数理解析研究所講究録 第2101巻 2019年 165-177

q-analogue of a system of equations from geometry

By

JUN YAMAMOTO* and HIROSHI YAMAZAWA**

Abstract

R.Bielawski [1] studied some systems of partial-differential equations and gave the unique existence theorem of holomorphic solutions. In this paper we investigate that a system of q-difference-differential equations under some conditions which Yamazawa have given in [3]. Our purpose in this paper is to obtain the same results in [3]. We show for a general equations an existence of holomorphic solutions and for an example case we consider an existence of singular solutions.

§ 1. Introduction and Result

Throughout this paper, let suppose $q>1$ and define $|x| = \max_{1\leqslant i\leqslant n}|x_i|$ for $x = (x_1, \cdots, x_n) \in \mathbb{C}_x^n$. We denote

- $\mathbb{N} = \{0, 1, \cdots\}$, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$,
- $D_R = \{x = (x_1, \cdots, x_n) \in \mathbb{C}_x^n; |x| < R\}$,
- $D_T = \{t \in \mathbb{C}_t; |t| < T\}$,
- $D_T \times D_R = \{(t, x) \in \mathbb{C}_t \times \mathbb{C}_x^n; t \in D_T, x \in D_R\}$,
- $\mathcal{O}(D_R)$ be the set of all holomorphic functions on D_R,
- $\mathcal{O}(D_T)$ be the set of all holomorphic functions on D_T,
- $\mathcal{O}(D_R \times D_T)$ be the set of all holomorphic functions on $D_R \times D_T$.

First we define the q-difference operator D_q for a function $f(t, x)$ by

$$D_q f(t,x) = \frac{f(qt,x) - f(t,x)}{qt - t}.$$

2010 Mathematics Subject Classification(s): Primary 39A13; Secondary 35C10.
Key Words: Formal solutions, Holomorphic solutions, Singular solutions, q-difference-differential equations

*Graduate School of Engineering and Science, Shibaura Institute of Technology, Saitama 337-8570, Japan.
**College of Engineering and Design, Shibaura Institute of Technology, Saitama 337-8570, Japan.

Let m_1 and m_2 be positive integers. In this paper we investigate the following system of q-difference-differential equations:

$$\begin{cases} tD_q u = F(t,x,tD_q u,\{\partial_x^\alpha u\}_{|\alpha|\leqslant m_1},v) \\ D_q v = c(t,x) + \sum_{|\alpha|\leqslant m_2} d_\alpha(x)\partial_x^\alpha u \end{cases} \tag{1.1}$$

where $(t,x)=(t,x_1,\cdots,x_n)\in\mathbb{C}_t\times\mathbb{C}_x^n$, $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$, $c(t,x)\in\mathcal{O}(D_T\times D_{R_0})$ for some $T,R_0>0$, $F(t,x,u,V,W)$ with $V=\{V_\alpha\in\mathbb{C};|\alpha|\leqslant m_1\}$ is a function defined in some polydisk Δ centered at the origin of $\mathbb{C}_t\times\mathbb{C}_x^n\times\mathbb{C}_u\times\mathbb{C}_V^\delta\times\mathbb{C}_W$ and δ is the cardinal of $\{\alpha\in\mathbb{N}^n;|\alpha|\leqslant m_1\}$. Let us denote $\Delta_0=\Delta\cap\{t=0,u=0,V=0,W=0\}$.

Yamazawa [3] considered the following nonlinear q-difference-differential equation

$$tD_q u = F(t,x,\{\partial_x^\alpha u\}_{|\alpha|\leqslant m}), \tag{1.2}$$

defined the q-difference-differential equations of Briot-Bouquet type and constructed the holomorphic and singular solutions for the equation (1.2). In [3], Yamazawa gave the definition of q-Briot-Bouquet equation as below:

Definition 1.1 (H. Yamazawa [3]). The equation (1.2) is called a q-analogue of the Briot-Bouquet type with respect to t or simply q-Briot-Bouquet type with respect to t if the equation (1.2) satisfies the following three conditions:

A_1: $F(t,x,V)$ is holomorphic in Δ,

A_2: $F(0,x,0)=0$ in Δ_0,

A_3: $\partial_{V_\alpha}F(0,x,0)=0$ in Δ_0 for all $1\leqslant|\alpha|\leqslant m$,

where Δ is a polydisk centered at the origin of $\mathbb{C}_t\times\mathbb{C}_x^n\times\mathbb{C}_V^\delta$, δ is the cardinal of $\{\alpha\in\mathbb{N}^n;|\alpha|\leqslant m\}$ and $\Delta_0=\Delta\cap\{t=0,V=0\}$.

Definition 1.2 (R. Gérard and H. Tahara [2]). Set

$$\rho(x)=\frac{\partial F}{\partial u}(0,x,0)$$

then the holomorphic function $\rho(x)$ is called the characteristic exponent of the equation (1.2).

