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A method for computing generic Lê numbers associated with non-isolated hypersurface singulrities

By

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Abstract

Lê cycles and Lê numbers introduced by D. Massey are considered in the context of symbolic computation. A method for computing generic Lê numbers is proposed. Keys of the proposed method are the use of parametric saturations in polynomial rings and of parametric local cohomology systems.

§ 1. Introduction

In 1991, D. Massey studied non-isolated hypersurface singularities and introduced the concept of Lê cycles and that of Lê numbers ([8], [9]). The Lê numbers are generalization of the Milnor number. D. Massey showed, among other things, in particular that the alternating sum of Lê numbers is equal to the reduced Euler characteristic of the Milnor fibre. He also gave in [8], [9], a method for computing Lê cycles and Lê numbers. However, as Lê numbers depend on the choice of coordinate systems used in computation, they are not invariants of singularities. In contrast, generic Lê numbers are complex analytic invariants of singularities (remark 9.1 in [10]). A problem comes from fact that no effective way for computing generic Lê numbers is known.

In a series of papers, by using the langage of derived category and the theory of perverse sheaf and micro-support, D. Massey has developed and generalised the theory of Lê cycles and Lê numbers in more general context. Nowadays, Lê cycles and Lê numbers are extensively studied by several authors ([1], [2], [4], [6]). Note in particular, as T. Gaffney pointed out, that Lê cycles and generic Lê numbers are closely related with holonomic D-modules associated with hypersurface singularities ([3], [10]) It is therefore desirable to establish an effective method for computing generic Lê numbers.

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We propose in this paper an effective method for computing generic Lê numbers. The main idea of our approach is the use of a familly of coordinate systems. Key tools are parametric Gröbner systems [13], [19] and parametric local cohomology systems [15]. We show that these two tools allow us to compute generic Lê numbers without choosing a generic coordinate system.

§ 2. Polar variety and Lê cycle

In this section, we recall some basics on polar varieties and Lê cycles.

Let X be an open neighbourhood of the origin $\mathcal{O}$ in $\mathbb{C}^{n+1}$. Let h be a holomorphic function defined on X, S the hypersurface $S = \{x \in X \mid h(x) = 0\}$ defined by h. Let Σ_h denote the singular set of S :

$$\Sigma_h = \{x \in S \mid h(x) = \frac{\partial h}{\partial x_0}(x) = \frac{\partial h}{\partial x_1}(x) = \cdots = \frac{\partial h}{\partial x_n}(x) = 0\}.$$

Let s be the dimension at $\mathcal{O}$ of the singular set Σ_h.

Now let us breafly recall a method given by D. Massey for computing Lê cycles and Lê numbers. Suppose that a system of coordinates $z = (z_1, z_1, ..., z_n)$ is given. Assume that it is generic enough.

Remark D. Massey introduced in [8] several notion of genericity. We refer the reader to [8], [9] for details.

For $s < k \leq n$, set

$$J_{h,z}^{(k)} = (\frac{\partial h}{\partial z_k}, \frac{\partial h}{\partial z_{k+1}}, ..., \frac{\partial h}{\partial z_n}), I_{\Gamma_{h,z}}^{(k)} = J_{h,z}^{(k)} \subset \mathcal{O}_X$$

$$Z_{h,z}^{(k)} = V(J_{h,z}^{(k)}), \quad \Gamma_{h,z}^{(k)} = Z_{h,z}^{(k)}.$$

For $k = s$, set

$$J_{h,z}^{(s)} = (\frac{\partial h}{\partial z_s}, I_{\Gamma_{h,z}}^{(s+1)}), \quad I_{\Gamma_{h,z}}^{(s)} = J_{h,z}^{(s)} : I_{\Sigma_h}^{\infty}, \text{ (saturation)}$$

$$Z_{h,z}^{(s)} = V(J_{h,z}^{(s)}), \quad \Gamma_{h,z}^{(s)} = V(I_{\Gamma_{h,z}}^{(s)})$$

and

$$I_{h,z}^{(s)} = J_{h,z}(s) : (I_{\Gamma_{h,z}}^{(s)})^{\infty}, \quad \Lambda_{h,z}^{(s)} = V(I_{h,z}^{(s)}).$$

