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Schreier coset graphs of Baumslag-Solitar groups

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1. Introduction

This article is an extended version of the talk given at the RIMS Meeting on Set Theoretic and Geometric Topology held in Kyoto University from June 20 to June 22, 2018.

Let m and n be non-zero integers. The group which has the presentation $\langle A, B \mid AB^m = B^n A\rangle$ is called the *Baumslag-Solitar group* and denoted by $BS(m, n)$. In 1962, G. Baumslag and D. Solitar [1] introduced these groups and showed that $BS(3, 2)$ is the first example of non-Hopfian groups with one defining relation. Since then these groups have served as a proving ground for many new ideas in combinatorial and geometric group theory (see for example, [2, 3]).

Schreier coset graphs are generalizations of the Cayley graph of a group G, which are constructed for each choice of a subgroup of G and a generating set of G. In general, to construct the Cayley graph of a group or Schreier coset graphs is difficult. However once we have the appropriate Cayley or Schreier graphs, we can use them as discrete models and may learn, from combinatorial and geometric viewpoints, some properties of the original group or its subgroups. Recently, in [5, 6], D. Savchuk constructed Schreier graphs of Thompson's group F from a motivation to study amenability of the group.

In this article, we focus on the group $BS(1, n)$, where $n \geq 2$, which has an action on the real line, and we explicitly construct Schreier graphs of the group for stabilizers of all points under this action. As its consequence, we classify the Schreier graphs up to isomorphism. This leads to a relevance to presentations for the stabilizers which turn out to be infinite index subgroups in the group $BS(1, n)$.

2. Definitions and notations

Let m and n be nonzero integers. The group which has the presentation $\langle A, B \mid AB^m = B^n A\rangle$ is called the *Baumslag-Solitar group* and it is denoted by $BS(m, n)$. In the case of $m = 1$ and $n \geq 2$ the group $BS(1, n)$ has a geometric representation. That is, we define two affine maps a and b of the real line $\mathbb{R}$ by

$a(x) = nx$ and $b(x) = x + 1$ respectively. Then $BS(1, n)$ is isomorphic to the group $\langle\{a, b\}\rangle$. We note that

$$\langle\{a, b\}\rangle = \{g : \mathbb{R} \to \mathbb{R} \mid g(x) = n^i x + j/n^k,\ i, j, k \in \mathbb{Z}\}.$$

A *labelled directed graph* denoted by $(V, E, L, \alpha, \beta, l)$ consists of a nonempty set V of vertices, a set E of edges, a set L of labels and three mappings $\alpha : E \to V$, $\beta : E \to V$, and $l : E \to L$. The vertices $\alpha(e)$ and $\beta(e)$ are called the *initial* and the *terminal vertices* of the edge e, respectively.

A *marked labelled directed graph* denoted by $(V, E, L, \alpha, \beta, l, v_0)$ is a labelled directed graph with a distinguished vertex v_0 called the *marked vertex.*

For $i \in \{1, 2\}$ let Γ_i be labelled directed graph $(V_i, E_i, L_i, \alpha_i, \beta_i, l_i)$. Γ_1 is said to be *isomorphic* to Γ_2 if there exist bijections $f : V_1 \to V_2$, $\psi : E_1 \to E_2$, and $\gamma : L_1 \to L_2$ such that $\alpha_2(\psi(e)) = f(\alpha_1(e))$, $\beta_2(\psi(e)) = f(\beta_1(e))$, and $l_2(\psi(e)) = \gamma(l_1(e))$ for all $e \in E_1$. In particular, if $L_1 = L_2 = L$ and $\gamma = 1$, Γ_1 is said to be *L-isomorphic* to Γ_2.

For $i \in \{1, 2\}$ let Γ_i be marked labelled directed graph. Γ_1 is said to be *isomorphic* to Γ_2 if Γ_1 is isomorphic to Γ_2 as labelled directed graphs and the mapping between vertices preserves the marked vertices.

