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L^q-L^r estimate of a generalized Oseen evolution operator, with applications to the Navier-Stokes flow past a rotating obstacle

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1 Introduction

Let us consider the motion of a viscous incompressible fluid in 3D exterior domains when the obstacle (rigid body) moves with prescribed time-dependent translational and angular velocities, while the fluid is at rest at spatial inifinity. In a frame attached to the moving obstacle, the problem is reduced to

$$\begin{aligned}
&\partial_t u + u\cdot\nabla u = \Delta u - \nabla p_u + (\eta(t)+\omega(t)\times x)\cdot\nabla u - \omega(t)\times u,\\
&\mathrm{div}\ u = 0,\\
&u|_{\partial D} = \eta(t)+\omega(t)\times u,\\
&u\to 0 \quad \text{as } |x|\to\infty,\\
&u(\cdot,0)=u_0,
\end{aligned} \tag{1.1}$$

in a fixed exterior domain $D\subset\mathbb{R}^3$ with smooth boundary ∂D (see Galdi [7] for details), where $u=u(x,t)$ and $p_u=p_u(x,t)$ are unknown velocity and pressure of the fluid. The translational and angular velocities of the obstacle are denoted by $\eta(t)$ and $\omega(t)$. Suppose they converge to some constant vectors η_∞, ω_∞, respectively, as $t\to\infty$. The existence of a unique global solution to (1.1) with small $u_0\in L^3(D)$ for the case $\{\eta_\infty,\omega_\infty\}=\{0,0\}$ has been studied by the present author [16]. In this article we discuss more relevant case $\{\eta_\infty,\omega_\infty\}\neq\{0,0\}$. When they are small enough, there is a steady flow $\{u_s,p_{u_s}\}$ (unique in the small $\|u_s\|_{3,\infty}$) with respect to a frame attached to the moving obstacle with the rigid motion $\eta_\infty+\omega_\infty\times x$, see section 2 for the steady problem (2.1).

The linearization around the steady flow $\{u_s,p_{u_s}\}$ leads to the non-autonomous system

$$\begin{aligned}
\partial_t u + u_s\cdot\nabla u + u\cdot\nabla u_s &= \Delta u - \nabla p_u + (\eta(t)+\omega(t)\times x)\cdot\nabla u - \omega(t)\times u,\\
\mathrm{div}\ u &= 0.
\end{aligned} \tag{1.2}$$

For our goal mentioned above, analysis of large time behavior of solutions to the initial value problem for (1.2) subject to the homogeneous Dirichlet boundary condition is the essential step. This is done under the assumption

$$\eta,\ \omega \in C^\theta([0,\infty);\mathbb{R}^3)\cap L^\infty(0,\infty;\mathbb{R}^3) \tag{1.3}$$

for some $\theta\in(0,1)$ as well as smallness of the steady flow u_s in $L^{3,\infty}(D)$ (which is accomplished for small η_∞, ω_∞). We show that the linearized operator generates an evolution operator $\{T(t,s)\}_{t\geq s\geq 0}$ on $L^q_\sigma(D)$ by using evolution operators in the whole space and in a bounded domain near the boundary. The construction of a parametrix of the evolution operator in exterior domains is performed along the approach introduced by Hansel and Rhandi [14], who studied the case $u_s=0$, and it is based on a certain iteration combined with a cut-off procedure. This procedure provides us with the L^q-L^r smoothing rates of $T(t,s)$ near the initial time $t=s$ although it is out of class of parabolic evolution operators in the sense of Tanabe [22] on account of the drift term whose coefficient is the rigid motion. We then follow more or less the idea of [16] based on the duality argument with use of the first energy relation as well as a cut-off technique to develop the L^q-L^r decay estimates of the evolution operator $T(t,s)$ and its adjoint $T(t,s)^*$ as $(t-s)\to\infty$. Unfortunately our approach does not work for deduction of the pointwise decay of the gradient of the evolution operator, nevertheless, we can get around this difficulty in construction of the Navier-Stokes flow by making use of the energy relation combined with the L^q-L^r estimate. This way was proposed first by the present author [16, Lemma 5.1].

As a result, under suitable conditions, one finds a global solution to (1.1) which goes to the steady flow u_s as $t\to\infty$, see Theorem 2.1 in the next section, however, in this article we concentrate ourselves on development of analysis of the linearized operator. Section 3 is devoted to construction of the evolution operator generated by (1.2). In section 4 we give the outline of the proof of its decay properties.

