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An attempt to compute holonomic systems for Feynman integrals in two-dimensional space-time

By

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Abstract

We present some examples of holonomic systems for Feynman integrals associated with Feynman diagrams by using integration algorithms for D-modules.

§ 1. Introduction

We consider Feynman integrals associated with Feynman diagrams (see e.g., [1]). Microlocal analysis of Feynman integrals was initiated by M. Sato, T. Kawai, H.P. Stapp, M. Kashiwara, T. Oshima, and others in the 1970's. See e.g., [13], [7], [5], [6]. In their investigation, the theory of microfunctions and (holonomic systems of) microdifferential equations played a decisive role.

Recently, N. Honda and T. Kawai studied the geometry of Landau-Nakanishi surfaces systematically and discovered interesting phenomena in the 2-dimensional space-time in a series of papers, e.g., [2], [3], [4]. Inspired by their work, we will report on actual computation of holonomic systems for Feynman integrals associated with very simple Feynman diagrams by computer.

Let G be a connected Feynman graph (diagram); i.e., G consists of

- vertices $V_1, \cdots, V_{n'}$,
- oriented line segments $L_1, \ldots, L_N$ called internal lines,
- oriented half-lines $L_1^e, \ldots, L_n^e$ called external lines.

The end-points of each internal line L_l are two distinct vertices, and each external line has only one end-point, which coincides with one of the vertices.

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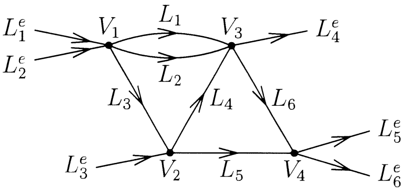


We associate ν-dimensional vector $\mathbf{p}_r = (p_{r,0}, p_{r,1}, \ldots, p_{r,\nu-1})$ to each external line L_r^e $(1 \leq r \leq n')$; a ν-dimensional vector $\mathbf{k}_l = (k_{l,0}, k_{l,1}, \ldots, k_{l,\nu-1})$ and a *positive* real number m_l to each internal line L_l $(1 \leq l \leq N)$. For a vertex V_j and an internal or external line L_l, the incidence number $[j : l]$ is defined as follows:

$$[j : l] = 1 \text{ if } L_l \text{ ends at } V_j,$$
$$[j : l] = -1 \text{ if } L_l \text{ starts from } V_j,$$
$$[j : l] = 0 \text{ otherwise.}$$

The Feynman integral associated with G is defined to be

$$F_G(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \int_{\mathbb{R}^{\nu N}} \frac{\prod_{j=1}^{n'} \delta^\nu \left(\sum_{r=1}^{n} [j : r] \mathbf{p}_r + \sum_{l=1}^{N} [j : l] \mathbf{k}_l \right)}{\prod_{l=1}^{N} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}0)} \prod_{l=1}^{N} d^\nu \mathbf{k}_l.$$

Here δ^ν denotes the ν-dimensional delta function,

$$\mathbf{k}_l^2 := k_{l,0}^2 - k_{l,1}^2 - \cdots - k_{l,\nu-1}^2$$

is the Minkowski norm of $\mathbf{k}_l = (k_{l,0}, k_{l1}, \cdots, k_{l,\nu-1})$, and $d^\nu \mathbf{k}_l$ is the ν-dimensional volume element.

However, the Feynman integral is not necessarily well-defined since it involves the product and the integral of generalized functions. In order to bypass this difficulty without what is called renormalization or compactification of the domain of integration, we consider it as a microfunction defined on a certain subset of the cotangent space following the work by M. Sato and others mentioned above. This point of view has a close connection with what is called the Landau-Nakanishi variety associated with G as is explained in [6].

Our purpose is to compute a holonomic system which the Feynman integral satisfies, in the two-dimensional space-time, by using algorithms and computer programs for D-modules. We also compute the Landau-Nakanishi variety for comparison with the holonomic system.

I would like to thank Professors Takahiro Kawai and Naofumi Honda for helpful suggestions and comments. In actual computation, I made use of a computer algebra system Risa/Asir [8] developed by Masayuki Noro, originally at Fujitsu Laboratories Limited.

In particular, the integration of a D-module was computed by using a Risa/Asir library file 'nk_restriction.rr' coded by Hiromasa Nakayama; decomposition of a variety into irreducible components was done by using a library file `noro_pd.rr` coded by Noro.

§ 2. A recipe for computing a holonomic system for the Feynman integral

In what follows, we assume that for each vertex V_j, there exists a unique external line, which we may assume to be L_j^e, that ends at V_j and that no external line starts from V_j. Then $n = n'$ holds and the Feynman integral is given by

$$F_G(\mathbf{p}_1, \dots, \mathbf{p}_n) = \int_{\mathbb{R}^{\nu N}} \frac{\prod_{j=1}^n \delta^\nu \left(\mathbf{p}_j + \sum_{l=1}^N [j:l]\mathbf{k}_l\right)}{\prod_{l=1}^N (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}0)} \prod_{l=1}^N d^\nu \mathbf{k}_l. \tag{2.1}$$

§ 2.1. Rewriting the Feynman integral

The delta factors of the integrand of the Feynman integral (2.1) correspond to the linear equations (momentum preservation)

$$p_j + \sum_{l=1}^N [j:l]k_l = 0 \qquad (1 \le j \le n)$$

for indeterminates p_j and k_l which correspond to the vectors $\mathbf{p}_j$ and $\mathbf{k}_l$. These equations define an N-dimensional linear subspace of $\mathbb{R}^{n+N}$, which is contained in the hyperplane $p_1 + \cdots + p_n = 0$ since $\sum_{j=1}^n [j:l] = 0$.