Let S be a generating set of a group G. The generating set S is *symmetric* if $S = S^{-1}$.

Definition 1. Let G be a finitely generated group, S be a symmetric finite generating set of G and M be a set. Let $\varphi : G \to \mathrm{Aut}(M)$ be a homomorphism, where $\mathrm{Aut}(M)$ is the set of all bijections of M onto itself. The *Schreier graph* denoted by (M, S, φ) is a labelled directed graph $(M, M \times S,\ S,\ \alpha,\ \beta,\ l)$ such that $\alpha(m, s) = m$, $l(m, s) = s$, and $\beta(m, s) = \varphi(s)(m)$. The *Schreier graph with a marked vertex* denoted by (M, S, φ, m_0) is a Schreier graph with a marked vertex $m_0 \in M$.

Let H be a subgroup of a group G with a symmetric finite generating set S and G/H be the set of all left cosets of H in G. The *Schreier coset graph* denoted by $(G/H, S)$ is a Schreier graph $(G/H, S, \varphi_0)$ where $\varphi_0 : G \to \mathrm{Aut}(G/H)$ is defined by $\varphi_0(x)(gH) = xgH$.

The next proposition will help us to describe Schreier graphs explicitly in the later sections.

Proposition 1. *Let G be a finitely generated group, S be a symmetric finite generating set of G and M be a set. Let $\varphi : G \to \mathrm{Aut}(M)$ be a homomorphism. For an element $x_0 \in M$ the Schreier graph $(\mathrm{Orb}(x_0), S, \varphi, x_0)$ with the marked vertex x_0 is S-isomorphic to the Schreier coset graph $(G/H, S, H)$ with the marked vertex $H = \mathrm{Stab}(x_0)$ as marked labelled directed graphs.*

3. The Schreier graph of the action ϕ_x

We consider the Baumslag-Soliter group $BS(1,n) = \langle\{a,b\}\rangle$, where $n \geq 2$. For any $x \in \mathbb{R}$ the inclusion $\rho : \langle\{a,b\}\rangle \hookrightarrow \mathrm{Aut}(\mathbb{R})$ induces the action $\phi_x : \langle\{a,b\}\rangle \to \mathrm{Aut}(\mathrm{Orb}(x))$ given by $\phi_x(g) = \rho(g)| = g|$. We will consider the Schreier graph $(\mathrm{Orb}(x), \{a,b\}^{\pm}, \phi_x)$. From now on, this Schreier graph is denoted by $(\mathrm{Orb}(x), \{a,b\}^{\pm})$.

Let $X = \{0, 1, \ldots n-1\}$. The set of all finite words over X and the set of all infinite words over X are denoted by X^* and X^ω respectively. Let $\tilde{X} = X^* \setminus \{\varepsilon\}$, where ε denotes the *empty word*. For a word $w = w_1 w_2 \ldots w_n$ in X^* the *length* of the word w, denoted by $\mathrm{lh}(w)$, is the number n. Note that the length of the empty word ε is zero.

Let $\sigma : X^\omega \to X^\omega$ be the sift map given by $\sigma(w_1 w_2 w_3 \ldots) = w_2 w_3 w_4 \ldots$. The i-th letter of the infinite word $\sigma^{k-1}(w)$, where $k \geq 1$ and $w \in X^\omega$ is denoted by $\sigma^{k-1}(w)_i (= w_{k-1+i})$.

For any $v \in X^\omega$ put $D_v = \mathbb{Z} + \sum_{i\geq 1} v_i/n^i$, $D_v^t = n\mathbb{Z} + t + \sum_{i\geq 1} v_i/n^i$, where $t \in X$. Note that $0 \leq \sum_{i \geq 1} v_i/n^i \leq 1$ and $D_v = \bigsqcup_{t\in X} D_v^t$.

