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The confluent hypergeometric function and WKB solutions

By

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Abstract

Explicit formulae which describe relations between the confluent hypergeometric function and WKB solutions are found.

§ 1. Introduction

The purpose of this paper is to describe a relation between the confluent hypergeometric function and WKB solutions when the parameters live in a specific region and an independent variable lives in a specific Stokes region. As is well known, the confluent hypergeometric function, which is denoted ${}_1F_1(a, c, ; z)$ below, is an entire function and is a solution to the following Kummer's equation:

$$\frac{d^2w}{dz^2} + (c - z)\frac{dw}{dz} - aw = 0 \tag{1.1}$$

The equation we mainly consider in this paper is the following differential equation:

$$x\frac{d^2w}{dx^2} + (\gamma + \gamma_0\eta^{-1} - x)\eta\frac{dw}{dx} - \eta^2(\alpha + \alpha_0\eta^{-1})w = 0, \tag{1.2}$$

which is obtained by setting

$$a = \alpha_0 + \eta\alpha,\ c = \gamma_0 + \eta\gamma,\ z = \eta x \tag{1.3}$$

in (1.1). On the other hand, WKB solutions $\psi_\pm$ are defined as solutions (which are formal power series in η^{-1}) for the differential equation

$$(-\frac{d^2}{dx^2} + \eta^2 R)\psi = 0, \tag{1.4}$$

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where R is specified in the subsequent section. We call (1.4) the Whittaker's equation with a large parameter. The equation (1.4) is obtained by eliminating the first order term of (1.2). The WKB solution is Borel summable under some suitable assumptions. The Borel sums are analytic solutions of (1.4). Here the following natural question arises: What is the relation between the solution of (1.4) coming from the confluent hypergeometric function and the Borel sums of WKB solutions.

In [21], a similar question for the Gauss hypergeometric function and WKB solutions has been treated and partially solved by M. Tanda. She compared the monodromy matrices, and as a result, obtained a relation between a basis of the solution space consisting of hypergeometric functions and another basis given by the Borel sums of WKB solutions up to multiplicative constants. After her work, we treated the same question and succeeded in determining the constants in [4]. In that work, we made full use of the WKB solutions normalized at one of the regular singular points. We use a similar method in this work. To obtain a relation between the confluent hypergeometric function and WKB solutions normalized at the turning point, we have to compute the Voros coefficients. Although an explicit forms of the Voros coefficients for the confluent hypergeometric differential equation with a large parameter are given in [5], it is done only for the case where $\alpha_0 = 1/2$ and $\gamma_0 = 1$ in (1.3). We shall show that the Voros coefficients for the Whittaker equation (1.4) can be computed if we suppose the parameters (1.3). To this end, the definition of the Voros coefficients is slightly modified.

The study on this topic is done by the author in [19]. All proofs of theorems that will appear below are given there.

§ 2. Kummer's equation with a large parameter

We consider the following differential equation:

$$x\frac{d^2w}{dx^2} + (\gamma + \gamma_0\eta^{-1} - x)\eta\frac{dw}{dx} - \eta^2(\alpha + \alpha_0\eta^{-1})w = 0, \tag{2.1}$$

which is obtained by setting $a = \alpha_0 + \eta\alpha,\ c = \gamma_0 + \eta\gamma,\ z = \eta x$ in the Kummer equation:

$$\frac{d^2w}{dz^2} + (c - z)\frac{dw}{dz} - aw = 0. \tag{2.2}$$

Here $\alpha_0, \gamma_0, \alpha$ and γ are complex parameters. As is well known, the equation (2.1) has a regular singular point at the origin and an irregular singular point at ∞, which are represented by b_j $(j = 0, 2)$, namely, $b_0 = 0, b_2 = \infty$. In order to study (2.1) By eliminating the first order term of (2.1) by $\psi = x^{\frac{\gamma_0+\gamma\eta}{2}}\exp(-\frac{x\eta}{2})w$, we get the following Whittaker-type equation:

$$(-\frac{d^2}{dx^2} + \eta^2 R)\psi = 0, \tag{2.3}$$

where R is a polynomial in η^{-1} of the form $R = R_0 + \eta^{-1}R_1 + \eta^{-2}R_2$ with

$$R_0 = \frac{x^2 + 2x(2\alpha - \gamma) + \gamma^2}{4x^2}, \quad R_1 = \frac{(2\alpha_0 - \gamma_0)x + \gamma(\gamma_0 - 1)}{2x^2}, \quad R_2 = \frac{\gamma_0^2 - 2\gamma_0}{4x^2}.$$

We set the following conditions:

$$\alpha\gamma(\alpha - \gamma) \neq 0, \tag{2.4}$$

$$\mathrm{Re}\ \alpha \mathrm{Re}\ \gamma \mathrm{Re}\ (\alpha - \gamma) \neq 0. \tag{2.5}$$

The first condition ensures that there exist two distinct turning points a_0 and a_1 and the second condition implies that there is no Stokes curve which connects turning points. The turning points are understood to be simple zeros of $R_0 dx^2$ which are different from b_0 and b_2.

