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Hyperfunctions and Čech-Dolbeault cohomology in the microlocal point of view

By

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Abstract

In this note, we explain how to construct the boundary value map $b_\Omega : \mathscr{O}(\Omega) \to \mathscr{B}(M)$ of Sato's hyperfunction in the framework of Čech-Dolbeault cohomology.

§ 1. Introduction

The boudnary value map b_Ω is the most important morphism in the hyperfunction theory, by which we can understand a hyperfunction to be the sum of boundary values of holomorphic functions defined on wedges along a real analytic manifold. As was explained in [2] of this volume, the theory of Čech-Dolbeault cohomology brings several benefits to the treatment of a hyperfunction, and as such an important example, we here explain how to construct the boundary value morphism in our framework. For an application to the theory of Laplace hyperfunctions, see [3] in this volume.

This is a joint work with Takashi Izawa and Tatsuo Suwa.

§ 2. Boundary value morphism

Let M be a real analytic manifold assumed to be orientable and countable at infinity, and let X be its complexification. Note that, through the note, we use the same notations as those in [2] of this volume.

Let Ω be an open subset in X, for which we introduce the following two conditions:

(B_1) $\overline{\Omega} \supset M$.

(B_2) The inclusion $(X \smallsetminus \Omega) \smallsetminus M \hookrightarrow X \smallsetminus \Omega$ gives a homotopy equivalence.

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For example, a usual convex wedge Ω along M like a left shape in Fig. 1 satisfies the condition (B_2). However, the right shape in the same figure that is a wedge along M in which the smaller one is removed violates (B_2) because $(X \smallsetminus \Omega) \smallsetminus M$ consists of two connected components while $X \smallsetminus \Omega$ is connected.

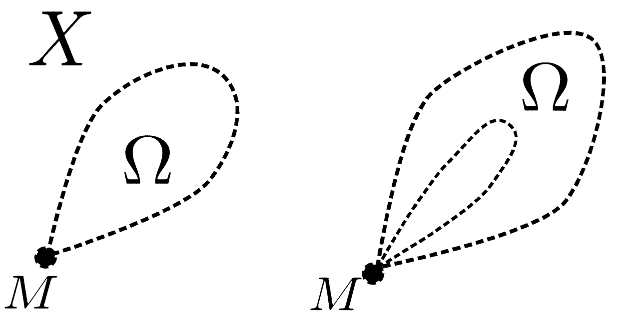

Figure 1. A good case (left) and bad case (right).

Set $\mathcal{W} = \{V_0 = X \smallsetminus M, V_1 = X\}$ and $\mathcal{W}' = \{V_0\}$. We also set $V_{01} = V_0 \cap V_1$ as usual. In what follows, we always assume that Ω satsifies the above two conditions. Then we will define the following boundary value map

$$\begin{aligned} b_\Omega : \mathscr{O}(\Omega) \longrightarrow \mathscr{B}(M) &= H^n_M(X; \mathscr{O}) \otimes_{\mathbb{Z}_M(M)} or_{M/X}(M) \\ &\simeq H^{0,n}_{\bar\vartheta}(\mathcal{W}, \mathcal{W}') \otimes_{\mathbb{Z}_M(M)} or_{M/X}(M) \end{aligned}$$

using Čech-Dolbeault cohomology.

As M is orientable, we can take a global section $\mathbb{1}$ in $or_{M/X}(M)$ which generates each stalk of the sheaf $or_{M/X}$ over $\mathbb{Z}$. We fix such a section $\mathbb{1}$ hereafter.

The canonical sheaf morphism $\mathbb{Z}_X \to \mathbb{C}_X$ induces the morphism of $\mathbb{Z}$-modules

$$or_{M/X}(M) = H^n_M(X; \mathbb{Z}_X) \hookrightarrow H^n_M(X; \mathbb{C}_X) = H^n_D(\mathcal{W}, \mathcal{W}'),$$

which is clearly injective. The image in $H^n_M(X; \mathbb{C}_X)$ of $\mathbb{1}$ by this morphism is still denoted by the same symbol in what follows. The following lemma is crucial to our construction of b_Ω.

Lemma 2.1. *Under the conditions* (B_1) *and* (B_2), *the* $\mathbb{1} \in H^n_D(\mathcal{W}, \mathcal{W}')$ *has a representative*

$$(\nu_1, \nu_{01}) \in \mathscr{E}^{(n)}(V_1) \oplus \mathscr{E}^{(n-1)}(V_{01}) = \mathscr{E}^{(n)}(\mathcal{W}, \mathcal{W}')$$

which satisfies $\mathrm{Supp}_{V_1}(\nu_1) \subset \Omega$ *and* $\mathrm{Supp}_{V_{01}}(\nu_{01}) \subset \Omega$ *(see Fig. 2 also).*

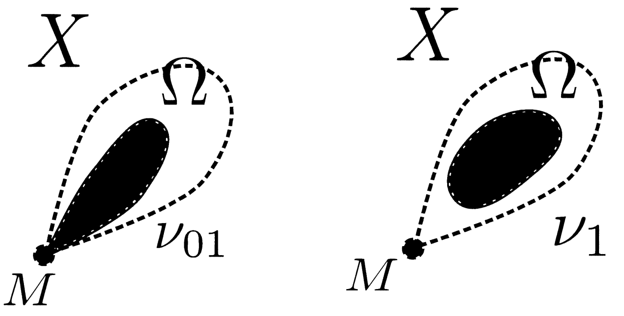

Figure 2. The support of ν_{01} and ν_1 indicated by black regions.