Let $w \in X^\omega$. Put

$$M_w = \bigcup_{j\geq 1} D_{\sigma^j(w)} \cup \bigcup_{u\in X^*} D_{uw} \cup \bigcup_{j\geq 1} \bigcup_{u\in X^*} \bigcup_{t\in X, t\neq w_j} D_{ut\sigma^j(w)}.$$

For any $x \in \mathbb{R}$ there exist $y \in [0,1)$ and $n \in \mathbb{Z}$ such that $x = b^n(y)$ and the orbit $\mathrm{Orb}(x)$ equals the orbit $\mathrm{Orb}(y)$. Thus it suffices to consider only the Schreier graph $(\mathrm{Orb}(y), \{a,b\}^{\pm})$ for $y \in [0,1)$.

Proposition 2. *Suppose that $y \in [0,1)$ can be written by $y = \sum_{i\geq 1} w_i/n^i$ for some $w \in X^\omega$. Then the orbit* $\mathrm{Orb}(y)$ *coincides with the set M_w.*

4. The Schreier graph of the action ϕ_q

We say that a pair (x,y) of words satisfies (A) if $x \in X^*$ and $y \in \tilde{X}$ satisfy the following two conditions.

(1) For any $k \geq 2$ and any $t \in \tilde{X}$, $\quad y \neq t^k$.
(2) $x \neq \varepsilon \;\Rightarrow\; x_{\mathrm{lh}(x)} \neq y_{\mathrm{lh}(y)}$.

In this section we will construct Schreier graphs for all rational numbers.

Let q be a rational number in $\mathbb{Q} \cap [0,1)$. Then there exist words $u \in X^*$ and $v \in \tilde{X}$ such that $q = \sum_{i\geq 1} (uv^\infty)_i/n^i$, $\mathrm{lh}(v) \geq 1$, and the pair (u,v) satisfies (A).

Theorem 1. *The Schreier graph* $(\mathrm{Orb}(q), \{a,b\}^{\pm}) = (M_{uv^\infty}, \{a,b\}^{\pm})$ *has the following structure.*

(1) *The set* M_{uv^∞} *coincides with the following set* $\tilde{M}_{uv^\infty}$, *where*

$$\tilde{M}_{uv^\infty} = \bigsqcup_{j=\mathrm{lh}(u)}^{\mathrm{lh}(u)+\mathrm{lh}(v)-1} D_{\sigma^j(uv^\infty)} \sqcup \bigsqcup_{j=\mathrm{lh}(u)+1}^{\mathrm{lh}(u)+\mathrm{lh}(v)} \bigsqcup_{s\in X^*} \bigsqcup_{t\in X,\, t\neq (uv^\infty)_j} D_{st\sigma^j(uv^\infty)}.$$

(2) *For any rational element* w *in* X^ω *with any one of the forms* $\sigma^j(uv^\infty)$ *and* $st\sigma^j(uv^\infty)$, *there is a labelled directed graph consisting of* D_w *and* $D_w \times \{b\}^\pm$ *as the set of vertices and edges respectively. For any edge* e *in* $D_w \times \{b\}^\pm$ *with the label* b, *there exists an integer* $j \in \mathbb{Z}$ *such that* $\alpha(e) = j + \sum_{i\geq 1} w_i/n^i$ *and* $\beta(e) = j + 1 + \sum_{i\geq 1} w_i/n^i$.

(3) *For any rational element* w *in* X^ω *with any one of the forms* $\sigma^j(uv^\infty)$ *and* $st\sigma^j(uv^\infty)$, *there exists the set of edges* $D_w \times \{a\}$ *labelled by* a *such that each element in the set* D_w *is the initial vertex of an edge in* $D_w \times \{a\}$ *and the terminal vertex of the edge lies in* $D^{w_1}_{\sigma(w)} \subset D_{\sigma(w)}$. $D^{w_1}_{\sigma(w)} \times \{a^{-1}\}$ *is the set of inverse edges of* $D_w \times \{a\}$.