On the other hand, R.Bielawski [1] studied the conditions that the given Kähler metric h extends to a Ricci-flat Kähler metric on a line bundle L in a manifold M such that the canonical S^1-action on L is Hamiltonian. The necessary condition which he gave is to solve the following Cauchy problem:

$$\begin{cases} t\partial_t v = -1 + ce^{-v}\det[g_{ij}] \\ \partial_{x_i x_j} v + \partial_{y_i y_j} v + c\partial_t g_{ij} = 0 \\ (g_{ij})|_{t=0} = h_{ij}, \quad (e^v)|_{t=0} = c\det h. \end{cases}$$

In order to study this solvability for this Cauchy problem, R.Bielawski considered the following system of partial differential equations:

$$(1.3)\qquad \begin{cases} t\partial_t u = F(t,x,u,\partial_x u, t\partial_t u, \{v_i\}_{i=1}^N) \\ \partial_t v_i = c_i(t,x) + \sum_{|\alpha|\leqslant 2} d_{i,\alpha}(x)\partial_x^\alpha u \quad \textit{for } i=1,\cdots,N, \end{cases}$$

where $c_i(t,x) \in \mathcal{O}(D_T \times D_{R_0})$ and $d_{i,\alpha}(x) \in \mathcal{O}(D_{R_0})$. In [1], R.Bielawski gave the following assumptions for (1.3):

$B_1:$ $F(t,x,u,V,W,Z)$ is holomorphic in Δ,
$B_2:$ $F(0,x,0,0,0,0) = 0$ in Δ_0,
$B_3:$ $\partial_{V_i}F(0,x,0,0,0,0) = 0$ in Δ_0 for $i=1,\cdots,n$,

where Δ is a polydisk centered at the origin of $\mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_u \times \mathbb{C}_V^n \times \mathbb{C}_W \times \mathbb{C}_Z^N$ and $\Delta_0 = \Delta \cap \{t=0, u=0, V=0, W=0, Z=0\}$.

Under these assumptions, R.Bielawski gave the unique exicetence theorem of holomorphic solutions of (1.3) provided that the characteristic exponent $\rho(x)$ satisfies $\rho(0) \notin \mathbb{N}^*$.

In this paper we assume the following assumptions for (1.1):

q-$B_1:$ $F(t,x,u,V,W)$ is holomorphic in Δ,
q-$B_2:$ $F(0,x,0,0,0) = 0$ in Δ_0,
q-$B_3:$ $\partial_{V_\alpha}F(0,x,0,0,0) = 0$ in Δ_0 for all $1 \leqslant |\alpha| \leqslant m_1$.

We concern the following Yamazawa's result [3]:

Theorem 1.3 (H. Yamazawa [3]). *If* (1.2) *is of the q-Briot-Bouquet type and* $\rho(0) \neq (q^i-1)/(q-1)$ *for* $i=1,2,\cdots$, *then we have*:

(1) (Holomorphic solutions) *The equation* (1.2) *has a unique solution* $u_0(t,x)$ *holomorphic near the origin of* $\mathbb{C}_t \times \mathbb{C}_x^n$ *satisfying* $u_0(t,x) \equiv 0$.

(2) (Singular solutions) *Set* $\rho_q(x) = \log\{1+(q-1)\rho(x)\}/\log q$. *When* $\Re\rho(0) > 0$ *for any* $\varphi(x) \in \mathbb{C}\{x\}$ *there exists an* $\tilde{\mathcal{O}}_+$*-solution* $U(\varphi)$ *of* (1.2) *having an expansion of the following form*:

$$U(\varphi) = \sum_{i=1}^{\infty} u_i(x)t^i + \sum_{k\leqslant i+2m(j-1),\, j\geqslant 1} \varphi_{i,j,k}(x)t^{i+\rho_q(x)j}(\log t)^k,$$

where the coefficients $\{u_i(x) \in \mathbb{C}\{x\}; i \geqslant 1\}$ *and* $\{\varphi_{i,j,k}(x) \in \mathbb{C}\{x\}; j \geqslant 1, k \leqslant i+2m(j-1)\}$ *are determined by* $\varphi(x)$.

In the above theorem, $\mathbb{C}\{x\}$ is the ring of germs of holomorphic functions at the origin of $\mathbb{C}^n$ and for the definition of $\tilde{\mathcal{O}}_+$, see Definition 5.2. In relation to this result, we aim to get the structure of holomorphic and singular solutions for (1.1). Our main result is as follows:

Theorem 1.4. *If the first equation of* (1.1) *satisfies the conditions* q-B_1,q-B_2 *and* q-B_3 *and* $\rho(0) \neq (q^i - 1)/(q-1)$ *for* $i \in \mathbb{N}^*$, *then the system* (1.1) *has a pair of unique holomorphic solutions* (u, v) *satisfying* $u(0, x) = v(0, x) \equiv 0$.

This paper is organized as follows. In section 2 and 3 we prepare lemmas in order to show Theorem 1.4. In section 4 we show Theorem 1.4 and in section 5 give a proof of an existence of formal solutions as singular solutions for the particular case of (1.1).