2 Result

We start with the corresponding steady problem with the rigid motion $\eta_\infty+\omega_\infty\times x$, that is,

$$\begin{aligned}
&-\Delta u_s+\nabla p_{u_s}-(\eta_\infty+\omega_\infty\times x)\cdot\nabla u_s+\omega_\infty\times u_s+u_s\cdot\nabla u_s=0,\\
&\mathrm{div}\ u_s=0,\\
&u_s|_{\partial D}=\eta_\infty+\omega_\infty\times x,\\
&u_s\to 0\quad\text{as } |x|\to\infty,
\end{aligned} \tag{2.1}$$

in the exterior domain $D \subset \mathbb{R}^3$. When $\omega_\infty \neq 0$, the Mozzi-Chasles transform, see [11, Section 2] and [9, Chapter VIII], makes it possible to reduce the problem to the particular case when the translational and angular velocities are parallel each other. In fact, the drift term can be rewritten as

$$(\eta_\infty + \omega_\infty \times x) \cdot \nabla u = \left\{ \frac{(\omega_\infty \cdot \eta_\infty)\omega_\infty}{|\omega_\infty|^2} + \omega_\infty \times \left(x - \frac{\omega_\infty \times \eta_\infty}{|\omega_\infty|^2} \right) \right\} \cdot \nabla u.$$

Thus, by the change $\widetilde{x} = x - \omega_\infty \times \eta_\infty / |\omega_\infty|^2$, the new translational velocity is indeed parallel to ω_∞. In view of this observation, let us consider the cases

$$\text{(i)} \quad \omega_\infty \neq 0,\ \omega_\infty \cdot \eta_\infty = 0 \qquad \text{(ii)} \quad \omega_\infty \neq 0,\ \omega_\infty \cdot \eta_\infty \neq 0 \qquad \text{(iii)} \quad \omega_\infty = 0,\ \eta_\infty \neq 0$$

separately. We know from [3], [4], [8], [11], [12] and [15] that there is a constant $\delta_0 = \delta_0(D) > 0$ with the following property: if $|\eta_\infty| + |\omega_\infty| \leq \delta_0$, then problem (2.1) admits a solution $\{u_s, p_{u_s}\} \in C^\infty(D)$ which is unique in the small (concerning the uniqueness, we have even more, see [11], [15]). In any case, the solution is of class

$$\sup_{x \in D} (1 + |x|)|u_s(x)| < \infty, \qquad \{\nabla u_s, p_{u_s}\} \in L^{3/2,\infty}(D) \cap H^1(D).$$

In addition, we have

$$\sup_{x \in D} (1 + |x|)^2 |\nabla u_s(x)| < \infty \tag{2.2}$$

for the case (i), while an anisotropic decay structure with wake region leads to

$$u_s \in L^q(D) \quad \forall q \in (2, \infty]; \qquad \nabla u_s \in L^r(D) \quad \forall r \in (4/3, \infty]$$

for the cases (ii), (iii). All the quantities above are bounded by $|\eta_\infty|$ and $|\omega_\infty|$. Let us collect the properties of u_s we need in what follows (the last condition in (2.3) follows actually from (2.2) for the case (i)):

$$\begin{cases} \{u_s,\ |x|u_s,\ \nabla u_s,\ \nabla^2 u_s,\ |x|\nabla^2 u_s\} \in L^\infty(D), \\ \nabla u_s \in L^{3/2,\infty}(D), \qquad |x|\nabla u_s \in L^{3,\infty}(D), \end{cases} \tag{2.3}$$

$$\|u_s\|_{3,\infty} \leq c_\dagger(|\eta_\infty| + |\omega_\infty|) \leq c_\dagger \delta_0. \tag{2.4}$$

Here and throughout this article, L^q and $L^{q,\infty}$ denote the Lebesgue and Lorentz spaces endowed with norms $\|\cdot\|_q$ and $\|\cdot\|_{q,\infty}$. If the region over which those function spaces are defined is different from the exterior domain D under consideration, we indicate it, for instance, $\|\cdot\|_{q,\mathbb{R}^3}$.

We find the solution to (1.1) of the form $u = u_s + v,\ p_u = p_{u_s} + p_v$, where the perturbation shoud obey

$$
\begin{aligned}
&\partial_t v + v\cdot\nabla v + u_s\cdot\nabla v + v\cdot\nabla u_s = \Delta v - \nabla p_v + (\eta+\omega\times x)\cdot\nabla v - \omega\times v + f,\\
&\mathrm{div}\ v = 0,\\
&v|_{\partial D} = (\eta-\eta_\infty) + (\omega-\omega_\infty)\times x\\
&v\to 0 \quad \text{as } |x|\to\infty,\\
&v(\cdot,0) = v_0 := u_0 - u_s,
\end{aligned}
\tag{2.5}
$$

with

$$
f = \{(\eta(t)-\eta_\infty) + (\omega(t)-\omega_\infty)\times x\}\cdot\nabla u_s - (\omega(t)-\omega_\infty)\times u_s.
$$

