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A FACTORIZATION THEOREM FOR WEAK α-FAVOURABILITY

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All spaces are assumed to be Tychonoff and all topological groups are assumed to be Hausdorff.

1. What is weak α-favourability? The Banach-Mazur game

The notion of weak α-favourability is defined in terms of a topological game.

For a topological space X, the *Banach-Mazur game* on X is played between two players. At round 1, Player A selects a non-empty open subset A_1 of X, and Player B responds with choosing a non-empty open subset B_1 inside of A_1. At round 2, Player A selects a non-empty open subset $A_2 \subseteq B_1$, and Player B responds by choosing a non-empty open subset $B_2 \subseteq A_2$. The game continues to infinity producing a decreasing sequence

$$A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \dots \tag{1}$$

of non-empty open subsets of X. Player B wins if

$$\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} B_n \neq \emptyset; \tag{2}$$

otherwise Player A wins.

Player A tries to make the intersection $\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} B_n$ empty, while Player B tries to make it non-empty. The vague word "tries" corresponds to the rigorous notion of a winning strategy.

Let $\mathcal{O}$ denotes the family of all non-empty open subsets of X. Let $\mathrm{Seq}(\mathcal{O})$ denotes the set of all finite sequences $(U_0, \dots, U_n)$ of elements of $\mathcal{O}$. A *strategy* is a function $\sigma : \mathrm{Seq}(\mathcal{O}) \to \mathcal{O}$ such that

$$\sigma(U_0, \dots, U_n) \subseteq U_n \quad \text{for every} \quad (U_0, \dots, U_n) \in \mathrm{Seq}(\mathcal{O}).$$

A strategy β is a *winning strategy for Player B* provided that Player B wins the game defined by

$$A_1 = \alpha(\emptyset), B_1 = \beta(A_1), \dots, A_n = \alpha(B_1, \dots, B_{n-1}), B_n = \beta(A_1, \dots, A_n), \dots$$

for *every* strategy α for Player A; that is, (2) holds.

Definition 1.1. The space X is called *weakly α-favourable* if Player B has a winning strategy in the Banach-Mazur game on X.

It is known that

$$\text{Čech-complete} \to \text{Oxtoby complete} \to \text{weakly } \alpha\text{-favourable} \to \text{Baire}.$$

This talk was presented at the conference by the second listed author.

2. Factorization theorems for weak α-favourability

We are going to study weak α-favourability and related completeness properties by means of a factorization theorem.

Let $\mathcal{P}$ and $\mathcal{Q}$ be topological properties. Assume that $h : X \to Z$ is a continuous map such that X has property $\mathcal{P}$ and Z has property $\mathcal{Q}$. Can we factorize this map h through an intermediate space Y which has both properties $\mathcal{P}$ and $\mathcal{Q}$? In other words, can we find a topological space Y having both properties $\mathcal{P}$ and $\mathcal{Q}$, a continuous surjection $g : X \to Y$ and a continuous map $f : Y \to Z$ such that $h = f \circ g$? If this is possible for *every* continuous map h as above, then we say that the *factorization theorem holds for the pair of topological properties* $(\mathcal{P}, \mathcal{Q})$.

The notion of a factorization theorem can be appropriately defined also for other categories, for example, for the category of topological groups and their continuous homomorphisms.

Our main result is the following factorization theorem, in which $w(X)$ denotes the weight of a topological space X, i.e. the minimal cardinality of a base for X.

Theorem 2.1 (A general factorization theorem for weak α-favourability)**.** *Let $h : X \to Z$ be a continuous map from a weakly α-favourable space X to a topological space Z. Then there exist a weakly α-favourable topological space Y and continuous maps $g : X \to Y$, $f : Y \to Z$ such that $h = f \circ g$, $Y = g(X)$ and $w(Y) \leq w(Z)$.*

Since a metric weakly α-favourable space Y contains a dense completely metrizable subspace [8, Theorem 3(11)], from this theorem we get the following corollary.

Corollary 2.2. *For every continuous map $h : X \to Z$ from a weakly α-favourable space X to a separable metric space Z, there exist a separable metric space Y, a continuous surjection $g : X \to Y$ and a continuous map $f : Y \to Z$ such that $h = f \circ g$ and Y contains a dense completely metrizable (thus, Polishable) subspace.*

The next corollary provides a factorization meta-theorem for a class $\mathbf{P}$ of spaces.