§ 2. Lemma

In this section we give some lemmas. Set $\|u\|_R = \sup_{x\in D_R} |u(x)|$ for $u(x) \in \mathcal{O}(D_{R_0})$ and $0 < R_0 < R$.

Lemma 2.1 (Nagumo's lemma). *Assume* $u(x)$ *be holomorphic on* D_R. *If for any* $0 < r < R$,

$$\|u\|_r \leqslant \frac{C}{(R-r)^p}$$

holds for some $p \geqslant 0$, *then we have*

$$\|\partial_{x_j} u\|_r \leqslant \frac{Ce(p+1)}{(R-r)^{p+1}} \quad \text{for any } \ 0 < r < R \ \text{ and } \ j = 1, \cdots, n.$$

Lemma 2.2. *There exists a constant* M *such that for any* $i \in \mathbb{N}^*$

$$\frac{(m^* ie)^{m^*}}{q^i} \leqslant M$$

where $m^* = \max\{m_1, m_2\}$.

Proof. Let consider a function $f(x) = (m^* xe)^{m^*}/q^x = (m^* e)^{m^*} x^{m^*} q^{-x}$, then we have

$$f'(x) = x^{m^*}\{(m^* e)^{m^*}(m^* x^{-1} - \log q)\}.$$

Therefore, this function $f(x)$ takes the maximum value at $x = m^*/\log q$. Setting $x_0 = m^*/\log q$ then we have

$$f(x) \leqslant f(x_0)$$

and this $f(x_0)$ means the constant M of the statement in Lemma 2.2. □

Lemma 2.3 (H. Yamazawa [3]). *Let* $q > 1$. *Suppose*

$$\lambda(x) \neq q^i \quad \text{for } \ x \in D_r$$

Then there exists a constant $\sigma > 0$ *such that*

$$|q^i - \lambda(x)| \geqslant \sigma q^i \quad \text{for } \ x \in D_r. \tag{2.1}$$

§ 3. Reduction equations

In this section we reduct the system (1.1) into the following system of q-difference-differential equations under the assumptions q-B_1,q-B_2 and q-B_3 to show Theorem 1.4.

$$
\tag{3.1}
\begin{cases}
(\sigma_q - \lambda(x))u = a(x)t + b(x)v + G_2(x)(t, \sigma_q u, \{\partial_x^\alpha u\}_{|\alpha|\leqslant m_1}, v) \\
\sigma_q v = \sum_{i\geqslant 1} c_i(x)t^i + \sum_{|\alpha|\leqslant m_2} d_\alpha(x) t\partial_x^\alpha u
\end{cases}
$$

where $\sigma_q u(t,x) = u(qt,x)$, $a(x), c_i(x), d_\alpha(x) \in \mathcal{O}(D_{R_0})$ and the function $G_2(x)(t,u,V,W)$ has the following expansion:

$$
\tag{3.2}
G_2(x)(t,u,V,W) = \sum_{p+\mu+|\nu|+\xi\geqslant 2} g_{p,\mu,\nu,\xi}(x)t^p u^\mu V^\nu W^\xi
$$

where $V^\nu = \prod_{|\alpha|\leqslant m_1}\{V_\alpha\}^{\nu_\alpha}$.

Lemma 3.1. *If the system* (1.1) *satisfies the condition q-B_1,q-B_2 and q-B_3, then we can reduct* (1.1) *into the system* (3.1) *with* (3.2).

Proof. We multiply the both side of (1.1) by $q-1$. Then we get $\lambda(x) = 1+(q-1)\rho(x)$ and (3.1). □

Remark. The assumption $\rho(0) \neq (q^i - 1)/(q-1)$ for $i \in \mathbb{N}^*$ implies $\lambda(0) \neq q^i$ for $i \in \mathbb{N}^*$.

§ 4. Proof of Theorem 1.4

In this section we will prove Theorem 1.4. This proof is divided into two steps; the first step is to construct a pair of formal power series solutions $(\hat{u}, \hat{v})$ of the system (1.1) and the other one is about the convergence of the pair of the formal power series solutions $(\hat{u}, \hat{v})$.

§ 4.1. Formal power series solutions

Let us show the system (1.1) has a pair of formal power series solutions of the form

$$
\tag{4.1}
\hat{u} = \sum_{i\geqslant 1} u_i(x)t^i, \ \hat{v} = \sum_{i\geqslant 1} v_i(x)t^i.
$$