Lemma 2.1. *Let A be the $n \times N$ matrix whose (j,l)-element is $[j:l]$. Then the rank of A is $n-1$.*

In view of this lemma, we can choose a set of indices

$$J = \{l_1, \dots, l_{N-n+1}\} \subset \{1 \dots, N\}$$

and integers a_{lr} and b_{lj} so that the system

$$p_j + \sum_{l=1}^N [j:l]k_l = 0 \qquad (1 \le j \le n)$$

of linear equations is equivalent to

$$\sum_{j=1}^n p_j = 0, \quad k_l - \psi_l(p_1, \dots, p_{n-1}, k_{l_1}, \dots, k_{l_{N-n+1}}) = 0 \quad (l \in J^c)$$

with

$$\psi_l(p_1, \dots, p_{n-1}, k_{l_1}, \dots, k_{l_{N-n+1}}) = \sum_{r=1}^{n-1} a_{lr} p_r + \sum_{j=1}^{N-n+1} b_{lj} k_{l_j}$$

and that the $(n-1) \times (n-1)$-matrix (a_{lr}) is non-singular. These data can be computed by row operations on the matrix A augmented by ${}^t(p_1, \dots, p_n)$, which produce a matrix with a row $(0, \dots, 0, p_1 + \cdots + p_n)$.

Then the Feynman integral is written in the form

$$\begin{aligned} F_G(\mathbf{p}_1, \dots, \mathbf{p}_n) &= \int_{\mathbb{R}^{N\nu}} \delta(\mathbf{p}_1 + \cdots + \mathbf{p}_n) \prod_{l \in J^c} \delta(\mathbf{k}_l - \psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})) \\ &\quad \times \prod_{l=1}^{N} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}0)^{-1} \prod_{l=1}^{N} d\mathbf{k}_l \\ &= \delta(\mathbf{p}_1 + \cdots + \mathbf{p}_n) \tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}) \end{aligned}$$

with the amplitude function

$$\begin{aligned} \tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}) = \int_{\mathbb{R}^{(N-n+1)\nu}} & \prod_{l \in J} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}0)^{-1} \\ \times & \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2 + \sqrt{-1}0)^{-1} \prod_{l \in J} d\mathbf{k}_l. \end{aligned}$$

Note that the functions F_G and $\tilde{F}_G$ are invariant under the action of the Lorentz group: Let T be a $\nu \times \nu$ matrix such that

$${}^tT \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ 0 & 0 & \cdots & -1 \end{pmatrix} T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ 0 & 0 & \cdots & -1 \end{pmatrix}.$$

Then one has

$$\begin{aligned} F_G(T\mathbf{p}_1, \dots, T\mathbf{p}_{n-1}, T\mathbf{p}_n) &= F_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{p}_n), \\ \tilde{F}_G(T\mathbf{p}_1, \dots, T\mathbf{p}_{n-1}) &= \tilde{F}_G(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}). \end{aligned}$$

§ 2.2. Holonomic systems for integrands

In general, since dk_l $(l \in J)$ and $d\psi_l$ $(l \in J^c)$ are linearly independent, the integrand

$$\begin{aligned} &\Phi(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}}) \\ &= \prod_{l \in J} (\mathbf{k}_l^2 - m_l^2 + \sqrt{-1}0)^{-1} \prod_{l \in J^c} (\psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2 + \sqrt{-1}0)^{-1} \end{aligned}$$

of the amplitude $\tilde{F}_G$ is well-defined as a hyperfunction on $\mathbb{R}^N$, represented as the boundary value of the rational function

$$\begin{aligned}\tilde{\Phi}(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})\\ = \prod_{l\in J}(\mathbf{k}_l^2 - m_l^2)^{-1} \prod_{l\in J^c} (\psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})^2 - m_l^2)^{-1}\end{aligned}$$

defined on

$$\begin{aligned}\{(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}}) \in \mathbb{C}^{\nu N} \mid \operatorname{Im} \mathbf{k}_l^2 > 0\ (l \in J),\\ \operatorname{Im} \psi_l(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}})^2 > 0\ (l \in J^c)\},\end{aligned}$$

whose closure contains $\mathbb{R}^{\nu N}$ in view of the linear independence above; here the assumption $m_l > 0$ is essential.