(4) *For any rational element* w *in* X^ω *with any one of the forms* $\sigma^j(uv^\infty)$ *and* $st\sigma^j(uv^\infty)$,

$$D_w = \bigsqcup_{t\in X} D^t_w.$$

5. The Schreier graph of the action ϕ_α

An element $w \in X^\omega$ is called a *rational element* in X^ω if there exist $u \in X^*$ and $v \in \tilde{X}$ such that $w = uv^\infty$. An element $w \in X^\omega$ is called an *irrational element* in X^ω if w is not a rational element in X^ω. Let α be an irrational number in $(\mathbb{R}\setminus\mathbb{Q}) \cap [0,1)$. Then there exists an irrational element $w \in X^\omega$ such that $\alpha = \sum_{i\geq 1} w_i/n^i$.

In this section we will describe the Schreier graph $(\mathrm{Orb}(\alpha), \{a,b\}^\pm) = (M_w, \{a,b\}^\pm)$. We notice that the Schreier graph is $\{a,b\}^\pm$-isomorphic to the Cayley graph of $BS(1,n) = \langle\{a,b\}\rangle$ relative to the generators $\{a,b\}^\pm$ by Proposition 1 since the stabilizer of α is trivial. However in the previous section we have constructed the Schreier graphs $(\mathrm{Orb}(q), \{a,b\}^\pm)$ for rational elements q and will compare those descriptions in the later section (see Theorem 2). Therefore we employ the Schreier graph $(\mathrm{Orb}(\alpha), \{a,b\}^\pm)$. We construct the Schreier graph $(\mathrm{Orb}(\alpha), \{a,b\}^\pm)$ by an arrangement of elements in the orbit $\mathrm{Orb}(\alpha)$. The construction of the Cayley graph of $BS(1,n) = \langle\{a,b\}\rangle$ given in [4] depends on the fact that the word problem for $BS(1,n)$ is solvable.

PROPOSITION 3. *The Schreier graph* $(\mathrm{Orb}(\alpha), \{a,b\}^\pm) = (M_w, \{a,b\}^\pm)$ *has the following structure.*

(1) *The set M_w coincides with the disjoint union*

$$\bigsqcup_{j\geq 1} D_{\sigma^j(w)} \sqcup \bigsqcup_{u\in X^*} D_{uw} \sqcup \bigsqcup_{j\geq 1}\bigsqcup_{u\in X^*}\bigsqcup_{t\in X, t\neq w_j} D_{ut\sigma^j(w)}.$$

(2) *For any irrational element v in X^ω with any one of the forms $\sigma^j(w)$, uw, and $ut\sigma^j(w)$, there is a labelled directed graph consisting of D_v and $D_v \times \{b\}^{\pm}$ as the set of vertices and edges respectively. For any edge e in $D_v \times \{b\}^{\pm}$ with the label b, there exists an integer $j \in \mathbb{Z}$ such that $\alpha(e) = j + \sum_{i\geq 1} v_i/n^i$ and $\beta(e) = j + 1 + \sum_{i\geq 1} v_i/n^i$.*

(3) *For any irrational element v in X^ω with any one of the forms $\sigma^j(w)$, uw, and $ut\sigma^j(w)$, there exists the set of edges $D_v \times \{a\}$ labelled by a such that each element of the set D_v is the initial vertex of an edge in $D_v \times \{a\}$ and the terminal vertex of the edge lies in the set $D^{v_1}_{\sigma(v)} \subset D_{\sigma(v)}$. $D^{v_1}_{\sigma(v)} \times \{a^{-1}\}$ is the set of inverse edges of $D_v \times \{a\}$.*

(4) *For any irrational element v in X^ω with any one of the forms $\sigma^j(w)$, uw, and $ut\sigma^j(w)$,*

$$D_v = \bigsqcup_{t\in X} D^t_v.$$

6. Applications

In this section, first we classify Schreier graphs described in the previous sections.

