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A survey on the restriction problem of p-adic unitary group for some non-generic L-parameter

Jaeho Haan

Research Fellow, Center for Mathematical Challenges

Abstract

The local Gan-Gross-Prasad conjecture of unitary groups, which is now settled by the works of Beuzart-Plessis, Gan and Ichino, says that for a pair of generic L-parameters of $(U(n+1), U(n))$, there is a unique pair of representations in their associated Vogan L-packets which produces the Bessel model. In this survey article, we report that the conjecture does not hold for a non-generic case.

1 Introduction

The local *Gan-Gross-Prasad* (GGP) conjecture concerns the restriction problem of real or p-adic Lie groups. Though the GGP conjecture is now formulated for all classical groups, we will restrict ourselves only to unitary groups in this survey article.

Let E/F be a quadratic extension of local fields of characteristic zero. Let V_{n+1} be a Hermitian space of dimension $n+1$ over E and W_n a skew-Hermitian space of dimension n over E. Let $V_n \subset V_{n+1}$ be a nondegenerate subspace of codimension 1 and we set

$$G_n = \mathrm{U}(V_n) \times \mathrm{U}(V_{n+1}) \quad \text{or} \quad \mathrm{U}(W_n) \times \mathrm{U}(W_n)$$

and

$$H_n = \mathrm{U}(V_n) \quad \text{or} \quad \mathrm{U}(W_n).$$

Then we have a diagonal embedding

$$\Delta : H_n \hookrightarrow G_n.$$

Let π be an irreducible smooth representation of G_n. In the Hermitian case, one is interested in computing

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n}(\pi, \mathbb{C})$$

and it is called the *Bessel* case (B) of the GGP conjecture. To describe the GGP conjecture for the skew-Hermitian case, we need another data, that is a Weil representation ω_{ψ,χ,W_n}.

(Here, ψ is a nontrivial additive character of F and χ is a character of $E^\times$ whose restriction to $F^\times$ is the non-trivial quadratic character associated to E/F by local class field theory.) In this case, one is interested in computing

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n}(\pi, \omega_{\psi,\chi,W_n})$$

and we call this the *Fourier–Jacobi* case (FJ) of the GGP conjecture. To treat them simultaneously, we use the notation $\nu = \mathbb{C}$ or ω_{ψ,χ,W_n} in the respective cases.

By the results of [1] and [9], it is known

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n}(\pi, \nu) \leq 1.$$

So our next task should be specifying irreducible smooth representations π such that

$$\mathrm{Hom}_{\Delta H_n}(\pi, \nu) = 1.$$

In a seminal paper [5], Gan, Gross and Prasad proposed a conjecture which contains both mulitiplicity one theorem (for generic case) and the answer to the above question. To explain it, we need the notion of relevant pure inner forms of G_n and relevent Vogan L-packets. A pure inner form of G_n is a group of the form

$$G_n' = \mathrm{U}(V_{n+1}') \times \mathrm{U}(V_n') \quad \text{or} \quad \mathrm{U}(W_n') \times \mathrm{U}(W_n')$$

where $V_n' \subset V_{n+1}'$ are hermitian spaces over E whose dimensions are n and $n+1$ respectively and W_n' is a n-dimensional skew-hermitian spaces over E.

Furthermore, if

$$V_{n+1}'/V_n' \cong V_{n+1}/V_n \quad \text{or} \quad W_n' = W_n,$$

we say that G_n' is a relevant pure inner form of G_n.

If G_n' is relevant of G_n, we set

$$H_n' = \mathrm{U}(V_n') \quad \text{or} \quad \mathrm{U}(W_n')$$

so that we have a diagonal embedding

$$\Delta : H_n' \hookrightarrow G_n'.$$

For an L-parameter ϕ of G_n, there is the associated (relevant) Vogan L-packet Π_ϕ which consists of certain irreducible smooth representations of G_n and its (relevant) pure inner forms G_n' whose corresponding L-parameter is ϕ. We denote the relevant Vogan L-packet of ϕ by Π_ϕ^R.

Now we can loosely state the GGP conjecture as follows:

Gan–Gross–Prasad conjecture. *For a generic L-parameter ϕ of G_n, the followings hold:*

(i) $\sum_{\pi' \in \Pi_\phi^R} \dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n'}(\pi', \nu) = 1.$

(ii) *Using the local Langlands correspondence for unitary group, we can pinpoint $\pi' \in \Pi_\phi^R$ such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n'}(\pi', \nu) = 1.$$

2 Current status of the GGP conjecture

Following the strategy of Waldspurger ([11]–[14]) for orthogonal groups, Beuzart-Plessis [2],[3],[4] established (B) of the GGP conjecture for tempered L-parameter ϕ. Building upon Beuzart-Plessis's work, Gan and Ichino [6] proved (FJ) for tempered case first by establishing the precise local theta correspondence for almost equal rank unitray groups and then extended both (B) and (FJ) to generic cases. Because the generic case is now completely settled, it is natural to turn our attention to the non-generic case.

3 Main Theorem

In [8], the author considered a non-generic case of (B) when $n = 2$. We extended the reult to all $n \geq 2$ when an L-parameter of G_n involves some non-generic L-parameter of $U(V_{n+1})$. We can roughly state our main result in the following.

Main Theorem. *For all $n \geq 1$, let ϕ^{NG} be a special non-generic L-parameter of $U(V_{n+2})$ whose L-packet consisting of only supercuspidal representations and ϕ^T be a tempered L-parameter of $U(V_{n+1})$. Then for the L-parameter $\phi = \phi^{NG} \otimes \phi^T$ of $G_{n+1} = U(V_{n+2}) \times U(V_{n+1})$, we have*

(i) *If the L-parameter ϕ^T does not contain χ_W^{-1},*

$$\sum_{\pi' \in \Pi_\phi^R} \dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H'_{n+1}}(\pi', \mathbb{C}) = 0$$

(ii) *Suppose that ϕ^T contains χ_W^{-1}. Then*

$$\sum_{\pi' \in \Pi_\phi^R} \dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H'_{n+1}}(\pi', \mathbb{C}) \geq 1.$$

(iii) *If the multiplicity of χ_W^{-1} in ϕ^T is one, we have*

$$\sum_{\pi' \in \Pi_\phi^R} \dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H'_{n+1}}(\pi', \mathbb{C}) = 1.$$

Furthermore, using the local Langlands correspondence, we can explicitly describe $\pi' \in \Pi_\phi^R$ such that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H'_{n+1}}(\pi', \mathbb{C}) = 1.$$

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Research Fellow of CMC

E-mail address: jaehohaan@gmail.com