Let $D_{\nu N}$ be the ring of differential operators with polynomial coefficients in $\mathbf{p}_1$, $\dots$, $\mathbf{p}_{n-1}$, $\mathbf{k}_{l_1}$, $\dots$, $\mathbf{k}_{l_{N-n+1}}$, and $\mathcal{B}_{\mathbb{R}^{\nu N}}$ the sheaf of hyperfunctions on $\mathbb{R}^{\nu N}$. Then the annihilator (left ideal of $D_{\nu N}$)

$$\operatorname{Ann}_{D_{\nu N}} \Phi = \{P \in D_{\nu N} \mid P\Phi = 0 \text{ in } \mathcal{B}_{\mathbb{R}^{\nu N}}(\mathbb{R}^{\nu N})\}$$

of Φ coincides with the annihilator

$$\operatorname{Ann}_{D_{\nu N}} \tilde{\Phi} = \{P \in D_{\nu N} \mid P\tilde{\Phi} = 0 \text{ as rational function}\}$$

of $\tilde{\Phi}$ by virtue of the injectivity of the boundary map in the theory of hyperfunctions.

There exists a general algorithm to compute the annihilator of an arbitrary rational function. However, since the denominator of $\tilde{\Phi}$ is the product of polynomials whose differentials are linearly independent at each point, the annihilator of $\tilde{\Phi}$ is generated by first order differential operators, which are much easier to compute.

§ 2.3. Landau-Nakanishi varieties for amplitudes

Let $\mathbf{u}_r = (u_{r,0}, u_{r,1}, \dots, u_{r,\nu-1})$ be a ν-dimensional vector and set

$$\begin{aligned}\Lambda(G) = \{&(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \dots, \mathbf{k}_{l_{N-n+1}}; \mathbf{u}_1, \dots, \mathbf{u}_{n-1}; \alpha_1, \dots, \alpha_N)\\ &\in \mathbb{R}^{\nu N} \times \mathbb{R}^{\nu(n-1)} \times \mathbb{R}^N \mid\\ &\alpha_{l_j}(\mathbf{k}_{l_j}^2 - m_{l_j}^2) = 0\ (1 \le j \le N-n+1), \quad \alpha_l(\psi_l^2 - m_l^2) = 0\ (l \in J^c),\\ &\alpha_{l_j}\mathbf{k}_{l_j} + \sum_{l\in J^c} \alpha_l b_{lj} \psi_l = 0\ (1 \le j \le N-n+1),\\ &\mathbf{u}_r = \sum_{l\in J^c} \alpha_l a_{lr} \psi_l\ (1 \le r \le n-1), \quad \alpha_l \ge 0\ (1 \le l \le N)\}\end{aligned}$$

and

$$\begin{aligned}\Lambda_+(G) = \{(&\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{k}_{l_1}, \ldots, \mathbf{k}_{l_{N-n+1}}; \mathbf{u}_1, \ldots, \mathbf{u}_{n-1}; \alpha_1, \ldots, \alpha_N) \\ &\in \mathbb{R}^{\nu N} \times \mathbb{R}^{\nu(n-1)} \times \mathbb{R}^N \mid \\ &\alpha_{l_j}(\mathbf{k}_{l_j}^2 - m_{l_j}^2) = 0 \ (1 \le j \le N-n+1), \quad \alpha_l(\psi_l^2 - m_l^2) = 0 \ (l \in J^c), \\ &\alpha_{l_j}\mathbf{k}_{l_j} + \sum_{l \in J^c} \alpha_l b_{lj} \psi_l = 0 \ (1 \le j \le N-n+1), \\ &\mathbf{u}_r = \sum_{l \in J^c} \alpha_l a_{lr} \psi_l \ (1 \le r \le n-1), \quad \alpha_l > 0 \ (1 \le l \le N)\}.\end{aligned}$$

Let

$$\sqrt{-1}T^*\mathbb{R}^{\nu(n-1)} = \{(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}; \sqrt{-1}\mathbf{u}_1 d\mathbf{p}_1 + \cdots + \sqrt{-1}\mathbf{u}_{n-1} d\mathbf{p}_{n-1})\}$$

be the (purely imaginary) cotangent bundle of $\mathbb{R}^{\nu(n-1)}$ and let ϖ be the natural projection of $\Lambda(G)$ to $\sqrt{-1}T^*\mathbb{R}^{\nu(n-1)}$. Here we set

$$\mathbf{u}_j d\mathbf{p}_j = u_{j,0} dp_{j,0} - u_{j,1} dp_{j,1} - \cdots - u_{j,\nu-1} dp_{j,\nu-1}$$

in accordance with the Minkowski norm.

The amplitude $\tilde{F}_G$ is well-defined as a microfunction on the set

$$\sqrt{-1}T^*\mathbb{R}^{\nu(n-1)} \setminus \varpi\big(\Lambda(G) \setminus \Lambda_+(G)\big)$$

and its support is contained in $\varpi(\Lambda_+(G))$. This fact follows from the theory of integration of microfunctions (see e.g., Chapter 3 of [6]) and the non-singularity of the matrix (a_{lr}).

In practice, we can compute the complexifications $\Lambda^{\mathbb{C}}(G)$ and $\Lambda_+^{\mathbb{C}}(G)$ of $\Lambda(G)$ and $\Lambda_+(G)$ respectively allowing $\mathbf{k}_{l_j}$ and α_l to be complex and replacing the condition $\alpha_l > 0$ by $\alpha_l \neq 0$. This can be done by using Gröbner bases in the polynomial ring.