The canonical sheaf morphism $\iota : \mathbb{C}_X \to \mathscr{O}$ induces the canonical morphisms of $\mathbb{C}$-vector spaces:

$$H^k(\iota) : H^k_M(X;\mathbb{C}_X) \longrightarrow H^k_M(X;\mathscr{O}).$$

Its counterpart in the relative de Rham and relative Dolbeault cohomologies is, as was explained in Section 5 in [2], given by

$$\rho^k : \mathscr{E}^{(k)}(\mathcal{W},\mathcal{W}') \to \mathscr{E}^{(0,k)}(\mathcal{W},\mathcal{W}') \qquad \text{given by } \ (\omega_1,\omega_{01}) \mapsto (\omega_1^{(0,k)},\omega_{01}^{(0,k-1)}).$$

Hereafter we often write ρ instead of ρ^n. We give an example of a ν in the above lemma for a typical case.

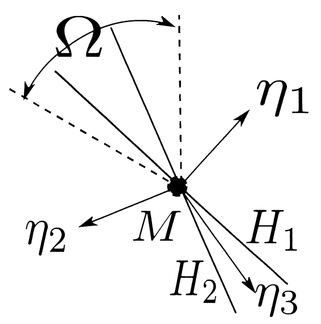

Figure 3. The typical picture of $n = 2$.

Example 2.2. Let $M = \mathbb{R}^n$, $X = \mathbb{C}^n$ and $\Omega = M \times \sqrt{-1}\Gamma$, where Γ is an open proper convex cone in $\mathbb{R}^n$. We first take n linearly independent unit vectors $\eta_1,\dots,\eta_n$ in $\mathbb{R}^n_y$ so that

$$\bigcap_{1\le k\le n} H_k \subset \Gamma$$

holds, where we set $H_k = \{ y \in \mathbb{R}^n_y \mid \langle y, \eta_k \rangle > 0 \}$. We also set

$$\eta_{n+1} = -(\eta_1 + \cdots + \eta_n).$$

Then, let φ_k, $k = 1, \dots, n+1$, be C^∞-functions on $X \setminus M$ which satisfy

(1) $\mathrm{Supp}_{X \setminus M}(\varphi_k) \subset M \times \sqrt{-1} H_k$ for any $k = 1, \dots, n+1$.

(2) $\displaystyle\sum_{k=1}^{n+1} \varphi_k = 1$ on $X \setminus M$.

Set

$$\nu_{01} = (-1)^n (n-1)!\, \hat{\chi}_{H_{n+1}}\, d\varphi_1 \wedge \cdots \wedge d\varphi_{n-1},$$

where $\hat{\chi}_{H_{n+1}}$ is the anti-characteristic function of the set H_{n+1}, that is,

$$\hat{\chi}_{H_{n+1}}(z) = \begin{cases} 0 & z \in H_{n+1}, \\ 1 & \text{otherwise}. \end{cases}$$

Then we can easily confirm that $\nu_{01} \in \mathscr{E}^{(n-1)}(X \setminus M)$ and

$$\mathrm{Supp}_{X \setminus M}(\nu_{01}) \subset M \times \sqrt{-1} \bigcap_{1 \le k \le n} H_k \subset \Omega.$$

Furthermore,

$$\nu = (0, \nu_{01}) \in \mathscr{E}^{(n)}(V_1) \oplus \mathscr{E}^{(n-1)}(V_{01}) = \mathscr{E}^{(n)}(\mathcal{W}, \mathcal{W}')$$

gives the image of a positively oriented generator of $or_{M/X}(M)$ under the standard orientations on M and X. Note that, by the definition of ρ, we have

$$\rho(\nu) = (0, (-1)^n (n-1)!\, \hat{\chi}_{H_{n+1}}\, \bar{\partial}\varphi_1 \wedge \cdots \wedge \bar{\partial}\varphi_{n-1}) \in \mathscr{E}^{(0,n)}(\mathcal{W}, \mathcal{W}').$$

Let us construct the boundary value map. We first take, thanks to Lemma 2.1, $\nu = (\nu_1, \nu_{01}) \in \mathscr{E}^n(\mathcal{W}, \mathcal{W}')$ which is a representative of $\mathbb{1} \in or_{M/X}(M)$ and satisfies

$$\mathrm{Supp}_X(\nu_1) \subset \Omega, \qquad \mathrm{Supp}_{X \setminus M}(\nu_{01}) \subset \Omega.$$

