions.
(2) There exists a constant $B>0$ such that for every $\varepsilon>0$ one can take a constant $C_\varepsilon>0$ for which

$$|a_n(z)|\le C_\varepsilon\frac{\varepsilon^n}{(n!)^{\frac{1}{q}}}\exp(B|z|^p) \qquad (n=0,1,2,...)$$

holds, where $\dfrac{1}{p}+\dfrac{1}{q}=1$ and $\dfrac{1}{q}=0$ when $p=1$.

If $P\in\mathcal{D}_{p,0}$, P acts on A_p as a continuous linear operator:

Theorem 1.2. ([1], Theorem 2.4., [2], Theorem 2.3.) *Let $P\in\mathcal{D}_{p,0}$ and let $f\in A_p$. Then $Pf\in A_p$ and P is continuous on A_p, that is $Pf\to 0$ as $f\to 0$. Here we set*

$$Pf=\sum_{n=0}^{\infty}a_n(z)\frac{d^nf}{dz^n}$$

for P of the form (1.1).

Conversely, let F be linear continuous endomorphism in A_p. Then the following natural question arises: Does there exist an operator $P\in\mathcal{D}_{p,0}$ for which

$$F(f)=Pf$$

holds for any $f\in A_p$? In this article, we shall show that, to give an answer to this question, we need to introduce a new class of operators which is slightly larger than $\mathcal{D}_{p,0}$:

Definition 1.3. Let p be a positive number. The set $\boldsymbol{D}_p$ consists of differential operators of infinite order of the form

$$P(z,\partial_z)=\sum_{n=0}^{\infty}a_n(z)\partial_z^n \tag{1.2}$$

satisfying:

(1) The coefficients $a_n(z)$ $(n=0,1,2,...)$ are entire functions.

(2) For every $\varepsilon > 0$ one can take constants $C_\varepsilon > 0$ and $B_\varepsilon > 0$ for which

$$|a_n(z)| \leq C_\varepsilon \frac{\varepsilon^n}{(n!)^{\frac{1}{q}}} \exp(B_\varepsilon |z|^p) \qquad (n = 0, 1, 2, ...)$$

holds, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $\dfrac{1}{q} = 0$ when $p = 1$.

Theorem 1.4. *Let $p > 1$. Let F be a linear continuous endomorphism in A_p. Then there exists a unique operator $P \in \boldsymbol{D}_p$ such that $F(f) = Pf$ for all $f \in A_p$. Conversely, if P belongs to $\boldsymbol{D}_p$, then P induces a linear continuous endomorphism $f \mapsto Pf$ in A_p.*

§ 2. Proof of Theorem 1.4

Definition 2.1. Let $P(z, \partial_z) = \sum_{n=0}^{\infty} a_n(z) \partial_z^n$ be a formal differential operator of infinite order. The symbol of P is the formal power series of ζ obtained by replacing ∂_z by a variable ζ:

$$P(z, \zeta) = \sum_{n=0}^{\infty} a_n(z) \zeta^n.$$

Remark. Formally we have $P(z, \zeta) = e^{-z\zeta} P(z, \partial_z) e^{z\zeta}$.

Lemma 2.2. *We asuume $p > 1$. Let $P(z, \partial_z)$ be an element in $\boldsymbol{D}_p$ and $P(z, \zeta)$ the symbol of $P(z, \partial_z)$. Then $P(z, \zeta)$ is an entire function of (z, ζ) satisfying the following condition:*

For each $\varepsilon > 0$, there exist $B_\varepsilon > 0$ and $C_\varepsilon > 0$ such that
$|P(z, \zeta)| \leq C_\varepsilon \exp(B_\varepsilon |z|^p + \varepsilon |\zeta|^q)$ holds for all (z, ζ).

Conversely, if $P(z, \zeta) = \sum_{n=0}^{\infty} a_n(z) \zeta^n$ is an entire function of (z, ζ) satisfying the above condition, then

$$\sum_{n=0}^{\infty} a_n(z) \partial_z^n$$

belongs to $\boldsymbol{D}_p$.

Proof. It follows from (2) of Definition 1.3 that $|P(z, \zeta)|$ is dominated by

$$\begin{aligned} |P(z, \zeta)| &\leq \sum_{n=0}^{\infty} |a_n(z)| |\zeta|^n \\ &\leq C_\varepsilon \exp(B_\varepsilon |z|^p) \sum_{n=0}^{\infty} \frac{(\varepsilon |\zeta|)^n}{(n!)^{\frac{1}{q}}}. \end{aligned}$$

By using the inequality $(n!)^{\frac{1}{q}} \geq \Gamma\Big(\frac{n}{q}+1\Big)$ and the properties of the Mittag-Leffler function ([3]), we find that there exists $B'>0$ and $C'>0$ such that

$$|P(z,\zeta)| \leq C_\varepsilon \exp(B_\varepsilon|z|^p)\; C' \exp(B'\varepsilon^q|\zeta|^q) = C''_{\varepsilon'} \exp(B_{\varepsilon'}|z|^p + \varepsilon'|\zeta|^q).$$