§ 2.4. Holonomic systems for amplitudes

Let $M = D_{\nu N}/\mathrm{Ann}_{D_{\nu N}}\Phi$ be the holonomic system for the integrand Φ of the Feynman integral (2.1). Let us denote by $D_{\nu(n-1)}$ the ring of differential operators with polynomial coefficients in the variables $\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}$. Then the integral $\int_{\varpi_{\mathbb{C}}} M$ of M along the fibers of the projection $\varpi_{\mathbb{C}} : \mathbb{C}^{\nu N} \to \mathbb{C}^{\nu(n-1)}$ is defined to be the left $D_{\nu(n-1)}$-module

$$\int_{\varpi_{\mathbb{C}}} M = M/(\partial_{\mathbf{k}_{l_1}} M + \cdots + \partial_{\mathbf{k}_{l_{N-n+1}}} M)$$

with the notation

$$\partial_{\mathbf{k}_l} M = \partial_{k_{l,0}} M + \partial_{k_{l,1}} M + \cdots + \partial_{k_{l,\nu-1}} M.$$

This is a holonomic $D_{\nu(n-1)}$-module since M is holonomic. Moreover, there is an algorithm for computing $\int_{\varpi_{\mathbb{C}}} M$ given a presentation of M (see [11], [12], [9]).

Theorem 2.2. *The Feynman amplitude $\tilde{F}_G$ satisfies the system $\int_{\varpi_{\mathbb{C}}} M$ of linear partial differential equations as a microfunction on the set*

$$\sqrt{-1}T^*\mathbb{R}^{\nu(n-1)} \setminus \varpi\big(\Lambda(G) \setminus \Lambda_+(G)\big).$$

In order to prove this theorem, we change the notation in the sequel and set $x = (x', x'')$ with $x' = (x_1, \dots, x_{n-d})$ and $x'' = (x_{n-d+1}, \dots, x_n)$ for the coordinate of the base space $\mathbb{R}^n$, and likewise $\xi = (\xi', \xi'')$ for the cotangential coordinate. Let $\mathcal{C}_{\mathbb{R}^n}$ be the sheaf on $\sqrt{-1}T^*\mathbb{R}^n$ of microfunctions (see [14], [6]). Let

$$\varpi : \sqrt{-1}T^*\mathbb{R}^n \ni (x, \sqrt{-1}\xi dx) \longmapsto (x', \sqrt{-1}\xi' dx') \in \sqrt{-1}T^*\mathbb{R}^{n-d}$$

be the projection and W be an open set of $\sqrt{-1}T^*\mathbb{R}^{n-d}$. Let us denote by $\mathcal{F}_W$ the set of the microfunctions u on $\varpi^{-1}(W)$ such that the restriction of ϖ to the set

$$\operatorname{supp} u \cap \varpi^{-1}(W) \cap \{(x, \sqrt{-1}\xi' dx') \mid \xi' \in \mathbb{R}^d\}$$

is proper.

Then for any $u \in \mathcal{F}_W$, the integral $\int_{\mathbb{R}^d} u(x)\, dx''$ is well-defined as a microfunction on W. We adopt a concrete definition by using defining functions following the arguments in Chapter 3 of [6].

Proposition 2.3. *Let u be an element of $\mathcal{F}_W$. Then the integral $\int_{\mathbb{R}^d} \partial_{x_j} u(x)\, dx''$ vanishes as a microfunction on W for any j such that $n - d + 1 \le j \le n$.*

Proof. Let $p' = (x'_0, \sqrt{-1}\xi'_0 dx')$ be a point of W. We may assume that W is a sufficiently small neighborhood of p'. If the support of an element of $\mathcal{F}_W$ is disjoint from $\{(x, \sqrt{-1}\xi' dx') \mid \xi' \in \mathbb{R}^d\}$, then its integral vanishes on W in view of the theory of integration of microfunctions. Hence we may assume that u is the spectrum of the hyperfunction defined as the boundary value $F(x + \sqrt{-1}V0)$ of a holomorphic function $F(z)$ on $(U \times \mathbb{R}^d) + \sqrt{-1}V0$, where U is an open neighborhood of x'_0 and V is an open convex cone of $\mathbb{R}^n$ with vertex at the origin such that $V' = V \cap (\mathbb{R}^{n-d} \times \{0\})$ is not empty. By the assumption, there exists $R > 0$ such that $F(z)$ continues analytically to $U \times (\mathbb{R}^d \setminus (-R/2, R/2)^d)$ if we take U to be small enough.