The WKB solutions normalized at a turning point a are defined as

$$\psi_{\pm} = \frac{1}{\sqrt{T_{\mathrm{odd}}}} \exp\Big(\pm \int_a^x T_{\mathrm{odd}}\, dx \Big),$$

where T_{odd} is the odd order part with respect to $\sqrt{R_0}$ of the formal solution T of the Riccati-type equation

$$\frac{dT}{dx} + T^2 = \eta^2 R.$$

Stokes curve emanating from a turning point a is defined as

$$\mathrm{Im} \int_a^x \sqrt{R_0} dx = 0.$$

Under the above condition, the WKB solutions normalized at the base point of the Stokes curve are Borel summable in each region surrounded by the Stokes curves emanating from the base point(Stokes regions). The Borel sums are analytic solutions of (2.3). In this paper, we consider only the case where the parameters α, γ are contained in

$$\Pi_1 := \omega_1 \cup \iota(\omega_1),$$

where

$$\omega_1 = \{(\alpha, \gamma) \in \mathbb{C}^2 | 0 < \mathrm{Re}\ \alpha < \mathrm{Re}\ \gamma\}$$

and

$$\iota(\omega_1) = \{(\alpha, \gamma) \in \mathbb{C}^2 | \mathrm{Re}\ \gamma < \mathrm{Re}\ \alpha < 0\}$$

and the branch of $\sqrt{R_0}$ is chosen so that, if Re $\gamma > 0$ (< 0), we have

$$\sqrt{R_0} \sim \frac{\gamma}{2x} \qquad x = b_0,$$

$$\sqrt{R_0} \sim \frac{1}{2} \qquad x = b_2. \tag{2.6}$$

$$(\sqrt{R_0} \sim \frac{-1}{2})$$

and the Stokes curve has the following form (Figure 1):

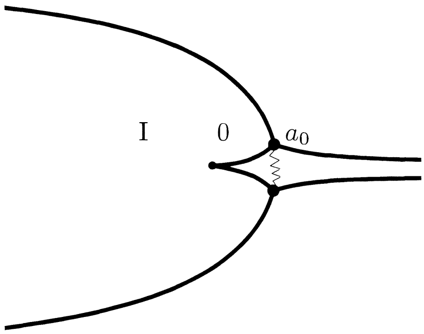

Figure 1

§ 3. Voros coefficients and their Borel sums

Let C_j $(j = 0, 2)$ be a path of integration starting from b_j, going around a_j in a counterclockwise manner and going back to the point of departure. We also have $T_{\mathrm{odd},\leq 0}$ denote the sum of the first two terms of T_{odd}. Since b_j $(j = 0, 2)$ are singular points, the series T_{odd} is not integrable on C_j. However, since the principal parts of T_{odd} and that of $T_{\mathrm{odd},\leq 0}$ coincide, $T_{\mathrm{odd}} - T_{\mathrm{odd},\leq 0}$ is integrable. For that reason, we define the Voros coefficients as follows:

Definition 3.1. Let $W_j = W_j(\alpha, \gamma, \eta)$ $(j = 0, 2)$ denote the formal power series in η^{-1} defined by

$$\frac{1}{2}\int_{C_j}(T_{\mathrm{odd}} - T_{\mathrm{odd},\leq 0})dx. \tag{3.1}$$

We call W_j the Voros coefficient of (2.3) with respect to b_j for $j = 0, 2$.

The explicit forms of W_j $(j = 0, 2)$ are given by the following theorem.

Theorem 3.2. *The Voros coefficients W_j $(j = 0, 2)$ have the following forms:*

$$W_0 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)}\Big(\frac{B_n(\alpha_0)}{\alpha^{n-1}} + \frac{B_n(\gamma_0 - \alpha_0)}{(\gamma - \alpha)^{n-1}} - \frac{B_n(\gamma_0) + B_n(\gamma_0 - 1)}{\gamma^{n-1}}\Big), \tag{3.2}$$

$$W_2 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)}\Big(\frac{B_n(\alpha_0)}{\alpha^{n-1}} - \frac{B_n(\gamma_0 - \alpha_0)}{(\gamma - \alpha)^{n-1}}\Big). \tag{3.3}$$

Here $B_n(x)$ denotes the Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{3.4}$$

We can also compute their Borel sums by a streightforward calculation. To get the Borel sums of the Voros coefficients W_j $(j=0,2)$, we need to consider the Borel summability of them. In our case, they are Borel summable when the parameters belong to Π_1.