Here we set $\varepsilon' = B'\varepsilon^q$ and $C''_{\varepsilon'} = C_\varepsilon C'$. Conversely,

$$\begin{aligned}
\left|\partial_\zeta^n P(z,\zeta)\right| &= \left|\frac{n!}{2\pi i}\int_{|\xi-\zeta|=s|\zeta|} \frac{P(z,\xi)}{(\xi-\zeta)^{n+1}}d\xi\right| \\
&\leq n!\frac{C_\varepsilon}{(s|\zeta|)^n}\exp(B_\varepsilon|z|^p + \varepsilon(s+1)^q|\zeta|^q) \\
&\leq n!\frac{C_\varepsilon}{(s|\zeta|)^n}\exp(B_\varepsilon|z|^p)\exp(2^q\varepsilon|\zeta|^q)\exp(2^q\varepsilon s^q|\zeta|^q)
\end{aligned}$$

for all $s>0$. Taking the minimum of the right-hand side of the above estimate with respect to s, we get

$$\left|\partial_\zeta^n P(z,\zeta)\right| \leq n!\; C_\varepsilon \exp(B_\varepsilon|z|^p)\exp(2^q\varepsilon|\zeta|^q)\Big(\frac{2^q\varepsilon q}{n}e\Big)^{\frac{n}{q}}. \tag{2.1}$$

Hence,

$$|a_n(z)| = \left|\left.\frac{\partial_\zeta^n P(z,\zeta)}{n!}\right|_{\zeta=0}\right| \leq C_{\varepsilon'}\exp(B_{\varepsilon'}|z|^p)\frac{(\varepsilon')^n}{(n!)^{\frac{1}{q}}}.$$

□

Lemma 2.3. *If $F: A_p \to A_p$ is linear continuous operator, there exist $a_n(z) \in A_p$ $(n=0,1,2,\ldots)$ such that $F(f) = \sum_{n=0}^{\infty} a_n(z)\partial_z^n f$ holds for all $f \in A_p$.*

Proof. We define $\{a_k(z)\}$ $(k=0,1,2,\ldots)$ recursively by

$$\begin{aligned}
a_0(z) &:= F(1), \\
a_k(z) &:= \frac{1}{k}\Big(F(z^k) - a_0(z)z^k - \cdots - (k-1)!a_{k-1}(z)z\Big) \qquad (k \geq 1).
\end{aligned}$$

Then,

$$\begin{aligned}
F(1) &= a_0(z), \\
F(z^k) &= a_0(z)z^k + \cdots + (k-1)!a_{k-1}(z)z + k!a_k(z).
\end{aligned}$$

We set $A_p \ni f = \sum_{k=0}^{\infty} f_k z^k$. Since F is a linear continuous operator, we obtain

$$
\begin{aligned}
F(f) &= \sum_{k=0}^{\infty} f_k F(z^k) \\
&= \sum_{n=0}^{\infty} a_n(z) \sum_{k=n}^{\infty} f_k \frac{k!}{(n-k)!} z^{k-n} \\
&= \sum_{n=0}^{\infty} a_n(z) z^n \sum_{k=0}^{\infty} f_k z^k .
\end{aligned}
$$

□

Proof of Theorem 1.4. We assume $F : A_p \to A_p$ is a linear continuous operator. Then, for all $c > 0$ there exists c' $(\geq c)$, there exists $C_c > 0$ for which

$$||F(f)||_{c'} \leq C_c ||f||_c \qquad (\forall f \in A_{p,c})$$

hold for any $f \in A_{p,c}$. From Lemma 2.3, there exist $a_n(z) \in A_p$ $(n = 0, 1, 2, ...)$ such that $F(f) = P(z, \partial_z)f := \sum_{n=0}^{\infty} a_n(z)\partial_z^n f$ holds for all $f \in A_p$. Let $P(z, \zeta)$ be the symbol of $P(z, \partial_z)$. We regard ζ as a complex parameter and we take the norm $||\cdot||_{c'}$ of $P(z, \zeta)$ as a function of z. Then we have

$$
\begin{aligned}
||P(z,\zeta)||_{c'} &= ||e^{-z\zeta} P e^{z\zeta}||_{c'} \\
&\leq ||e^{-z\zeta}||_{\frac{c'}{2}} ||P e^{z\zeta}||_{\frac{c'}{2}} \\
&\leq ||e^{-z\zeta}||_{\frac{c'}{2}} C_{\frac{c}{2}} ||e^{z\zeta}||_{\frac{c}{2}} \\
&\leq C_{\frac{c}{2}} \left(\sup_{z\in\mathbb{C}} \exp\big(|z||\zeta|\big) \exp\Big(-\frac{c'}{2}|z|^p\Big)\right)\left(\sup_{z\in\mathbb{C}} \exp\big(|z||\zeta|\big) \exp\Big(-\frac{c}{2}|z|^p\Big)\right) \\
&\leq C_{\frac{c}{2}} \exp\left(\frac{2}{q}\Big(\frac{2}{pc}\Big)^{\frac{1}{p-1}} |\zeta|^q \right).
\end{aligned}
$$

For any $\varepsilon > 0$, we take c so that $\frac{2}{p}\Big(\frac{2}{\varepsilon q}\Big)^{p-1} \leq c$ holds and write $C_\varepsilon = C_{\frac{\varepsilon}{2}}$. Then we have

$$||P(z,\zeta)||_{c'} \leq C_{\frac{\varepsilon}{2}} \exp\left(\frac{2}{q}\Big(\frac{2}{pc}\Big)^{\frac{1}{p-1}} |\zeta|^q \right) \leq C_\varepsilon \exp(\varepsilon|\zeta|^q)$$

If we write $B_\varepsilon = c'$, then we get

$$|P(z,\zeta)| \leq C_\varepsilon \exp(\varepsilon|\zeta|^q + c'|z|^p) = C_\varepsilon \exp(\varepsilon|\zeta|^q + B_\varepsilon |z|^p)$$

Then implies $P \in \boldsymbol{D}_p$.

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