Then $\int_{\mathbb{R}^d} \partial_{x_n} u(x)\, dx''$ is the spectrum of the boundary value $G(x' + \sqrt{-1}V'0)$ of

$$\begin{aligned} G(z') &= \int_{[-R,R]^d} \partial_{x_n} F(z', x'')\, dx'' \\ &= \int_{[-R,R]^{d-1}} F(z', x_{n-d+1}, \dots, x_{n-1}, R)\, dx_{n-d+1} \cdots dx_{n-1} \\ &\quad - \int_{[-R,R]^{d-1}} F(z', x_{n-d+1}, \dots, x_{n-1}, -R)\, dx_{n-d+1} \cdots dx_{n-1}. \end{aligned}$$

Hence $G(z')$ is real analytic on U. This implies that $u(x) = \mathrm{sp}(F(x+\sqrt{-1}V0)) = 0$ on a neighborhood of p'. □

Now let D_n and D_{n-d} be the rings of differential operators with polynomial coefficients in x and in x' respectively. Theorem 2.2 is a special case of the following theorem, which follows immediately from the proposition above:

Theorem 2.4. *Let u be an element of $\mathcal{F}_W$ and let I be a left ideal of D_n such that $Pu = 0$ as microfunction on $\varpi^{-1}(W)$ for any $P \in I$. Let Q be an element of*

$$(\partial_{x_{n-d+1}} D_n + \cdots + \partial_{x_n} D_n + I) \cap D_{n-d}.$$

Then Q annihilates $\int_{\mathbb{R}^{n-d}} u(x)\, dx''$ as microfunction on W. More generally, the integration induces a linear map

$$\mathrm{Hom}_{D_n}(M, \mathcal{F}_W) \longrightarrow \mathrm{Hom}_{D_{n-d}}(M', \Gamma(W, \mathcal{C}_{\mathbb{R}^{n-d}}))$$

with $M = D_n/I$ and $M' = M/(\partial_{x_{n-d+1}} M + \cdots + \partial_{x_n} M)$.

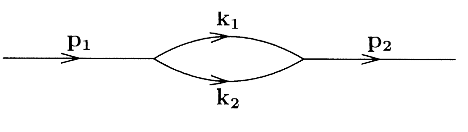
§ 3. Some examples in the two-dimensional space-time

In the sequel, we set $\nu = 2$ and consider Feynman integrals associated with some simple Feynman diagrams. In general, for a two-dimensional vector $\mathbf{p} = (p_0, p_1)$, we denote $\mathbf{p}^2 = p_0^2 - p_1^2$ for the Minkowski norm and $d\mathbf{p} = dp_0 dp_1$ for the volume element.

In actual computation in the sequel, we used a library file `nk_restriction.rr` of Risa/Asir [8] for integration of D-modules, and `noro_pd.rr` for decomposition of characteristic varieties into irreducible components.

We remark that holonomic systems for Cutkosky-type phase space integrals associated with the following Feynman diagrams are presented in [10].

Example 3.1. Let us study the Feynman diagram G below:



The associated Feynman integral is written in the form

$$\begin{aligned}
F_G(\mathbf{p}_1, \mathbf{p}_2) &= \int_{\mathbb{R}^4} \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)\delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2) \\
&\quad \times (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}0)^{-1}(\mathbf{k}_2^2 - m_2^2 + \sqrt{-1}0)^{-1}\, d\mathbf{k}_1 d\mathbf{k}_2 \\
&= \delta(\mathbf{p}_1 - \mathbf{p}_2)\tilde{F}_G(\mathbf{p}_1)
\end{aligned}$$

with the amplitude

$$\tilde{F}_G(\mathbf{p}_1) = \int_{\mathbb{R}^2} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}0)^{-1}((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1}0)^{-1}\, d\mathbf{k}_1.$$

The amplitude $\tilde{F}_G(\mathbf{p}_1)$ is well-defined as a microfunction on $\sqrt{-1}T^*\mathbb{R}^2 \setminus \mathbb{R}^2$, i.e., the whole cotangent bundle with the zero section removed. In other words, $\tilde{F}_G(\mathbf{p}_1)$ is well-defined as a section of the sheaf $\mathcal{B}_{\mathbb{R}^2}/\mathcal{A}_{\mathbb{R}^2}$ on $\mathbb{R}^2$ with $\mathcal{A}_{\mathbb{R}^2}$ being the sheaf of real analytic functions.

By the integration algorithm for D-modules, we know that $\tilde{F}_G(\mathbf{p}_1)$ satisfies a holonomic system $M = D_2/I$ with the left ideal I generated by three operators

$$\begin{aligned}
&p_{11}\partial_{p_{10}} + p_{10}\partial_{p_{11}},\\
&(p_{10} - m_1 - m_2)(p_{10} - m_1 + m_2)(p_{10} + m_1 - m_2)(p_{10} + m_1 + m_2)\partial_{p_{10}}\\
&\quad + p_{11}p_{10}(2p_{10}^2 - p_{11}^2 - 2m_1^2 - 2m_2^2)\partial_{p_{11}} + 2p_{10}^3 + (-2p_{11}^2 - 2m_1^2 - 2m_2^2)p_{10},\\
&(p_{10}^2 - p_{11}^2 - (m_1 + m_2)^2)(p_{10}^2 - p_{11}^2 - (m_1 - m_2)^2)\partial_{p_{11}}\\
&\quad - 2p_{11}p_{10}^2 + 2p_{11}^3 + (2m_1^2 + 2m_2^2)p_{11}.
\end{aligned}$$