Set $v = v_1(x)t + v^*$ with $v^* = \sum_{i\geqslant 2} v_i(x)t^i$. By substituting a pair of formal power series solutions $(\hat{u}, \hat{v})$ into the system (3.1), then we have the following recurrence formulas:

$$
\tag{4.2}
\begin{cases}
qv_1(x) = c_1(x) \\
(q - \lambda(x))u_1(x) = a(x) + b(x)v_1(x)
\end{cases}
$$

and for $i \geqslant 2$

(4.3)
$$\begin{cases} q^i v_i(x) = c_i(x) + \sum_{|\alpha| \leqslant m_2} d_\alpha(x) \partial_x^\alpha u_{i-1}(x) \\ (q^i - \lambda(x)) u_i(x) = b(x) v_i(x) \\ \qquad + \sum_{p+\mu+|\nu|+\xi=i} \sum^* g_{p,\mu,\nu,\xi}(x) \prod_{i=1}^{\mu} q_{l_i} u_{l_i}(x) \prod_{|\alpha| \leqslant m_1} \prod_{j=1}^{\nu_\alpha} \partial_x^\alpha u_{m_{\alpha,j}}(x) \prod_{k=1}^{\xi} v_{n_k}(x) \end{cases}$$

where

$$\sum^* = \sum_{|l|_\mu + |m|_{m_1,\nu} + |n|_\xi + p = i} \quad \text{with } |l|_\mu = l_1 + \cdots + l_\mu, |n|_\xi = n_1 + \cdots + n_\xi$$

$$\text{and} \quad |m|_{m_1,\nu} = \sum_{|\alpha| \leqslant m_1} \sum_{j=1}^{\nu_\alpha} m_{\alpha,j}.$$

Therefore, by the assumption we find out that the system (3.1) has a pair of formal power series solutions $(\hat{u}, \hat{v})$ whose coefficients are given by the above recurrence formulas.

§ 4.2. Convergence of the formal power series solutions

We will show that the pair of the formal power series solutions $(\hat{u}, \hat{v})$ converges in a neighborhood of the origin $t = 0$. Since the fact that $\hat{v} = \sum_{i \geqslant 1} v_i(x) t^i$ is written as the form $\hat{v} = v_1(x) t + v^*$, we can rewrite the system (3.1) by the following system of reduction equations:

$$\text{(4.4)} \qquad \begin{cases} (\sigma_q - \lambda(x)) u = a(x) t + b(x) v^* + G_2(x)(t, \sigma_q u, \{\partial_x u^\alpha\}_{|\alpha| \leqslant m_1}, v^*) \\ \sigma_q v^* = \sum_{i \geqslant 2} c_i(x) t^i + \sum_{|\alpha| \leqslant m_2} d_\alpha(x) t \partial_x^\alpha u. \end{cases}$$

Remark. The function $G_2(x)(t, u, V, W^*)$ of (4.4) differs from (3.1) in the following sense:

$$G_2(x)(t, u, V, W^*) = \sum_{p+\mu+|\nu|+\xi \geqslant 2} g^*_{p,\mu,\nu,\xi}(x) t^p u^\mu V^\nu W^{*\xi}$$

where

$$g^*_{p,\mu,\nu,\xi}(x) = \sum_{\xi_1+\xi_2=\xi} \frac{c_1(x)^{\xi_1} (\xi_1 + \xi_2)!}{\xi_1! \xi_2!} g_{p,\mu,\nu,\xi}(x).$$

This system of reduction equations (4.4) has a pair of formal power series solutions $(\hat{u}, \hat{v}^*)$ of the form

$$\text{(4.5)} \qquad \hat{u} = \sum_{i \geqslant 1} u_i(x) t^i, \ \hat{v}^* = \sum_{i \geqslant 2} v_i(x) t^i$$

and the coefficients $u_i(x)$ for $i \geqslant 1$ and $v_i(x)$ for $i \geqslant 2$ satisfy the recurrence formulas (4.2) and (4.3).

Let us consider the following system of analytic equations:

(4.6)
$$\begin{cases} \sigma X = \sigma A t + \dfrac{M}{(R-r)^{m^*}} \left\{ BY + \displaystyle\sum_{p+\mu+|\nu|+\xi \geqslant 2} \dfrac{t^p G^*_{p,\mu,\nu,\xi}}{(R-r)^{m^*(p+\mu+|\nu|+2\xi-2)}} X^{\mu+|\nu|} Y^{\xi} \right\} \\ Y = \displaystyle\sum_{i\geqslant 2} \dfrac{C_i}{(R-r)^{m^*(i-2)}} t^i + \sum_{|\alpha|\leqslant m_2} D_\alpha t X \end{cases}$$

where M is in Lemma 2.2, $A = \|a\|_R$, $B = \|b\|_R$, $G^*_{p,\mu,\nu,\xi} = \|g^*_{p,\mu,\nu,\xi}\|_R$, $C_i = \|c_i\|_R$ for any $i \in \mathbb{N}^*$ and $D_\alpha = \|d_\alpha\|_R$ for $|\alpha| \leqslant m_2$.

Lemma 4.1. *The system of analytic equations* (4.6) *has the pair of holomorphic solutions* (X, Y) *of the form*

(4.7)
$$X(t,r) = \sum_{i\geqslant 1} X_i(r) t^i, \quad Y(t,r) = \sum_{i \geqslant 2} Y_i(r) t^i,$$

moreover the coefficients $X_i(r), Y_i(r)$ *have the following forms*:

$$X_i = \frac{E_i}{(R-r)^{m^*(i-1)}} \quad \textit{for} \quad i \geqslant 1$$
$$Y_i = \frac{F_i}{(R-r)^{m^*(i-2)}} \quad \textit{for} \quad i \geqslant 2$$

with constants $E_1 = A$ *and* $E_i, F_i \geqslant 0$ *for* $i \geqslant 2$.