Theorem 3.3. *The Voros coefficients* W_j $(j=0,2)$ *are Borel summable in* ω_1 *and in* $\iota(\omega_1)$. *The Borel sums* W_j^1 *of* W_j *in* ω_1 *have the following form:*

$$(3.5)\quad W_0^1=\frac{1}{2}\log\frac{\Gamma(\gamma_0+\gamma\eta)\Gamma(\gamma_0-1+\gamma\eta)\alpha^{\alpha_0+\alpha\eta-\frac{1}{2}}(\gamma-\alpha)^{\gamma_0-\alpha_0+(\gamma-\alpha)\eta-\frac{1}{2}}}{\Gamma(\alpha_0+\alpha\eta)\Gamma(\gamma_0-\alpha_0+(\gamma-\alpha)\eta)\gamma^{2(\gamma_0+\gamma\eta-1)}\eta^{\gamma_0+\gamma-1}}+\frac{\gamma\eta}{2},$$

$$(3.6)\quad W_2^1=\frac{1}{2}\log\frac{\Gamma(\gamma_0-\alpha_0+(\gamma-\alpha)\eta)\alpha^{\alpha_0+\alpha\eta-\frac{1}{2}}\eta^{2\alpha_0-\gamma_0+(2\alpha-\gamma)\eta}}{\Gamma(\alpha_0+\alpha\eta)(\gamma-\alpha)^{\gamma_0-\alpha_0+(\gamma-\alpha)\eta-\frac{1}{2}}}-\frac{2\alpha-\gamma}{2}\eta.$$

§ 4. Statements of the main results

Now we state the main theorem.

Theorem 4.1. *Let* $\Psi_\pm^{\mathrm{I}}$ *be the Borel sums of* $\psi_\pm$ *normalized at* a_0 *in the Stokes region* I. *Under the above condition, the following relations hold:*

(*i*) *If* $(\alpha,\gamma)\in\omega_1$, *we have*

$${}_1F_1(a,c;\eta x)=\frac{\Gamma(c)e^{-\frac{\pi}{2}(c-a-\frac{1}{2})}}{\sqrt{2}\Gamma(a)^{\frac{1}{2}}\Gamma(c-a)^{\frac{1}{2}}\eta^{\frac{c-1}{2}}}x^{-\frac{c}{2}}e^{\frac{x\eta}{2}}\Psi_+^1$$

$(\mathrm{Im}(\gamma-\alpha)<0)$.

(*ii*) *If* $(\alpha,\gamma)\in\iota(\omega_1)$, *we have*

$${}_1F_1(a,c;\eta x)=x^{-\frac{c}{2}}e^{\frac{\eta x}{2}}(A_0^{\iota 1}\Psi_+^{\mathrm{I}}+A_1^{\iota 1}\Psi_-^{\mathrm{I}}),$$

where

$$A_0^{\iota 1}=i\frac{\sin(c-a)\pi\sin\frac{c\pi}{2}\Gamma(c)\eta^{\frac{1}{2}(a-c+1)}}{\sqrt{2\pi}\sin^{\frac{1}{2}}a\pi\sin\frac{(2a-c)\pi}{2}\Gamma(a)^{\frac{1}{2}}e^{\frac{\pi i}{2}(a-\frac{1}{2})}}\left(e^{c\pi i\epsilon}+\frac{\sin a\pi}{\sin(c-a)\pi}\right),$$

$$A_1^{\iota 1}=-iA_0^{\iota 1}+\frac{\sin^{\frac{1}{2}}a\pi\Gamma(c)\Gamma(1+a-c)^{\frac{1}{2}}e^{\frac{\pi i}{2}(a-2c+1)}}{\sqrt{2\pi}\Gamma(a)^{\frac{1}{2}}\eta^{\frac{1-c}{2}}}$$

$(\epsilon=\mathrm{sgn}(\mathrm{Im}\ x)$, $\mathrm{Im}\ \alpha>0$ *and* $\mathrm{Im}\ \gamma>0)$.

In Theorem 4.1, a and c denote $\alpha_0+\alpha\eta$ and $\gamma_0+\gamma\eta$ respectively.

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