The characteristic variety of M is

$$\begin{aligned}
\mathrm{Char}(M) = &\{(p_{10}, p_{11}; \sqrt{-1}(u_{10}dp_{10} + u_{11}dp_{11}) \mid u_{10} = u_{11} = 0\}\\
&\cup \{p_{10}^2 - p_{11}^2 - (m_1 + m_2)^2 = u_{11}p_{10} + u_{10}p_{11} = 0\}\\
&\cup \{p_{10}^2 - p_{11}^2 - (m_1 - m_2)^2 = u_{11}p_{10} + u_{10}p_{11} = 0\}
\end{aligned}$$

with each component of multiplicity one if $m_1 \neq m_2$ and

$$\begin{aligned}
\mathrm{Char}(M) = &\{(p_{10}, p_{11}; \sqrt{-1}(u_{10}dp_{10} + u_{11}dp_{11}) \mid u_{10} = u_{11} = 0\}\\
&\cup \{p_{10}^2 - p_{11}^2 - 4m^2 = u_{11}p_{10} + u_{10}p_{11} = 0\}\\
&\cup \{p_{10} - p_{11} = u_{10} + u_{11} = 0\} \cup \{p_{10} + p_{11} = u_{10} - u_{11} = 0\}\\
&\cup \{p_{10} = p_{11} = 0\}
\end{aligned}$$

with each component of multiplicity one if $m_1 = m_2 = m$.

In view of the invariance under Lorentz transformations, let us set $\mathbf{p}_1 = (x, 0)$ with $x \neq 0$. Then $\tilde{F}_G(x, 0)$ satisfies

$$\begin{aligned}
\{(x - m_1 - m_2)(x - m_1 + m_2)(x + m_1 - m_2)(x + m_1 + m_2)\partial_x&\\
+ 2x(x^2 - m_1^2 - m_2^2)\}\tilde{F}_G(x, 0) &= 0.
\end{aligned}$$

Hence the support of the microfunction $\tilde{F}_G(x, 0)$ is contained in the set

$$\{(x; \sqrt{-1}u dx) \mid x = \pm(m_1 + m_2), \pm(m_1 - m_2)\}$$

and one has, for example

$$\tilde{F}_G((x,0)) = C(x - m_1 + m_2)^{-1/2}(x + m_1 - m_2)^{-1/2} \times (x + m_1 + m_2)^{-1/2}(x - m_1 - m_2 + \sqrt{-1}0)^{-1/2}$$

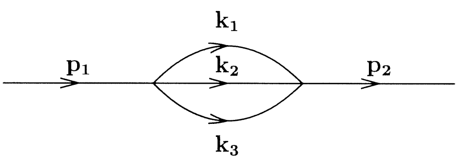
with a constant C as a microfunction at $(m_1 + m_2; \sqrt{-1}dx)$ if $m_1 \neq m_2$.

If $m_1 = m_2 = m$, then the support of $\tilde{F}_G((x,0))$ is contained in $\{x = 0, \pm 2m\}$ and one has

$$\tilde{F}_G((x,0)) = Cx^{-1}(x + 2m)^{-1/2}(x - 2m + \sqrt{-1}0)^{-1/2}$$

at $(2m, \sqrt{-1}dx)$.

Example 3.2. The Feynman integral associated with the graph G below



is given by

$$F_G(\mathbf{p}_1, \mathbf{p}_2) = \delta(\mathbf{p}_1 - \mathbf{p}_2)\tilde{F}_G(\mathbf{p}_1)$$

with

$$\tilde{F}_G(\mathbf{p}_1) = \int_{\mathbb{R}^4} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}0)^{-1}(\mathbf{k}_2^2 - m_2^2 + \sqrt{-1}0)^{-1} \times ((\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)^2 - m_3^2 + \sqrt{-1}0)^{-1}\, d\mathbf{k}_1 d\mathbf{k}_2.$$

We can confirm that $\tilde{F}_G(\mathbf{p}_1)$ is well-defined as a microfunction on $\sqrt{-1}T^*\mathbb{R}^2 \setminus \mathbb{R}^2$ and its support (singularity spectrum) is contained in

$$\begin{aligned}
&\{p_{10}^2 - p_{11}^2 - (-m_1 + m_2 + m_3)^2 = u_{11}p_{10} + u_{10}p_{11} = 0\} \\
&\cup \{p_{10}^2 - p_{11}^2 - (m_1 - m_2 + m_3)^2 = u_{11}p_{10} + u_{10}p_{11} = 0\} \\
&\cup \{p_{10}^2 - p_{11}^2 - (m_1 + m_2 - m_3)^2 = u_{11}p_{10} + u_{10}p_{11} = 0\} \\
&\cup \{p_{10}^2 - p_{11}^2 - (m_1 + m_2 + m_3)^2 = u_{11}p_{10} + u_{10}p_{11} = 0\}
\end{aligned}$$

for generic m_1, m_2, m_3.

We compute holonomic systems for $\tilde{F}_G((x,0))$ by assigning some special values to m_1, m_2, m_3 since the computation for general m_1, m_2, m_3 (as parameters) is intractable.