Proof. First we will show that the system of analytic equations (4.6) has a pair of formal power series solutions (X, Y) of the form (5.4). By substituting (X, Y) into the system (4.6), then we have the following recurrence formulas for $X_i(r), Y_i(r)$:

$$X_1 = A$$

and for $i \geqslant 2$

$$\begin{cases} \sigma X_i = \dfrac{M}{(R-r)^{m^*}} \Bigg\{ BY_i \\ \qquad + \displaystyle\sum_{p+\mu+|\nu|+\xi = i} \sum{}^{*} \dfrac{G^*_{p,\mu,\nu,\xi}}{(R-r)^{m^*(p+\mu+|\nu|+2\xi-2)}} \prod_{i=1}^{\mu} X_{l_i} \prod_{|\alpha|\leqslant m_1} \prod_{j=1}^{\nu_\alpha} X_{m_{\alpha,j}} \prod_{k=1}^{\xi} Y_{n_k} \Bigg\} \\ Y_i = \dfrac{C_i}{(R-r)^{m^*(i-2)}} + \displaystyle\sum_{|\alpha| \leqslant m_2} D_\alpha X_{i-1}. \end{cases}$$

This means that there exists a pair of unique formal power series solutions (X, Y). Let us show that this pair of formal power series solutions is holomorphic in a neighborhood of the origin $t = 0$. Let consider the following functions:

$$F(t,X,Y) := \sigma(X-At) - \frac{M}{(R-r)^{m^*}}\left\{BY + \sum_{p+\mu+|\nu|+\xi\geqslant 2} \frac{t^p G^*_{p,\mu,\nu,\xi} X^{\mu+|\nu|}Y^{\xi}}{(R-r)^{m^*(p+\mu+|\nu|+2\xi-2)}}\right\}$$

$$G(t,X,Y) := Y - \sum_{i\geqslant 2} \frac{C_i}{(R-r)^{m^*(i-2)}} t^i - \sum_{|\alpha|\leqslant m_2} D_\alpha t X.$$

Then it follows

$$F(0,0,0) = G(0,0,0) = 0, \ \det\begin{pmatrix} \partial_X F(0,0,0) & \partial_X G(0,0,0) \\ \partial_Y F(0,0,0) & \partial_Y G(0,0,0)\end{pmatrix} = \sigma \neq 0.$$

Therefore, by the implicit function's theorem we get holomorphic solutions (X,Y) satisfying $(X(0,r),Y(0,r)) = (0,0)$. We will prove the latter in the statements by induction on i. Since $X_1 = A$ holds from the recurrence formulas, it is clear for $i=1$. For the general case $i \geqslant 2$ we have

$$\begin{aligned} Y_i &= \frac{C_i}{(R-r)^{m^*(i-2)}} + \sum_{|\alpha|\leqslant m_2} D_\alpha X_{i-1} \\ &= \frac{C_i}{(R-r)^{m^*(i-2)}} + \sum_{|\alpha|\leqslant m_2} D_\alpha \frac{E_{i-1}}{(R-r)^{m^*(i-2)}} \\ &= \frac{C_i + \sum_{|\alpha|\leqslant m_2} D_\alpha E_{i-1}}{(R-r)^{m^*(i-2)}} \end{aligned}$$