Theorem 2. *Let $S = \{a, b\}^{\pm}$.*

(1) *For any irrational numbers $\alpha_1, \alpha_2 \in \mathbb{R}\setminus\mathbb{Q}$ the Schreier graph $(\mathrm{Orb}(\alpha_1), S, \alpha_1)$ is S-isomorphic to the Schreier graph $(\mathrm{Orb}(\alpha_2), S, \alpha_2)$ as marked labelled directed graphs.*

(2) *For any rational number $q \in \mathbb{Q}$ and any irrational number $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ the Schreier graph $(\mathrm{Orb}(q), S)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}(\alpha), S)$ as labelled directed graphs.*

(3) *Let q_1, q_2 be any rational numbers in $\mathbb{Q}$. Suppose that there exist $m_i \in \mathbb{Z}$, $u_i \in X^*$, and $v_i \in \tilde{X}$ such that $q_i = m_i + \sum_{j\geq 1}(u_i v_i^\infty)_j/n^j$ for each i, where the pair (u_i, v_i) satisfies (A). Then the following conditions are equivalent.*

   (a) *The Schreier graph $(\mathrm{Orb}(q_1), S)$ is isomorphic to the Schreier graph $(\mathrm{Orb}(q_2), S)$ as labelled directed graphs.*

   (b) $\mathrm{Orb}(q_1) = \mathrm{Orb}(q_2)$ *or* $\mathrm{Orb}(-q_1) = \mathrm{Orb}(q_2)$.

   (c) *There exists a nonnegative integer j with $j < \mathrm{lh}(v_1)$ such that $v_2^\infty = \sigma^j(v_1^\infty)$ or there exists a nonnegative integer j with $j < \mathrm{lh}(v_1)$ such that*

$v_2^\infty = \sigma^j(\overline{v_1}^\infty)$, *where put* $\overline{t} = n-1-t$ *for* $t \in X$ *and* $\overline{v} = \overline{v_1} \dots \overline{v_{\mathrm{lh}(v)}}$ *for* $v \in \tilde{X}$.

COROLLARY 1. *Let* $S = \{a, b\}^{\pm}$. *Let* q_1, q_2 *be any rational numbers in* $\mathbb{Q}$. *Then the followings are equivalent.*

(*a*) *The Schreier graph* $(\mathrm{Orb}(q_1), S, q_1)$ *is isomorphic to the Schreier graph* $(\mathrm{Orb}(q_2), S, q_2)$ *as marked labelled directed graphs.*

(*b*) $|q_1| = |q_2|$.

By noting a closed edge path in the Schreier graph $(\mathrm{Orb}(q), S, q)$ which has a non-trivial sequence of labels in $BS(1, n)$, we have next proposition.

PROPOSITION 4. *For any rational number* $q \in \mathbb{Q}$ *the stabilizer* $\mathrm{Stab}(q)$ *is isomorphic to* $\mathbb{Z}$.

Next we introduce the definition of presentation isomorphic subgroups in order to translate the graphical expression of the Schreier graphs into the algebraic expression of subgroups. Consequently, we get a relevance to presentations for the stabilizers from the previous result about the classification of the Schreier graphs(see Proposition 6).

For any $i \in \{1, 2\}$ let G_i be a group and T_i be a generating set of G_i. Let $T_i^{-1} = \{t^{-1} \,|\, t \in T_i\}$ and $T_i^{\pm} = T_i \cup T_i^{-1}$. We assume that