For $0 < k < s$, set

$$J_{h,z}^{(k)} = (\frac{\partial h}{\partial z_k}, I_{\Gamma_{h,z}}^{(k+1)}), \quad I_{\Gamma_{h,z}}^{(k)} = J_{h,z}^{(k)} : I_{\Sigma_h}^{\infty},$$

$$Z^{(k)}_{h,z} = V(J^{(k)}_{h,z}), \quad \Gamma^{(k)}_{h,z} = V(I^{(k)}_{\Gamma_{h,z}})$$

and

$$I^{(k)}_{h,z} = J_{h,z}(k) : (I^{(k)}_{\Gamma_{h,z}})^{\infty}, \quad \Lambda^{(k)}_{h,z} = V(I^{(k)}_{h,z}).$$

For $k = 0$, set

$$J^{(0)}_{h,z} = (\frac{\partial h}{\partial z_0}, I^{(1)}_{\Gamma_{h,z}}), \quad Z^{(0)}_{h,z} = V(J^{(0)}_{h,z}),$$

and

$$I^{(0)}_{h,z} = J_{h,z}(0), \quad \Lambda^{(0)}_{h,z} = V(I^{(0)}_{h,z}).$$

$\Gamma^{(k)}_{h,z}$ and $\Lambda^{(k)}_{h,z}$ are called polar variety and Lê cycles (or Lê variety) respectively.

Under the genericity condition, we have

Proposition 2.1. *([8], [9])*

(i) $\dim\Lambda^{(k)}_{h,z} = k$

(ii) $\Gamma^{(k+1)}_{h,z} = \cup_{i\leq k}\Lambda^{(i)}_{h,z}$

(iii) $\Sigma_h = \cup_{k\leq s}\Lambda^{(k)}_{h,z}$

The intersection numbers at the origin $\mathcal{O}$

$$\gamma^{(k)}_{h,z} = (V(z_0, z_1, ..., z_{k-1}) \cdot \Gamma^{(k)}_{h.z})_{\mathcal{O}}, \quad \lambda^{(k)}_{h,z} = (V(z_0, z_1, ..., z_{k-1}) \cdot \Lambda^{(k)}_{h,z})_{\mathcal{O}},$$

are called polar multiplicity and Lê number respectively.

Note that if we define

$$\zeta^{k)}_{h,z} = (V(z_0, z_1, ..., z_{k-1}) \cdot Z^{(k)}_{h,z})_{\mathcal{O}}$$

then we have

Proposition 2.2. *([10])*

(i) $\zeta^{(k)}_{h,z} = \gamma^{(k)}_{h,z} + \lambda^{(k)}_{h,z}, \quad 1 \leq k$

(ii) $\zeta^{(0)}_{h,z} = \lambda^{0)}_{h,z}$

The result above will be used in the next section for computing generic Lê numbers.

The following example is taken from a paper of A. Zaharia [20].

Example 2.3.

Let $h(x_1, x_2, y_1, y_2) = y_1^2(y_1 + x_1^3 + x_2^2) + y_2^2$ and set $S = \{x \in \mathbb{C}^4 \mid h(z) = 0\}$, where $x = (x_1, x_2, y_1, y_2)$. The singular locus $\Sigma_{h,x}$ of the hypersurface S is

$$\Sigma_{h,x} = \{(x_1, x_2, 0, 0) \mid x_1, x_2 \in \mathbb{C}\} \cong \mathbb{C}^2 \subset \mathbb{C}^4.$$

The dimension s of $\Sigma_{h,x}$ is equal to 2.