and

$$\begin{aligned} &X_i \\ &= \frac{M}{\sigma(R-r)^{m^*}}\Bigg(B\frac{F_i}{(R-r)^{m^*(i-2)}} + \sum_{p+\mu+|\nu|+\xi=i} {\sum}^* \frac{G^*_{p,\mu,\nu,\xi}}{(R-r)^{m^*(p+\mu+|\nu|+2\xi-2)}} \\ &\qquad \times \prod_{i=1}^{\mu} \frac{E_{l_i}}{(R-r)^{m^*(l_i-1)}} \prod_{|\alpha|\leqslant m_1}\prod_{j=1}^{\nu_\alpha} \frac{E_{m_{\alpha,j}}}{(R-r)^{m^*(m_{\alpha,j}-1)}} \prod_{k=1}^{\xi} \frac{F_{n_k}}{(R-r)^{m^*(n_k-2)}}\Bigg) \\ &= \frac{M}{\sigma(R-r)^{m^*}}\Bigg(\frac{BF_i}{(R-r)^{m^*(i-2)}} \\ &+ \sum_{p+\mu+|\nu|+\xi=i} {\sum}^* \frac{G^*_{p,\mu,\nu,\xi}}{(R-r)^{m^*(p+\mu+|\nu|+2\xi-2)}} \frac{\prod_{i=1}^{\mu} E_{l_i} \prod_{|\alpha|\leqslant m_1}\prod_{j=1}^{\nu_\alpha} E_{m_{\alpha,j}} \prod_{k=1}^{\xi} F_{n_k}}{(R-r)^{m^*(|l|_\mu + |m|_{m_1,\nu} + |n|_\xi - (\mu+|\nu|+2\xi))}}\Bigg) \\ &= \frac{M}{\sigma(R-r)^{m^*}}\Bigg(\frac{BF_i}{(R-r)^{m^*(i-2)}} \\ &\qquad + \sum_{p+\mu+|\nu|+\xi=i} {\sum}^* \frac{G^*_{p,\mu,\nu,\xi} \prod_{i=1}^{\mu} E_{l_i} \prod_{|\alpha|\leqslant m_1}\prod_{j=1}^{\nu_\alpha} E_{m_{\alpha,j}} \prod_{k=1}^{\xi} F_{n_k}}{(R-r)^{m^*(i-p)+m^*(p-2)}}\Bigg) \\ &= \frac{M/\sigma(BF_i + \sum_{p+\mu+|\nu|+\xi=i} {\sum}^* G^*_{p,\mu,\nu,\xi} \prod_{i=1}^{\mu} E_{l_i} \prod_{|\alpha|\leqslant m_1}\prod_{j=1}^{\nu_\alpha} E_{m_{\alpha,j}} \prod_{k=1}^{\xi} F_{n_k})}{(R-r)^{m^*(i-1)}} \end{aligned}$$

□

We give the following proposition in order to show the convergence of the pair of the formal power series solutions $(\hat{u}, \hat{v}^*)$.

Proposition 4.2. *For any* $0 < r < R$ *we have*

$$\|q^i u_i\|_r,\ \|\partial_x^\alpha u_i\|_r \leqslant X_i \quad \text{for} \quad i \geqslant 1,\ |\alpha| \leqslant m^*, \tag{4.8}$$

$$\|v_i\|_r \leqslant Y_i \quad \text{for} \quad i \geqslant 2. \tag{4.9}$$

Proof. We prove the evaluations by induction on i. For the case $i = 1$ it follows by the definition of A. For the case $i = 2$ by recurrence formulas and induction hypothesis we have

$$\begin{aligned}\|q^2 v_2\|_r &= \left\| c_2 + \sum_{|\alpha| \leqslant m_2} d_\alpha \partial_x^\alpha u_1 \right\|_r \\ &\leqslant C_2 + \sum_{|\alpha| \leqslant m_2} D_\alpha X_1 \\ &= Y_2\end{aligned}$$

therefore we have

$$\|v_2\|_r \leqslant X_2.$$

For $i \geqslant 3$ we have in the same manner

$$\begin{aligned}\|q^i v_i\|_r &= \left\| c_i + \sum_{|\alpha| \leqslant m_2} d_\alpha \partial_x^\alpha u_{i-1} \right\|_r \\ &\leqslant C_i + \sum_{|\alpha| \leqslant m_2} D_\alpha X_{i-1} \\ &\leqslant \frac{C_i}{(R-r)^{m^*(i-2)}} + \sum_{|\alpha| \leqslant m_2} D_\alpha X_{i-1} \\ &= Y_i\end{aligned}$$

thus we get

$$\|v_i\|_r \leqslant Y_i.$$

Hence on $v_i(x)$ for $i \geqslant 2$ we get the inequalitiy (4.9) in Proposition 4.2. On the other hand, for $i \geqslant 2$ we have

$$
\begin{aligned}
&\|(q^i-\lambda)u_i\|_r \\
&= \left\| bv_i + \sum_{p+\mu+|\nu|+\xi=i} \sum\nolimits^* g^*_{p,\mu,\nu,\xi} \prod_{i=1}^{\mu} q_{l_i} u_{l_i} \prod_{|\alpha|\leqslant m_1} \prod_{j=1}^{\nu_\alpha} \partial_x^\alpha u_{m_{\alpha,j}} \prod_{k=1}^{\xi} v_{n_k} \right\|_r \\
&\leqslant BY_i + \sum_{p+\mu+|\nu|+\xi=i} \sum\nolimits^* G^*_{p,\mu,\nu,\xi} \prod_{i=1}^{\mu} X_{l_i} \prod_{|\alpha|\leqslant m_1} \prod_{j=1}^{\nu_\alpha} X_{m_{\alpha,j}} \prod_{k=1}^{\xi} Y_{n_k} \\
&\leqslant BY_i + \sum_{p+\mu+|\nu|+\xi=i} \sum\nolimits^* \frac{G^*_{p,\mu,\nu,\xi}}{(R-r)^{m^*(p+\mu+|\nu|+2\xi-2)}} \prod_{i=1}^{\mu} X_{l_i} \prod_{|\alpha|\leqslant m_1} \prod_{j=1}^{\nu_\alpha} X_{m_{\alpha,j}} \prod_{k=1}^{\xi} Y_{n_k} \\
&= \frac{(R-r)^{m^*}}{M}\sigma X_i
\end{aligned}
$$

therefore by Lemma 2.3 we have

$$\|u_i\|_r \leqslant \frac{1}{\sigma q^i}\frac{(R-r)^{m^*}}{M}\sigma X_i \leqslant \frac{X_i}{q^i}. \tag{4.10}$$