By tracing the image of $\mathbb{1}$ in the diagram below, we obtain $\rho(\nu)$ in $H^{0,n}_{\bar{\vartheta}}(\mathcal{W}, \mathcal{W}')$.

$$\begin{array}{ccccc}
or_{M/X} = H^n_M(X;\, \mathbb{Z}_X) \longrightarrow & H^n_M(X;\, \mathbb{C}_X) & \xrightarrow{H^n(\iota)} & H^n_M(X;\, \mathscr{O}) \\
& \wr\| & & \wr\| \\
& H^n_D(\mathcal{W}, \mathcal{W}') & \xrightarrow{\rho} & H^{0,n}_{\bar{\vartheta}}(\mathcal{W}, \mathcal{W}') \\
\Psi & \Psi & & \Psi \\
\mathbb{1} & \nu & & \rho(\nu)
\end{array}$$

Then, using $\rho(\nu)$, we define the boundary map

$$b_\Omega : \mathscr{O}(\Omega) \to H^{0,n}_{\bar\vartheta}(\mathcal{W}, \mathcal{W}') \underset{\mathbb{Z}(M)}{\otimes} or_{M/X}(M) = \mathscr{B}(M) \tag{2.1}$$

by, for $f \in \mathscr{O}(\Omega)$,

$$b_\Omega(f) := [f\rho(\nu)] \otimes \mathbb{1} \in H^{0,n}_{\bar\vartheta}(\mathcal{W}, \mathcal{W}') \underset{\mathbb{Z}(M)}{\otimes} or_{M/X}(M). \tag{2.2}$$

Lemma 2.3. *The above b_Ω is well-defined. That is, b_Ω does not depend on the choice of $\mathbb{1}$ and ν.*

Remark. Thus constructed $b_\Omega(\bullet)$ coincides with the original boundary value map by Sato-Kawai-Kashiwara [1].

The proposition below immediately comes from the definition:

Proposition 2.4. *Let $\Omega' \subset \Omega$ be an open subset in X. Assume that Ω' also satisfies the conditions* (B_1) *and* (B_2). *Then we have*

$$b_{\Omega'}(f|_{\Omega'}) = b_\Omega(f), \qquad f \in \mathscr{O}(\Omega).$$

We can easily estimate the microlocal analyticity of the hyperfunction $b_\Omega(f)$. Before stating the estimate, we give the characterization of microlocal analyticity of a hyperfunction in our framework. For an open subset V in X, we set

$$\mathcal{W}_V = \{V_0 = V \smallsetminus M,\, V_1 = V\} \quad \text{and} \quad \mathcal{W}'_V = \{V_0\}.$$

Proposition 2.5. *Let u be a hyperfunction at $x_0 \in M$. Then u is microlocally analytic at $p_0 = (x_0, \sqrt{-1}\xi_0) \in \sqrt{-1}T^*M$ if and only if there exist a closed cone $G \subset \mathbb{R}^n$ with the condition*

$$G \smallsetminus \{0\} \subset \{\, y \in \mathbb{R}^n \mid \mathrm{Re}\,\langle \sqrt{-1}y,\, \sqrt{-1}\xi_0 \rangle > 0 \,\} = \{\, y \in \mathbb{R}^n \mid \langle y,\, \xi_0 \rangle < 0 \,\},$$

an open neighborhood V of x_0 and a representative

$$(\tau_1, \tau_{01}) \in \mathscr{E}^{(0,n)}(V_1) \oplus \mathscr{E}^{(0,n-1)}(V_{01}) = \mathscr{E}^{(0,n)}(\mathcal{W}_V, \mathcal{W}'_V)$$

of u near x_0 which satisfies $\tau_1 = 0$ and $\mathrm{Supp}_{V\smallsetminus M}(\tau_{01}) \subset \mathbb{R}^n_x \times \sqrt{-1}G$.

Then it follows from the above two propositions that we have:

Theorem 2.6. *Let M be an open subset in $\mathbb{R}^n_x$ and $X = M \times \sqrt{-1}\mathbb{R}^n_y$. Assume that $\Omega \cap (\{x_0\} \times \sqrt{-1}\mathbb{R}^n_y)$ is a non-empty convex cone for any $x_0 \in M$. Then we have*

$$\mathrm{SS}(b_\Omega(f)) \subset \Omega^\circ, \qquad f \in \mathscr{O}(\Omega),$$

where Ω° is the polar set of Ω defined by

$$\bigsqcup_{x \in M} \{\, \sqrt{-1}\xi \in (T^*_M X)_x \mid \langle \xi, y \rangle \geq 0 \text{ for any } y \text{ with } (x, \sqrt{-1}y) \in \Omega \,\} \subset T^*_M X.$$

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[2] T. Suwa, *Relative Dolbeault cohomology and Sato hyperfunctions*, in this volume.
[3] K. Umeta, *Laplace hyperfunctions from the viewpoint of Čech-Dolbeault cohomology*, in this volume.