To treat this external force, the last condition in (2.3) is needed. As in [16, Section 5.2], the lifting function

$$
b = \frac{1}{2}\nabla\times\left[\phi(x)\left\{(\eta(t)-\eta_\infty)\times x - |x|^2(\omega(t)-\omega_\infty)\right\}\right]
$$

of the boundary data in (2.5) is harmless, where ϕ is a suitable cut-off function. Let us introduce the linearized operator $L(t)$ by

$$
\begin{cases}
D_q(L(t)) = \{u\in L^q_\sigma(D)\cap W^{1,q}_0(D)\cap W^{2,q}(D);\ (\omega(t)\times x)\cdot\nabla u\in L^q(D)\},\\
L(t)u = -P[\Delta u + (\eta(t)+\omega(t)\times x)\cdot\nabla u - \omega(t)\times u - u_s\cdot\nabla u - u\cdot\nabla u_s],
\end{cases}
\tag{2.6}
$$

where P denotes the Fujita-Kato projection associated with the Helmholtz decomposition of L^q-vector fields ([6], [20], [21]) and $L^q_\sigma(D)$ is the solenoidal L^q-space, $1<q<\infty$. Then the operator family $\{L(t)\}_{t\geq 0}$ generates an evolution operator $\{T(t,s)\}_{t\geq s\geq 0}$ on $L^q_\sigma(D)$, see Proposition 3.2. In terms of the adjoint evolution opertor $T(t,s)^*$ as well as $T(t,s)$, the equation for $w=v-b$ is described as

$$
\begin{aligned}
\langle w(t),\psi\rangle &= \langle T(t,0)w_0,\psi\rangle + \int_0^t \langle T(t,\tau)(f+g)(\tau),\psi\rangle d\tau\\
&\quad + \int_0^t \langle (w\otimes w + w\otimes b + b\otimes w)(\tau), \nabla T(t,\tau)^*\psi\rangle d\tau, \qquad \forall\psi\in C^\infty_{0,\sigma}(D),
\end{aligned}
\tag{2.7}
$$

where

$$
\begin{aligned}
&w_0 = v_0 - b(\cdot,0),\\
&g = \Delta b + (\eta+\omega\times x)\cdot\nabla b - \omega\times b - \partial_t b - b\cdot\nabla b - u_s\cdot\nabla b - b\cdot\nabla u_s.
\end{aligned}
$$

We are now in a position to state the main result. Note that the assumptions (2.8)–(2.9) below on $\{\eta,\omega\}$ imply (1.3), under which the linearized system (1.2) is analyzed. Why we take w_0 from $L^{3,\infty}_\sigma(D)$ rather than $L^3_\sigma(D)$ is that the steady flow u_s does not belong to $L^3(D)$ in general for the case (i) (that is, purely rotating case after the Mozzi-Chasles transform). Having the starting problem ([5], [10]) in mind, we prefer to cover the case

$u_0 = 0$ in (1.1). By (2.10) and (2.9) we conclude that $\lim_{t\to\infty} \|u(t) - u_s\|_r = 0$ for (1.1), where $r \in (3, 6)$.

Theorem 2.1 *Let*

$$\eta,\ \omega \in C^1([0,\infty); \mathbb{R}^3). \tag{2.8}$$

Suppose that there are $\{\eta_\infty, \omega_\infty\} \in \mathbb{R}^3 \times \mathbb{R}^3$ *and* $\gamma \in (1,\infty)$ *such that*

$$\begin{aligned} M_0 &:= \sup_{t\geq 0}\, (1+t)^\gamma(|\eta(t) - \eta_\infty| + |\omega(t) - \omega_\infty|) < \infty, \\ M_1 &:= \sup_{t\geq 0}\, (1+t)^{1/8}(|\eta'(t)| + |\omega'(t)|) < \infty, \end{aligned} \tag{2.9}$$

as well as (2.3) *and* (2.4)*. Then there is a constant* $\delta = \delta(D,\gamma) > 0$ *with the following property: If* $w_0 \in L^{3,\infty}_\sigma(D)$ *and if*

$$|\eta_\infty| + |\omega_\infty| + M_0 + M_1 + \|w_0\|_{3,\infty} \leq \delta,$$

then problem (2.7) *admits a unique global solution which enjoys*

$$\|w(t)\|_r = O(t^{-\mu}) \qquad \text{as } t \to \infty \tag{2.10}$$

for every $r \in (3,6)$ *with* $\mu := \min\{1/2 - 3/2r, 1/8\}$.

3 Evolution Operator

Let us start with the initial value problem for (1.2) in $\mathbb{R}^3 \times (s,\infty)$ subject to $u(\cdot, s) = f$, where the initial time $s \geq 0$ is a parameter. Here, the coefficient u_s is understood as extension by setting $\eta_\infty + \omega_\infty \times x$ in $\mathbb{R}^3 \setminus D$. By $U_0(t,s)$ we denote the evolution operator for the case $u_s = 0$ in the whole space. One can explicitly describe the formula of $U_0(t,s)$, see [16, (3.4), Lemma 3.1], which together with $\nabla U_0(t,s)$ satisfies the L^q-L^r decay estimates. The regularity of $U_0(t,s)$ in terms of the space

$$Y_q(\mathbb{R}^3) = \{u \in L^q_\sigma(\mathbb{R}^3) \cap W^{2,q}(\mathbb{R}^3);\ |x|\nabla u \in L^q(\mathbb{R}^3)\}$$