First let us set $m_1 = 1$, $m_2 = 2$, $m_3 = 4$ so that $(-m_1+m_2+m_3)^2$, $(m_1-m_2+m_3)^2$, $(m_1+m_2-m_3)^2$ are distinct. Then $\tilde{F}_G((x,0))$ is annihilated by the differential operator

$$\begin{aligned}P = {} & 30x(x-1)(x+1)(x-3)(x+3)(x-5)(x+5)(x-7)(x+7)\partial_x^3 \\ & + (-2x^{12} + 191x^{10} - 5340x^8 + 35954x^6 + 273082x^4 - 2071305x^2 + 661500)\partial_x^2 \\ & + (-10x^{11} + 675x^9 - 12108x^7 + 15454x^5 + 936462x^3 - 2692665x)\partial_x \\ & - 8x^{10} + 372x^8 - 3300x^6 - 36028x^4 + 457932x^2 - 356760.\end{aligned}$$

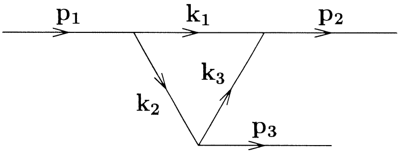
The singular points $x = 0, \pm 1, \pm 3, \pm 5, \pm 7$ of P are all regular and the indicial equations are all $s^2(s-1)$. This implies, e.g., $\tilde{F}_G((x,0)) = U \log(x + i0)$ at $(1, \sqrt{-1}dx)$ with a microdifferential operator U of order zero by virtue of Lemma 4.2.6 (p. 425) of Sato-Kawai-Kashiwara [14].

Next set $m_1 = m_2 = m_3 = 1$. Then $\tilde{F}_G((x,0))$ is annihilated by

$$Q = x(x-1)(x+1)(x-3)(x+3)\partial_x^2 + (5x^4 - 30x^2 + 9)\partial_x + 4x^3 - 12x.$$

The points $0, \pm 1, \pm 3$ are regular singular points of Q and its indicial equations at these points are all s^2. This implies $\tilde{F}_G((x,0)) = U \log(x - 1 + i0)$ e.g., at $(1, \sqrt{-1}dx)$ with a microdifferential operator U of order zero.

Example 3.3. The Feynman integral associated with the graph $G = T_1$ below



is given by

$$F_G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)\tilde{F}_G(\mathbf{p}_1, \mathbf{p}_2)$$

with

$$\begin{aligned}\tilde{F}_G(\mathbf{p}_1, \mathbf{p}_2) = {} & \int_{\mathbb{R}^2} (\mathbf{k}_1^2 - m_1^2 + \sqrt{-1}0)^{-1} \\ & \times ((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2 + \sqrt{-1}0)^{-1})^{-1}((\mathbf{p}_2 - \mathbf{k}_1)^2 - m_3^2 + \sqrt{-1}0)^{-1}\, d\mathbf{k}_1.\end{aligned}$$

Computation for general m_1, m_2, m_3 is intractable. So let us set $m_1 = m_2 = m_3 = 1$ in the sequel. In this situation, the Landau-Nakanishi variety was investigated by N. Honda and T. Kawai ([2],[3]) in detail.

The amplitude $\tilde{F}_G((x,0),(y,z))$ is well-defined on

$$\begin{aligned}&\{(x,y,z;\sqrt{-1}(udx+vdy+wdz) \mid (u,v,w)\neq(0,0,0)\}\\ &\quad\setminus\Big(\{(x-y)^2-z^2-4=wx-wy+vz=u+v=0\}\\ &\qquad\cup\{x-y=z=u+v=0\}\cup\{y^2-z^2-4=wy-vz=u=0\}\\ &\qquad\cup\{x^2-4=v=w=0\}\cup\{x=v=w=0\}\cup\{y=z=u=0\}\Big)\end{aligned}$$

as a microfunction and its support is contained in

$$\sqrt{-1}T^*_{\{f=0\}}\mathbb{R}^3\cup\sqrt{-1}T^*_{\{x=y=z=0\}}\mathbb{R}^3\cup\sqrt{-1}T^*_{\{x=y^2-z^2-4=0\}}\mathbb{R}^3$$

with

$$f=(y-z)(y+z)x^2-2(y-z)(y+z)yx+(y-z)^2(y+z)^2+4z^2,$$

where we denote by $T^*_S\mathbb{R}^3$ the closure of the conormal bundle of the regular part of a real analytic set S of $\mathbb{R}^3$.

We can compute a holonomic system $M=D_3/I$ for $\tilde{F}_G((x,0),(y,z))$, which is too complicated to show here. The characteristic variety of M is

$$\begin{aligned}&\mathbb{C}^3\cup T^*_{\{f=0\}}\mathbb{C}^3\cup T^*_{\{x=f=0\}}\mathbb{C}^3\cup T^*_{\{(x-y)^2-z^2-4=0\}}\mathbb{C}^3\cup T^*_{\{y^2-z^2-4=0\}}\mathbb{C}^3\cup T^*_{\{x-y-z=0\}}\mathbb{C}^3\\ &\cup T^*_{\{x-y+z=0\}}\mathbb{C}^3\cup T^*_{\{y-z=0\}}\mathbb{C}^3\cup T^*_{\{y+z=0\}}\mathbb{C}^3\cup T^*_{\{x=0\}}\mathbb{C}^3\cup T^*_{\{x-2=0\}}\mathbb{C}^3\cup T^*_{\{x+2=0\}}\mathbb{C}^3\\ &\cup T^*_{\{x=y^2-z^2-4=0\}}\mathbb{C}^3\cup T^*_{\{x=y-z=0\}}\mathbb{C}^3\cup T^*_{\{x=y+z=0\}}\mathbb{C}^3\cup T^*_{\{x-y=z=0\}}\mathbb{C}^3\cup T^*_{\{y=z=0\}}\mathbb{C}^3\\ &\cup T^*_{\{x=y=z=0\}}\mathbb{C}^3,\end{aligned}$$

where we denote by $T^*_Z\mathbb{C}^3$ the closure of the conormal bundle of the regular part of an analytic set Z of $\mathbb{C}^3$.