Let give estimates of the derivative of $u_i(x)$ for $i \geqslant 2$. By applying Lemma 2.1 to (4.10) with m_1-times, then we have for $|\alpha| \leqslant m_1$

$$
\begin{aligned}
\|\partial_x^\alpha u_i\|_r &\leqslant \frac{(m^*(i-2)+1)+\cdots+(m^*(i-2)+m^*)}{Mq^i} e^{m^*} \frac{E_i}{(R-r)^{m^*(i-2)+m^*}} \\
&\leqslant \frac{(m^*(i-2)+2m^*)+\cdots+(m^*(i-2)+2m^*)}{Mq^i} e^{m^*} \frac{E_i}{(R-r)^{m^*(i-1)}} \\
&= \frac{(m^* i e)^{m^*}}{Mq^i} X_i \\
&\leqslant X_i.
\end{aligned}
$$

Therefore we get the desired results. □

Thus, by summing up from Proposition 4.2 we have

$$|\hat{u}| \leqslant X, \ |\hat{v}| \leqslant Y.$$

Therefore the formal power series solutions $(\hat{u}, \hat{v})$ of (3.1) converges in a neighborhood of the origin $t = 0$.

§ 5. Formal solutions for the particular case of (1.1)

In this section we will construct formal solutions as singular solutions for the particular case of the system (1.1). Let us consider the following system:

$$
\begin{cases}
(\sigma_q - \lambda(x))u = a(x)t + v + u\partial_x u \\
\sigma_q v = b(x)t + t\partial_x u
\end{cases} \tag{5.1}
$$

where $(t,x) \in \mathbb{C}_t \times \mathbb{C}_x$, $\lambda(x)$, $a(x)$, $b(x) \in \mathcal{O}(D_{R_0})$.

Definition 5.1. Let us denote by

- $\mathcal{R}(\mathbb{C}\setminus\{0\})$ the universal covering space of $\mathbb{C}\setminus\{0\}$,
- S_θ the sector in $\mathcal{R}(\mathbb{C}\setminus\{0\})$; $\{t \in \mathbb{C}\setminus\{0\}; |\arg t| < \theta\}$,
- $S(\epsilon(s)) = \{t \in \mathbb{C}\setminus\{0\}; 0 < |t| < \epsilon(\arg t)\}$ for some positive-valued function $\epsilon(s)$ defined and continuous on $\mathbb{R}$.

Definition 5.2. We define the set $\tilde{\mathcal{O}}_+$ of all functions $u(t,x)$ satisfying the following conditions:

1. $u(t,x)$ is holomorphic in $S(\epsilon(s)) \times D_R$ for some ϵ and $R > 0$,
2. there is an $a > 0$ such that for any $\theta > 0$ and compact subset K of D_R

$$\max_{x\in K} |u(t,x)| = O(|t|^a) \quad \text{as } t \to 0 \text{ in } S_\theta.$$

Theorem 5.3. *Set $\lambda_q(x) = \log\lambda(x)/\log q$. If the holomorphic function $\lambda(x)$ satisfies $\lambda(0) \neq q^i$ for $i \in \mathbb{N}^*$, then there exists a pair of formal solutions of* (5.1) *having an expansion of the following forms*:

$$\left\{\begin{aligned} U(\varphi) &= \sum_{i\geqslant 1} u_i(x)t^i + \sum_{k\leqslant i+j-1, j\geqslant 1} \varphi_{i,j,k}(x)t^{i+j\lambda_q(x)}(\log t)^k \\ V(\varphi) &= \sum_{i\geqslant 1} v_i(x)t^i + \sum_{k\leqslant i+j-1, i,j\geqslant 1} \psi_{i,j,k}(x)t^{i+j\lambda_q(x)}(\log t)^k \end{aligned}\right. \tag{5.2}$$

where $\varphi = \varphi_{0,1,0}(x)$ is any holomorphic function on D_{R_0}, $u_i(x), v_i(x) \in \mathcal{O}(D_{R_0})$ for $i \geqslant 1$ and $\varphi_{i,j,k}(x), \psi_{i,j,k}(x) \in \mathcal{O}(D_{R_0})$ for $j \geqslant 1, k \leqslant i+j-1$.