has been also investigated, see Hansel and Rhandi [14] and the references therein. The linearized operator $L_{\mathbb{R}^3}(t)$ in the whole space is defined in the same way as in (2.6), then we note that $Y_q(\mathbb{R}^3) \subset D_q(L_{\mathbb{R}^3}(t))$ for all t. The following proposition provides a solution to the initial value problem above. For the proof, the regular solution and the decaying one to

$$u(t) = U_0(t,s)f - \int_s^t U_0(t,\tau)Bu(\tau)\, d\tau, \qquad Bu = P_{\mathbb{R}^3}(u_s \cdot \nabla u + u \cdot \nabla u_s), \tag{3.1}$$

are constructed independently, where $P_{\mathbb{R}^3}$ denotes the Fujita-Kato projection in the whole space, but they can be identified each other. In construction of the latter solution, a real interpolation technique developed by Yamazaki [23] plays a crucial role. This technique was also adopted in the paper [17, Theorem 2.2, Proposition 4.2] by Schonbek and the present author, where the situation was quite similar. Notice that this technique does not provide us with decay properties of the gradient of solutions, however, see (3.4).

Proposition 3.1 *Suppose* (1.3) *for some* $\theta \in (0,1)$. *Let* $1 < q < \infty$. *Then there is an operator family* $\{U(t,s)\}_{t\geq s\geq 0}$ *on* $L^q_\sigma(\mathbb{R}^3)$ *with the following properties.*

1. *$U(t,s)$ is a bounded linear operator from $L^q_\sigma(\mathbb{R}^3)$ into itself with the semigroup property*
$$U(t,\tau)U(\tau,s) = U(t,s) \quad (t \geq \tau \geq s \geq 0); \quad U(s,s) = I,$$
in $\mathcal{L}(L^q_\sigma(\mathbb{R}^3))$.

2. *For every $f \in L^q_\sigma(\mathbb{R}^3)$, the function $u(t) = U(t,s)f$ is of class $u \in C((s,\infty); L^q_\sigma(\mathbb{R}^3)\cap W^{1,q}(\mathbb{R}^3))$ and satisfies* (3.1) *as well as $\lim_{t\to s}\|u(t) - f\|_{q,\mathbb{R}^3} = 0$.*

3. *Let $\tau_* \in (0,\infty)$ and $r \in [q,\infty]$, then there is a constant $C_{\tau_*} = C_{\tau_*}(q,r,u_s) > 0$ such that*
$$\|\nabla^j U(t,s)f\|_{r,\mathbb{R}^3} \leq C_{\tau_*}(t-s)^{-(3/q-3/r)/2-j/2}\|f\|_{q,\mathbb{R}^3} \tag{3.2}$$
for all (s,t) with $0 \leq s < t \leq s+\tau_*$ and $f \in L^q_\sigma(\mathbb{R}^3)$, where $j = 0,1$.

4. *If in particular $q \in (3/2,\infty)$, then the following two assertions hold so that $U(t,s)$ is indeed an evolution operator in usual sense:*
Fix $s \geq 0$, then for every $f \in Y_q(\mathbb{R}^3)$ and $t \in [s,\infty)$ we have $U(t,s)f \in Y_q(\mathbb{R}^3)$ and $U(\cdot,s)f \in C^1([s,\infty); L^q_\sigma(\mathbb{R}^3))$ with
$$\partial_t U(t,s)f + L_{\mathbb{R}^3}(t)U(t,s)f = 0, \quad t \in [s,\infty),$$
in $L^q_\sigma(\mathbb{R}^3)$.
Fix $t \geq 0$, then for every $f \in Y_q(\mathbb{R}^3)$, we have $U(t,\cdot)f \in C^1([0,t]; L^q_\sigma(\mathbb{R}^3))$ with
$$\partial_s U(t,s)f = U(t,s)L_{\mathbb{R}^3}(s)f, \quad s \in [0,t],$$
in $L^q_\sigma(\mathbb{R}^3)$.

5. *For every $r_\dagger \in (q,\infty)$, there is a constant $\delta_* = \delta_*(q,r_\dagger) > 0$ with the following property. If $\|u_s\|_{3,\infty} \leq \delta_*$, then for every $r \in [q,r_\dagger)$ there is a constant $C = C(q,r_\dagger,r,u_s) > 0$ such that*
$$\|U(t,s)f\|_{r,\mathbb{R}^3} \leq C(t-s)^{-(3/q-3/r)/2}\|f\|_{q,\mathbb{R}^3} \tag{3.3}$$
for all $t > s \geq 0$ and $f \in L^q_\sigma(\mathbb{R}^3)$.