$$(*) \qquad t \in T_i \cap T_i^{-1} \iff t \in T_i, \quad t^2 = 1.$$

For any $i \in \{1, 2\}$ let $X_i = \{\, x_t \,|\, t \in T_i \,\}$. Put $X_i^{-1} = \{x_t^{-1} \,|\, t \in T_i\}$, where x_t^{-1} denotes a new symbol corresponding to the element x_t. We assume that $X_i \cap X_i^{-1} = \emptyset$ and that the expression $(x_t^{-1})^{-1}$ denotes the element x_t. For any $i \in \{1, 2\}$ the free group with the basis X_i is denoted by $F(X_i)$, and for a subset R_i of $F(X_i)$ the normal closure of the set R_i in $F(X_i)$ is denoted by $\langle\langle R_i \rangle\rangle$. Let G_i have the presentation $\langle X_i \,|\, R_i \rangle$ with respect to the epimorphism $\psi_i : F(X_i) \to G_i$ given by $\psi_i(x_t) = t$.

DEFINITION 2. For any $i \in \{1, 2\}$ let H_i be a subgroup of G_i. H_1 is *presentation isomorphic* to H_2 if there exists a bijection $\gamma : X_1^{\pm} \to X_2^{\pm}$ with $\gamma(x_t^{-1}) = \gamma(x_t)^{-1}$ such that $\tilde{\gamma}(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$ and $\tilde{\gamma}(\langle\langle R_1 \rangle\rangle) = \langle\langle R_2 \rangle\rangle$, where $\tilde{\gamma} : F(X_1) \to F(X_2)$ given by $\tilde{\gamma}(x_{t_1}^{\varepsilon_1} \cdots x_{t_k}^{\varepsilon_k}) = \gamma(x_{t_1})^{\varepsilon_1} \cdots \gamma(x_{t_k})^{\varepsilon_k}$, $\varepsilon_i = \pm 1$.

PROPOSITION 5. *Let* $\Gamma_i = (G_i/H_i, T_i^{\pm}, H_i)$, $t_j \in T_1$, *and* $\varepsilon_j = \pm 1$. *Then the followings are equivalent.*

(*a*) Γ_1 *is isomorphic to* Γ_2 *as marked labelled directed graphs by a bijection* $\gamma : T_1^{\pm} \to T_2^{\pm}$ *satisfying the condition*

$$(C) \qquad t_1^{\varepsilon_1} \cdots t_k^{\varepsilon_k} = 1_{G_1} \iff \gamma(t_1^{\varepsilon_1}) \cdots \gamma(t_k^{\varepsilon_k}) = 1_{G_2}.$$

(*b*) H_1 *is presentation isomorphic to* H_2.

By Proposition 5 and Corollary 1, we obtain the following proposition.

PROPOSITION 6. *Let q_1, $q_2 \in \mathbb{Q}$. Then the followings are equivalent.*

(*a*) $\mathrm{Stab}(q_1)$ *is presentation isomorphic to* $\mathrm{Stab}(q_2)$.
(*b*) $|q_1| = |q_2|$.

References

[1] G. BAUMSLAG, D. SOLITAR, Some two-generator one-relator non-Hopfian groups, Bull. Amer. Math. Soc. **68** (1962), 199–201.

[2] B. FARB, L. MOSHER, A rigidity theorem for the solvable Baumslag-Solitar groups, with an appendix by Daryl Cooper, Invent. Math. **131** (1998), no. 2, 419–451.

[3] B. FARB, L. MOSHER, Quasi-isometric rigidity for the solvable Baumslag-Solitar groups II, Invent. Math. **137** (1999), no. 3, 613–649.

[4] J. MEIER, *Groups, graphs and trees, An introduction to the geometry of infinite groups*, London Mathematical Society Student Texts 73, Cambridge University Press, Cambridge, 2008.

[5] D. SAVCHUK, Schreier graphs of actions of Thompson's group F on the unit interval and on the Cantor set, Geom. Dedicata **175** (2015), 355–372.

[6] D. SAVCHUK, Some graphs related to Thompson's group F , *Combinatorial and geometric group theory*, Trends Math. Birkhäuser/Springer Basel AG, Basel, 2010, 279–296.

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