$$J_{h,x}^{(3)} = (\frac{\partial h}{\partial y_2}) = (y_2), \quad Z_{h,x}^{(3)} = V(J_{h,x}^{(3)}),$$

$$I_{\Gamma_{h,x}}^{(3)} = (y_2), \quad \Gamma_{h,x}^{(3)} = \{(x_1, x_2, y_1, 0) \mid x_1, x_2.y_1 \in \mathbb{C}\}.$$

Since

$$J_{h,x}^{(2)} = (\frac{\partial h}{\partial y_1}, y_2) = (2(x_1^3 + x_2^2)y_1 + 3y_1^2, y_2),$$

we have

$$I_{\Gamma_{h,x}}^{(2)} = J_{h,x}^{(2)} : I_\Sigma^\infty = (2(x_1^3 + x_2^2) + 3y_1, y_2), \quad I_{h,x}^{(2)} = (y_1, y_2),$$

and

$$\Gamma_{h,x}^{(2)} = \{(x_1, x_2, y_1, 0) \mid 2(x_1^3 + x_2^2) + 3y_1 = 0\}, \quad \Lambda_{h,x}^{(2)} = \{(x_1, x_2, 0, 0) \mid x_1, x_2 \in \mathbb{C}\}.$$

From

$$(x_1, x_2, I_{\Gamma_{h,x}}^{(2)}) = (x_1, x_2, y_1, y_2), \quad (x_1, x_2, I_{h,x}^{(2)}) = (x_1, x_2, y_1, y_2),$$

the polar multiplicity $\gamma_{h,x}^{(2)}$ and the Lê number $\lambda_{h,x}^{(2)}$ are

$$\gamma_{h,x}^{(2)} = \dim_{\mathbb{C}}(\mathcal{O}_X/(x_1, x_2, y_1, y_2) = 1, \quad \lambda_{h,x}^{(2)} = \dim_{\mathbb{C}}(\mathcal{O}_X/(x_1, x_2, y_1, y_2) = 1.$$

As $J_{h,x}^{(1)} = (\frac{\partial h}{\partial x_2}, I_{\Gamma_{h,x}}^{(2)}) = (x_2 y_1^2, 2(x_1^3 + x_2^2) + 3y_1, y_2)$,

$$I_{\Gamma_{h,x}}^{(1)} = J_{h,x}^{(1)} : I_\Sigma^\infty = (2x_1^3 + 3y_1, x_2, y_2), \quad I_{h,x}^{(1)} = J_{h,x}^{(1)} : (I_{\Gamma_{h,x}}^{(1)})^\infty = (2x_1^3 + 2x_2^2 + 3y_1, y_1^2, y_2).$$

From

$$(x_1, x_2, 2x_1^3 + 3y_1, y_2) = (x_1, x_2, y_1, y_2), \quad (x_1, 2x_1^3 + x_2^2 + 3y_1, y_1^2, y_2) = (x_1, 2x_2^2 + 3y_1, y_1^2, y_2),$$

we have

$$\gamma_{h,x}^{(1)} = \dim_{\mathbb{C}}(\mathcal{O}_X/(x_1, x_2, y_1, y_2) = 1, \quad \lambda_{h,x}^{(1)} = \dim_{\mathbb{C}}(\mathcal{O}_X/(x_1, 2x_2^2 + 3y_1, y_1^2, y_2) = 4.$$

$$\Gamma_{h,x}^{(1)} = \{(x_1, 0, y_1, 0) \mid 2x_1^3 + 3y_1 = 0\}, \quad \Lambda_{h,x}^{(1)} = \{(x_1, x_2, 0, 0) \mid x_1^3 + x_2^2 = 0\}.$$

Finally, $J_{h,x}^{(0)} = (\frac{\partial h}{\partial x_1}, x_2, 2x_1^3 + 3y_1, y_2) = (x_1^2 y_1^2, x_2, 2x_1^3 + 3y_1, y_2)$ and $I_{h,x}^{(0)} = J_{h,x}^{(0)}$, we have

$$\lambda_{h,x}^{(0)} = 8$$

by direct computation.