Although optimal rate of decay of the gradient of the evolution operator is not available, combining (3.3) with (3.2) simply leads to

$$\|\nabla U(t,s)f\|_{r,\mathbb{R}^3} \le C_1\|U(t-1,s)f\|_{r,\mathbb{R}^3} \le C(t-s)^{-(3/q-3/r)/2}\|f\|_{q,\mathbb{R}^3} \tag{3.4}$$

for $t-s>2$ as long as u_s is sufficiently small in $L^{3,\infty}(D)$; indeed, (3.4) is needed for (4.8) and also enough for our aim. Why we need the condition $q>3/2$ to get the fourth assertion above is that the boundedness $\||x|\mathcal{R}f\|_{q,\mathbb{R}^3} \le C\||x|f\|_{q,\mathbb{R}^3}$ holds for such q by the Muckenhoupt theory, where $\mathcal{R}=\nabla(-\Delta)^{-1/2}$ is the Riesz transform. Since $P_{\mathbb{R}^3}=I+\mathcal{R}\otimes\mathcal{R}$, we find

$$\||x|\nabla P_{\mathbb{R}^3}f\|_{q,\mathbb{R}^3} \le C\||x|\nabla f\|_{q,\mathbb{R}^3}$$

for $q\in(3/2,\infty)$, which together with (2.3) implies that

$$\||x|\nabla Bu\|_{q,\mathbb{R}^3} \le C\|u\|_{Y_q(\mathbb{R}^3)}$$

for such q.

We fix $R>0$ such that $\mathbb{R}^3\setminus D\subset B_R$ and consider the initial value problem for (1.2) in $D_R=D\cap B_R$ subject to the homogeneous Dirichlet boundary condition as well as $u(\cdot,s)=f$. As in the case $u_s=0$ that was studied by Hansel and Rhandi [14], the non-autonomous system (1.2) can be still treated as a simple application of the Tanabe-Sobolevskii theory [22], so that the associated linear operator generates an evolution operator $\{V(t,s)\}_{t\ge s\ge 0}$ on $L^q_\sigma(D_R)$, $1<q<\infty$, which satisfies the L^q-L^r smoothing estimates near the initial time $t=s$. See [16, Lemma 3.2], in which it was further clarified that the constant $C=C_{\tau_*}$ in those estimates can be taken uniformly in (s,t) with $t-s\le\tau_*$ provided that η and ω fulfill (1.3).

We now proceed to the exterior problem, that is, the non-autonomous sysytem (1.2) subject to

$$u|_{\partial D}=0, \qquad u\to 0 \quad \text{as } |x|\to\infty, \qquad u(\cdot,s)=f. \tag{3.5}$$

Since the linear operator for fixed t is not a generator of analytic semigroups, the general theory of parabolic evolution operators [22] is no longer useful. Following the idea due to Hansel and Rhandi [14], let us sketch the construction of the evolution operator. Given $f\in L^q_\sigma(D)$, we take suitable two modifications of f which can be regarded as initial values in the whole space $\mathbb{R}^3$ and in the bounded domain D_R, respectively. Denoting them still by the same symbol f itself for simplicity, we set

$$W(t,s)f=(1-\phi)U(t,s)f+\phi V(t,s)f+\mathbb{B}[(U(t,s)f-V(t,s)f)\cdot\nabla\phi], \tag{3.6}$$

by using both evolution operators explained above and an approproate cut-off function $\phi(x)$, where $\mathbb{B}$ stands for the Bogovskii operator in a bounded domain that contains the support of $\nabla\phi$. The Bogovskii operator provides a particular solution, with several fine properties such as optimal regularity, to the boundary value problem for the divergence equation in a bounded domain, see [1], [2], [9] and [13], and is often useful to recover the solenoidal condition in cut-off procedures. Let us introduce the space

$$Y_q(D) = \{u \in L^q_\sigma(D) \cap W^{1,q}_0(D) \cap W^{2,q}(D);\ |x|\nabla u \in L^q(D)\}.$$

If in particular $f \in Y_q(D)$ with some $q \in (3/2, \infty)$, then the function defined by (3.6) solves

$$\partial_t W(t,s)f + L(t)W(t,s)f = PR(t,s)f$$

in $L^q_\sigma(D)$ subject to the initial condition $W(s,s)f = f$, where $R(t,s)f$ is the remainder which consists of $U(t,s)f$ and $V(t,s)f$ (together with the associated pressure) and whose support is a compact set in D. If $T(t,s)$ is an evolution operator generated by (2.6), we should have

$$W(t,s)f = T(t,s)f + \int_s^t T(t,\tau)PR(\tau,s)f\,d\tau.$$

Therefore, it is reasonable to find the evolution operator $T(t,s)$ by solving the integral equation

$$T(t,s)f = W(t,s)f - \int_s^t T(t,\tau)PR(\tau,s)f\,d\tau.$$

Let us define the successive approximations $T_j(t,s)$ $(j = 0, 1, 2, \cdots)$ by

$$T_{j+1}(t,s) = W(t,s) - \int_s^t T_j(t,\tau)PR(\tau,s)\,d\tau, \qquad T_0(t,s) = W(t,s).$$

Thanks to Lemma 5.2 of [14] on iterated convolutions, we obtain the strong convergence of $T_j(t,s)f$ in $L^q_\sigma(D)$ uniformly in (s,t) with $0 \le s \le t \le s + \tau_*$, where $\tau_* \in (0,\infty)$ is fixed arbitrarily. To show that the limes $T(t,s)f$ obtained here provides a strong solution especially for $f \in Y_q(D)$, one needs $q \in (3/2, \infty)$ not only because this condition is already required for the fourth assertion of Proposition 3.1 but because

$$\||x|\nabla Pf\|_q \le C\||x|\nabla f\|_q + C\|f\|_{W^{1,q}(D)}$$

holds for $q \in (3/2, \infty)$. Since one has the latter reason even for the case $u_s = 0$ discussed in the paper [14] by Hansel and Rhandi, the same restriction should be also needed there although it is not explicitly mentioned in [14].