Proof. Set for $I \geqslant 1$

$$u_I(t,x) = u_I(x)t^I,\ \varphi_I(t,x) = \sum_{i+j=I, j\geqslant 1}\ \sum_{k\leqslant i+j-1} \varphi_{i,j,k}(x)t^{i+j\lambda_q(x)}(\log t)^k \tag{5.3}$$

and for $I \geqslant 2$

$$v_I(t,x) = v_I(x)t^I,\ \psi_I(t,x) = \sum_{i+j=I, i,j\geqslant 1}\ \sum_{k\leqslant i+j-1} \psi_{i,j,k}(x)t^{i+j\lambda_q(x)}(\log t)^k. \tag{5.4}$$

Then we can rewrite the form (5.2) as follows:

$$U(\varphi) = \sum_{I\geqslant 1}(u_I + \varphi_I),\ V(\varphi) = v_1 + \sum_{I\geqslant 2}(v_I + \psi_I). \tag{5.5}$$

By substituting the pair of formal solutions (5.5) into the system (5.1), then we have the following recurrence formulas for u_I, v_I and φ_I, ψ_I:

$$\left\{\begin{aligned} & qv_1(x) = b(x) \\ & (q-\lambda(x))u_1(x) = a(x) + v_1(x) \end{aligned}\right.$$

and for $I \geqslant 2$

$$\begin{cases} q^I v_I(x) = \partial_x u_{I-1}(x) \\ (q^I - \lambda(x))u_I(x) = v_I(x) + \sum\limits_{I_1+I_2=I} u_{I_1}(x)\partial_x u_{I_2}(x) \end{cases}$$

and:

$$(\sigma_q - \lambda(x))\varphi_1(t,x) = 0$$

and for $I \geqslant 2$

$$\begin{cases} \sigma_q \psi_I(t,x) = \partial_x \varphi_{I-1}(t,x) \\ (\sigma_q - \lambda(x))\varphi_I(t,x) \\ = \psi_I(t,x) + \sum\limits_{I_1+I_2=I} (u_{I_1}(t,x) + \varphi_{I_1}(t,x))\partial_x(u_{I_2}(t,x) + \varphi_{I_2}(t,x)) \\ \qquad - \sum\limits_{I_1+I_2=I} u_{I_1}(t,x)\partial_x u_{I_2}(t,x). \end{cases} \tag{5.6}$$

By assumption, it is clear that $u_I(x)$ and $v_I(x)$ are determined for $I \in \mathbb{N}^*$. We take any holomorphic function $\varphi(x) \in \mathcal{O}(D_{R_0})$ and put $\varphi(x) = \varphi_{0,1,0}(x)$. For $I \geqslant 2$ we will show that φ_I and ψ_I are determined by induction. By the above recurrence formulas, it is obvious that ψ_I is determined for $I \geqslant 2$ if φ_I is determined for $I \geqslant 1$. Therefore, for verifying that it is sufficient to see that φ_I is determined for $I \geqslant 2$. We substitute the function φ_I of the form (5.3) into the second equation of (5.6), then we have

$$\begin{aligned} &(q^{i+j\lambda_q(x)} - \lambda(x))\varphi_{i,j,k}(x) + \sum_{k'=1}^{i+j-1-k} q^{i+j\lambda_q(x)} \frac{(k+k')!}{k!k'!} (\log q)^{k'} \varphi_{i,j,k+k'}(x) \\ &= \psi_{i,j,k}(x) + \sum_{\substack{i_1+i_2=i \\ i_1 \geqslant 1, i_2 \geqslant 0}} \Bigg[j\partial_x \lambda_q(x) u_{i_1}(x)\varphi_{i_2,j,i_2+j-1}(x) \\ &\qquad + \sum_{k \leqslant i_2+j-1} \Big\{ \partial_x(u_{i_1}(x)\varphi_{i_2,j,k}(x)) + j\partial_x\lambda_q(x)u_{i_1}(x)\varphi_{i_1,j,k-1}(x) \Big\} \Bigg] \\ &\qquad + \sum_{\substack{i_1+i_2=i \\ j_1+j_2=j, j_1,j_2 \geqslant 1 \\ k_1+k_2=k}} \Big\{ \varphi_{i_1,j_1,k_1}(x)(\partial_x \varphi_{i_2,j_2,k_2}(x) + j_2 \partial_x \lambda_q(x) \varphi_{i_2,j_2,k_2-1}(x)) \Big\} \end{aligned}$$

where

$$\sum_{\substack{i_1+i_2=i \\ j_1+j_2=j, j \geqslant 2 \\ k_1+k_2=i+j-1}} \varphi_{i_1,j_1,k_1}(x)\partial_x \varphi_{i_2,j_2,k_2}(x) \equiv 0.$$

The definition of $\lambda_q(x)$ and this recurrence formulas tell us that the system (5.1) has the pair of formal solutions in the form (5.2). $\square$

References

[1] R. Bielawski, Ricci-flat Kähler metrics on canonical bundles, *Math. Proc. Cambridge philos. Soc.* **132** (2002), no. 3, 471-479.

[2] R. Gérard and H. Tahara, *Singular Nonlinear Partial Differential Equations*, Vieweg, 1996.

[3] H. Yamazawa, Holomorphic and Singular Solutions of q-Difference-Differential Equations of Briot-Bouquet Type, *Fuckcialaj Ekvacioj,* **59** (2016), 185-197.