Lê numbers $\lambda_{h,x}^{(2)}, \lambda_{h,x}^{(1)}, \lambda_{h,x}^{(0)}$ are $1, 4, 8$. Note that since $\zeta_{h,x}^{(2)}, \zeta_{h,x}^{(1)}$ and $\zeta_{h,x}^{(0)}$, are equal to 2.5 and 8, it follows from $(\gamma_{h,x}^{(2)}, \gamma_{h,x}^{(1)}) = (1, 1)$ that $(\lambda_{h,x}^{(2)}, \lambda_{h,x}^{(1)}, \lambda_{h,x}^{(0)}) = (1, 4, 8)$ immediately. Note also, as a set we have $\Lambda_{h,x}^{(1)} = \Gamma_{h,x}^{(2)} \cap \Sigma_h$.

For a relation with holonomic D-modules associated with b-functions, we refer the readers to [18].

§ 3. algorithm

We give an outline of an algorithm for computing generic Lê numbers. The main idea of the proposed method is the use of a family of linear change of coordinate systems. Key tools utilized to realise the idea above are parametric Gröbner systems [13], [14] and parametric local cohomology systems [15], [17], [19].

For a given system of coordinates, $x = (x_0, x_1, .., x_n)$ in $\mathbb{C}^{n+1}$, we set $z = (z_0, z_1, ..., z_n)$ by

$$
\begin{aligned}
x_0 &= z_0 + t_{0,1}z_1 + t_{0,2}z_2 + \cdots + t_{0,n}z_n \\
x_1 &= z_1 + t_{1,2}z_2 + t_{1,3}z_3 + \cdots + t_{1,n}z_n \\
\cdots &= \cdots \\
\cdots &= \cdots \\
x_n &= z_n
\end{aligned}
$$

where $t_{i,j}$ are parameters.

Algorithm
input $h(x)$: polynomial
output $(\lambda^{(s)}, \lambda^{(s-1)}, ..., \lambda^{(1)}, \lambda^{(0)})$: generic Lê numbers

step1. compute the radical of the Jacobi ideal $I_{\Sigma.x} = \sqrt{(\frac{\partial h}{\partial x_0}, \frac{\partial h}{\partial x_1}, ..., \frac{\partial h}{\partial x_n})}$

step2. compute the dimension s at $\mathcal{O}$ of the singular set Σ.

step3. I_Σ : rewrite $I_{\Sigma,x}$ in terms of variable z.

step4. set $J_h^{(s+1)} = (\frac{\partial h}{\partial z_{s+1}}, \frac{\partial h}{\partial z_{s+2}}, \cdots \frac{\partial h}{\partial z_n}),\ I_{\Gamma_h}^{(s+1)} = J_h^{(s+1)}$

step5. for $k = s$ to 1,

$J_h^{(k)} = (\frac{\partial h}{\partial z_k}, I_{\Gamma_h^{(k+1)}}),\ I_{\Gamma_h}^{(k)} = J_h^{(k)} : I_\Sigma^\infty$ (saturation)

$\zeta^k = (V(z_0, z_1, ..., z_{k-1}) \cdot Z_h^{(k)})_\mathcal{O},\ \gamma^{(k)} = (V(z_0, z_1, ..., z_{k-1}) \cdot \Gamma_h^{(k)})_\mathcal{O},$

where

$Z_h^{(k)} = V(J_h^{(k)}),\ \Gamma_h^{(k)} = V(I_{\Gamma_h}^{(k)}).$

$\lambda^{(k)} = \zeta^{(k)} - \gamma^{(k)},\ |\Lambda^{(k)}| = \Gamma_h^{(k)} \cap \Sigma$

step6, set $J_h^{(0)} = (\frac{\partial h}{\partial z_0}, I_{\Gamma_h}^{(1)}),$

compute $\lambda^{(0)} = multiplicity_\mathcal{O}(J_j^{(0)}),\ |\Lambda^{(0)}|$

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