In this way, we obtain the desired evolution operator. Note that the constant m in (3.7) below can be large.

Proposition 3.2 *Suppose* (1.3) *for some* $\theta \in (0,1)$. *Let* $1 < q < \infty$. *Then* $\{L(t)\}_{t\ge 0}$ *given by* (2.6) *generates an operator family* $\{T(t,s)\}_{t\ge s\ge 0}$ *on* $L^q_\sigma(D)$ *with the same properties as in Proposition 3.1 except the last assertion (on large time behavior) there. Given*

$m \in (0,\infty)$ as well as $\tau_* \in (0,\infty)$, the constant $C_{\tau_*} = C_{\tau_*}(m,q,r,\theta,u_s,D)$ in L^q-L^r smoothing estimate, which corresponds to (3.2)*, is taken uniformly in $\{\eta,\omega\}$ which satisfies*

$$\sup_{t\geq 0}\big(|\eta(t)|+|\omega(t)|\big)+\sup_{t>s\geq 0}\frac{|\eta(t)-\eta(s)|+|\omega(t)-\omega(s)|}{(t-s)^\theta}\leq m. \tag{3.7}$$

The adjoint evolution operator $T(t,s)^*$ must be related to the backward system subject to the final condition at t, that is,

$$\partial_s T(t,s)^* g = L(s)^* T(t,s)^* g, \quad s\in[0,t]; \qquad T(t,t)^* g = g, \tag{3.8}$$

in $L^{q'}_\sigma(D)$, where $1/q'+1/q=1$. We should start with analysis of the adjoint evolution operator $U(t,s)^*$ in the whole space $\mathbb{R}^3$, and then we rigorously find the adjoint relation in $\mathcal{L}(L^q_\sigma(D))$ for every $q\in(1,\infty)$ as in subsection 2.3 of [16].

Large time behavior of $T(t,s)$ simultaneously with $T(t,s)^*$ is discussed in the next section.

4 L^q-L^r decay estimate

When the linearized system (1.2) is autonomous and $u_s=0$, L^q-L^r decay estimates of the semigroup and those of its gradient have been developed by several authors since Iwashita [18] studied such estimates for the Stokes semigroup ($\eta=\omega=0$), see the references in [16]. Most of those papers rely on spectral analysis. In fact, we have the resolvent for the autonomous case and analysis of the regularity of the resolvent near $\lambda=0$ plays a crucial role, where λ denotes the resolvent parameter. For the non-autonomous system, some particular cases such as the time-periodic case could be still discussed by means of spectral analysis, however in general, that is not the case.

We thus take another way with some devise to prove

Theorem 4.1 *Suppose* (1.3) *for some $\theta\in(0,1)$. For every $\{q,r\}$ with $1<q\leq r<\infty$, there is a constant $\delta_{**}=\delta_{**}(q,r)>0$ with the following property. If $\|u_s\|_{3,\infty}\leq\delta_{**}$, then for each $m\in(0,\infty)$ there is a constant $C=C(m,q,r,\theta,u_s,D)>0$ such that*

$$\|T(t,s)f\|_r\leq C(t-s)^{-(3/q-3/r)/2}\|f\|_q \tag{4.1}$$

$$\|T(t,s)^*g\|_r\leq C(t-s)^{-(3/q-3/r)/2}\|g\|_q \tag{4.2}$$

hold for all $t>s\geq 0$ and $f,\, g\in L^q_\sigma(D)$ whenever (3.7) *is fulfilled.*

The non-autonomous terms $(\eta + \omega \times x) \cdot \nabla u - \omega \times u$ are skew-symmetric, and so is $u_s \cdot \nabla u$ on account of div $u_s = 0$. By the Lorentz-Hoelder inequality and the Lorentz-Sobolev embedding relation, we know

$$|\langle u \cdot \nabla u_s,\, v\rangle| = |\langle u,\, -(\nabla v)^\top u_s\rangle| \le c_0 \|u_s\|_{3,\infty} \|\nabla u\|_2 \|\nabla v\|_2,$$

which leads to the energy inequalities

$$\|T(t,s)f\|_2^2 + \int_\tau^t \|\nabla T(\sigma,s)f\|_2^2\, d\sigma \le \|T(\tau,s)f\|_2^2 \tag{4.3}$$

and

$$\|T(t,s)^*g\|_2^2 + \int_s^\tau \|\nabla T(t,\sigma)^*g\|_2^2\, d\sigma \le \|T(t,\tau)^*g\|_2^2 \tag{4.4}$$