In order to guess the multiplicity and the exponent (order) of $\tilde{F}_G$ along the conormal bundle of $f=0$ at a non-singular point, we compute the restriction of the holonomic system M to a generic line. For example, we can take $L=\{(x,y,z)\mid y=1,z=2\}$. The restriction of f to L is $-3x^2+6x+25=-(3x^2-6x-25)$, which have two real roots α and $2-\alpha$. Then $F(x):=\tilde{F}_G((x,0),(1,2))$ is annihilated by a 5th order differential operator

$$\begin{aligned}P&=147316552073926635122538062595769976812320x(x-3)\\ &\quad\times(x-2)(x+1)(x+2)(x^2-2x-7)(3x^2-6x-25)\partial_x^5\\ &\quad+(2871432833964372040345167998282243508711x^{19}+\cdots)\partial_x^4+\cdots.\end{aligned}$$

The indicial polynomial at α is $s(s-1)(s-2)(s-3)(s+1)$. Hence we have

$$\tilde{F}_G((x,0),(1,2))=U(x-\alpha+\sqrt{-1}0)^{-1}$$

at $(\alpha,\sqrt{-1}dx)$ with a microdifferential operator U of order 0.

§ 4. Landau-Nakanishi surface associated with T_1 for general m_1, m_2, m_3

Let $\tilde{F}_G((x,0),(y,z))$ be the amplitude function associated with the triangle diagram T_1 with general m_1, m_2, m_3. As a microfunction, the support of $\tilde{F}_G((x,0),(y,z))$ is contained in, outside of $x=0$, the conormal bundle of the (Landau-Nakanishi) surface $f(x,y,z)=0$ with

$$\begin{aligned} f =& (y^2-z^2)x^4 + (-2y^3 + (2z^2 - 2m_1^2 + 2m_3^2)y)x^3 \\ &+ (y^4 + (-2z^2 + 4m_1^2 - 2m_2^2 - 2m_3^2)y^2 + z^4 \\ &+ (2m_2^2 + 2m_3^2)z^2 + m_1^4 - 2m_3^2m_1^2 + m_3^4)x^2 \\ &+ ((-2m_1^2 + 2m_2^2)y^3 + ((2m_1^2 - 2m_2^2)z^2 - 2m_1^4 \\ &+ (2m_2^2 + 2m_3^2)m_1^2 - 2m_3^2m_2^2)y)x \\ &+ (m_1^4 - 2m_2^2m_1^2 + m_2^4)y^2 + (-m_1^4 + 2m_2^2m_1^2 - m_2^4)z^2. \end{aligned}$$

By the coordinate transformation $(y+z, y-z) \to (y,z)$, f becomes

$$\begin{aligned} f =& zyx^4 - (y+z)(zy + m_1^2 - m_3^2)x^3 \\ &+ \{(z^2+m_1^2)y^2 + 2(m_1^2 - m_2^2 - m_3^2)zy + m_1^2z^2 + (m_1^2 - m_3^2)^2\}x^2 \\ &- (m_1-m_2)(m_1+m_2)(y+z)(zy + m_1^2 - m_3^2)x + (m_1-m_2)^2(m_1+m_2)^2zy. \end{aligned}$$

The set of the singular points of $f=0$ is given by

$$\begin{aligned} \{f = f_x = f_y = f_z = 0\} =& \{y - z = -zx^2 + (z^2 + m_1^2 - m_3^2)x + (-m_1^2 + m_2^2)z = 0\} \\ &\cup \{x = m_1 - m_2 = 0\}. \end{aligned}$$

For example, if $m_1 = 1$, $m_2 = 2$, $m_3 = 3$ (probably a generic case), then the local b-function $b_{f,p}(s)$ of f at

$$p = \pm(1,1,1),\ \pm(1,-2,-2),\ \pm(3,-1,-1),\ \pm(3,2,2)$$

is $(s+1)^2(2s+3)$, which is the same as that of the Whitney umbrella $x^2 - y^2z = 0$.

On the other hand, if $m_1 = 2$, $m_2 = m_3 = 1$, then the local b-function $b_{f,p}(s)$ of f at $p = \pm(\sqrt{3}, \sqrt{3}/2, \sqrt{3}/2)$ is $(s+1)^3(2s+3)$ in contrast to the b-function $(s+1)^2(2s+3)$ of the Whitney umbrella. This implies that the singularity at p of f is not analytically equivalent to the Whitney umbrella. The local b-functions above were computed by using a library file `nn_ndbf.rr` of Risa/Asir [8].

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