for all $f,\, g \in Y_2(D)$ and $t \ge \tau \ge s \ge 0$, provided that $\|u_s\|_{3,\infty} \le 1/(2c_0)$. Let $q \in (1,2]$. Under the same conditions as in Theorem 4.1, L^q-L^r estimates (4.1)–(4.2) combined with (4.3)–(4.4) imply the following estimates for $f,\, g \in C_{0,\sigma}^\infty(D)$ and, therefore, those being in $L_\sigma^q(D)$:

$$\int_t^\infty \|\nabla T(\sigma,s)f\|_2^2\, d\sigma \le C(t-s)^{-(3/q-3/2)}\|f\|_q^2,$$
$$\int_0^s \|\nabla T(t,\sigma)^*g\|_2^2\, d\sigma \le C(t-s)^{-(3/q-3/2)}\|g\|_q^2,$$

for all $t > s \ge 0$. The latter estimate is actually employed in the proof of Theorem 2.1. Since w_0 is taken from $L_\sigma^{3,\infty}(D)$ in this theorem, one also needs

$$\|T(t,s)f\|_r \le C(t-s)^{-1/2+3/2r}\|f\|_{3,\infty}$$

for $r \in (3,\infty)$, which follows from (4.1) by interpolation. We note that the energy relations (4.3)–(4.4) play an important role in the proof of Theorem 4.1 as well.

Since the only knowledge about the decay property is (3.3) for the whole space problem, it is reasonable to regard the exterior flow as a perturbation from (a modification of) $U(t,s)f$. To be precise, given $f \in C_{0,\sigma}^\infty(D)$, we describe $T(t,s)f$ in the form

$$T(t,s)f = (1-\phi)U(t,s)f + \mathbb{B}[(U(t,s)f)\cdot\nabla\phi] + v(t) \tag{4.5}$$

by using a suitable cut-off function $\phi(x)$ and the Bogovskii operator $\mathbb{B}$ (see the previous section) in a bounded domain that contains the support of $\nabla\phi$. Then the perturbation $v(t) = v(t;s)$ is of class $C^1([s,\infty); L_\sigma^q(D))$ and $v(t) \in Y_q(D)$ whenever $q \in (3/2,\infty)$; furthermore, it obeys

$$\partial_t v + L(t)v = F(t), \quad t \in [s,\infty); \qquad v(s) = \widetilde{f} := \phi f - \mathbb{B}[f\cdot\nabla\phi], \tag{4.6}$$

with a solenoidal forcing term F with compact support. Although the explicit form of F is omitted here, it consists of several terms all of which involve $U(t,s)f$.

We intend to show (4.1) for $2 \le q \le r < \infty$ simultaneously with (4.2) for $1 < q \le r \le 2$. Let $r \in (2,\infty)$, then our task is to find the uniform boundedness

$$\|v(t)\|_r \le C\|f\|_r \qquad (t-s>3). \tag{4.7}$$

Once we have (4.7) with some $r = r_0 \in (2,\infty)$, we get (4.2) for $r_0' \le q \le r \le 2$ as in Lemma 4.1 of [16], where $1/r_0' + 1/r_0 = 1$; by duality, we get (4.1) for $2 \le q \le r \le r_0$ as well. In fact, (4.7) with $r = r_0$ together with (3.3) ($q = r = r_0$) as well as the third assertion of Proposition 3.2 ($\tau_* = 3$) leads to (4.1) with $q = r = r_0$ and, therefore, (4.2) with $q = r = r_0'$. From this latter thing, an interpolation inequality, embedding and the energy relation of differential form corresponding to (4.4), we obtain a differential inequality for $\|T(t,s)^*g\|_2^2$ with $g \in C_{0,\sigma}^\infty(D)$ as in [19, Section 5]. Solving this inequality yields the conclusion above.

Given $2 < r < r_\dagger < \infty$ (where $r_\dagger$ is soon suitably chosen), the decay properties (3.3)–(3.4) imply that the forcing term F given by (4.6) decays like

$$\|F(t)\|_r \le C(m + \|\{u_s, \nabla u_s\}\|_\infty + 1)(t-s)^{-(3/r-3/r_\ddagger)/2}\|f\|_r \qquad (t-s\ge 1) \tag{4.8}$$

with $r_\ddagger \in (r, r_\dagger)$ provided $\|u_s\|_{3,\infty} \le \delta_*(r, r_\dagger)$, while (3.2) leads to

$$\|F(t)\|_r \le C(m + \|\{u_s, \nabla u_s\}\|_\infty + 1)(t-s)^{-1/2}\|f\|_r \qquad (0<t-s<1). \tag{4.9}$$

Instead of (4.6), as in [16], we use the duality formulation

$$\langle v(t), \psi\rangle = \langle \widetilde{f}, T(t,s)^*\psi\rangle + \int_s^t \langle F(\tau), T(t,\tau)^*\psi\rangle\, d\tau \tag{4.10}$$

for $\psi \in C_{0,\sigma}^\infty(D)$ because we have the advantage that we have only to take a local norm of the adjoint evolution operator on account of compactness of the supports of $\widetilde{f}$ and F. Let $t-s>3$ and let us concentrate ourselves on consideration of the integral

$$J := \int_{s+1}^{t-1} \langle F(\tau), T(t,\tau)^*\psi\rangle\, d\tau$$

since the other parts of the RHS of (4.10) are treated straightforward for every $r \in (2,\infty)$ by using (4.8)–(4.9) (with, say, $r_\dagger = 25$) together with (4.4) (and are thus omitted). To discuss the integral J, one needs four steps (while three steps are enough for the case $u_s = 0$ in [16] since $U_0(t,s)$, see (3.1), enjoys L^r-L^∞ decay estimate). Except for the fourth step, let us choose for instance $r_\dagger = 25$ and adopt (4.8) with $r_\ddagger = 24$ for $r \in (2,8)$; to be precise,

first step: $r < 8/3$, second step: $r < 4$, third step: $r < 8$, and then the case $r \in [8, \infty)$ is discussed in the last step by choosing another $r_\dagger$. Let us begin with the first step in which the only thing we know is the energy inequality (4.4). Since F is compactly supported and since $T(t,\tau)^*\psi \in Y_2(D)$ vanishes at the boundary ∂D, one can use the Poincaré inequality to obtain

$$|J| \le C\|f\|_r \int_{s+1}^{t-1} (\tau - s)^{-3/2r+1/16} \|\nabla T(t,\tau)^*\psi\|_2 \, d\tau. \tag{4.11}$$

Then (4.4) leads us to

$$|J| \le C\|f\|_r \|T(t,t-1)^*\psi\|_2 \left(\int_{s+1}^{t-1} (\tau - s)^{-3/r+1/8} \, d\tau \right)^{1/2}.$$

Recalling the third assertion with $\tau_* = 1$ in Proposition 3.2, we find

$$|J| \le C\|f\|_r \|\psi\|_{r'} \qquad (t - s > 3) \tag{4.12}$$

provided $r \in (2, 8/3)$, which implies (4.7) for such r and, thereby, (4.2) with $8/5 < q \le r \le 2$. This yields

$$\|T(t,s)^*\psi\|_2 \le C_\varepsilon (t - s - 1)^{-3/16+\varepsilon} \|\psi\|_{r'} \tag{4.13}$$

with arbitrary small $\varepsilon > 0$ for $t - s > 1$ by the backward semigroup property when $r' < 8/5$. With this better information (than the energy inequality) at hand, we proceed to the second step, in which the integral in (4.11) is splitted into

$$\int_{s+1}^{(s+t)/2} + \int_{(s+t)/2}^{t-1}.$$

Combining (4.13) with (4.4), we see that

$$\int_{s+1}^{(s+t)/2} \|\nabla T(t,\tau)^*\psi\|_2^2 \, d\tau \le C_\varepsilon (t - s - 2)^{-3/8+2\varepsilon} \|\psi\|_{r'}^2. \tag{4.14}$$

From this decay property we use Lemma 3.4 of [16] to find the growth estimate

$$\int_{(s+t)/2}^{t-1} \|\nabla T(t,\tau)^*\psi\|_2 \, d\tau \le C_\varepsilon (t - s - 2)^{5/16+\varepsilon} \|\psi\|_{r'}. \tag{4.15}$$

Employing these estimates in (4.11), we get (4.12) for $r \in (8/3, 4)$, which implies (4.2) with $4/3 < q \le r \le 2$. Hence, in the third step, we have (4.13) for $r' \le 4/3$, in which the rate $-3/16 + \varepsilon$ of decay is replaced by $-3/8 + \varepsilon$. Accordingly, the rate of decay of (4.14) and the one of growth of (4.15) are improved as $-3/4 + 2\varepsilon$ and $1/8 + \varepsilon$, respectively. We then repeat the previous argument to obtain (4.12) for $r \in [4, 8)$, which implies (4.2) with $8/7 < q \le r \le 2$.

Finally, suppose $8 \le r < \infty$. This time we choose, for instance, $r_\dagger = 2r + 1$ and use (4.8) with $r_\ddagger = 2r$ (thus, $\|u_s\|_{3,\infty}$ must be taken smaller for larger r). Then the rate $-3/2r + 1/16$ should be replaced by $-3/4r$ in the integrand of (4.11). We also know even better rate $-9/16 + \varepsilon$ of decay in (4.13), where it is sufficient to take ε such that $1/16 < \varepsilon < 1/16 + 3/4r$. By the same argument as in the second and third steps, we find (4.12) even if r is arbitrarily large. As a consequence, we conclude (4.1) with $2 \le q \le r < \infty$ as well as (4.2) with $1 < q \le r \le 2$.

The opposite case, that is, (4.1) with $1 < q \le r \le 2$ and (4.2) with $2 \le q \le r < \infty$, can be discussed by means of the similar method in which the role of $T(t,s)$ and the one of $T(t,s)^*$ are replaced each other. The remaining case $q < 2 < r$ is filled easily on account of the semigroup property of evolution operators.

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