Corollary 2.3. *Let $\mathbf{P}$ be any class of topological spaces having the following two properties:*

(i) *every space in the class $\mathbf{P}$ is weakly α-favourable;*
(ii) *if a separable metric space Y has a dense completely metrizable subspace, then Y belongs to the class $\mathbf{P}$.*

Then for every continuous map $h : X \to Z$ from a space X in the class $\mathbf{P}$ to a separable metric space Z, there exist a separable metric space Y in the class $\mathbf{P}$, a continuous surjection $g : X \to Y$ and a continuous map $f : Y \to Z$ such that $h = f \circ g$.

Proof. Let X, Z and h be as in the assumption of our corollary. Then X is weakly α-favourable by item (i). Use Corollary 2.2 to find Y, f and g as in the conclusion of this corollary. Then the space Y belongs to the class $\mathbf{P}$ by item (ii). □

Items (i) and (ii) of this corollary can be briefly summarized by the following line:

$$\exists \text{ dense complete metric subspace} \to \text{ belongs to } \mathbf{P} \to \text{weakly } \alpha\text{-favourable.}$$

From Corollary 2.3, we obtain nine concrete factorization theorems for different classes $\mathbf{P}$ of topological spaces.

Corollary 2.4. *Let $\mathbf{P}$ be one of the following classes of spaces considered (and some of them defined) in* [4]*:*

(i) *Sánchez-Okunev countably compact spaces,*
(ii) *Oxtoby countably compact spaces,*
(iii) *Todd countably compact spaces,*
(iv) *strong Sánchez-Okunev complete spaces,*
(v) *strong Oxtoby complete spaces,*
(vi) *strong Todd complete spaces,*
(vii) *Sánchez-Okunev complete spaces,*
(viii) *Todd complete spaces,*
(ix) *Oxtoby complete spaces.*

Then for every continuous map $h : X \to Z$ from a space X in the class $\mathbf{P}$ to a separable metric space Z, there exist a separable metric space Y in the class $\mathbf{P}$, a continuous surjection $g : X \to Y$ and a continuous map $f : Y \to Z$ such that $h = f \circ g$.

Proof. By [4, Diagram 2], every space in the class $\mathbf{P}$ is Todd complete, and by [3, Theorem 6.10], every Todd complete space is weakly α-favourable. Therefore, by [4, Theorem 4.1] and [1, Corollary 2.4], if a metric space Y has a dense completely metrizable subspace, then Y belongs to $\mathbf{P}$. Hence, $\mathbf{P}$ satisfies the hypothesis of Corollary 2.3. □

3. Weak α-favourability in ω-bounded topological groups

Definition 3.1. [4, Definition 4.3] A topological group X is *Polish factorizable* provided that for every continuous homomorphism $h : X \to Z$ from X to a separable metric group Z, there exist a Polish group Y and continuous homomorphisms $g : X \to Y$, $f : Y \to Z$ such that $h = f \circ g$ and $Y = g(X)$.

Fact 3.2. [4] Pseudocompact groups are Polish factorizable.

Proof. Let $h : X \to Z$ be a continuous homomorphism from a pseudocompact group X to a separable metric group Z. Then $h(X)$ is pseudocompact. Since pseudocompact metric spaces are compact, $Y = h(X)$ is compact. Thus, Y is a Polish group. So we can take $g = h$ and $f = \mathrm{id} : Y \to Z$. □

Definition 3.3. [6] A topological group X is called *ω-bounded* provided that for every open neighbourhood U of the identity of X, one can find an at most countable set S such that $X = SU$.

Clearly, Lindelöf groups are ω-bounded [6].

Fact 3.4. [6] A topological group is ω-bounded if and only if it is topologically and algebraically isomorphic to a subgroup of a suitable product of separable metric groups.

Theorem 3.5. *A weakly α-favourable ω-bounded group is Polish factorizable.*

The following is a restatement of Theorem 3.5 based on Fact 3.4.

Theorem 3.6. *Let $\{H_i : i \in I\}$ be a family of separable metric groups and let $H = \prod_{i\in I} H_i$ be its product. If a subgroup G of H is weakly α-favourable, then G is Polish factorizable.*

4. When is a dense subgroup of a product of separable metric groups weakly α-favourable?

Our next theorem characterizes weak α-favourability of *dense* subgroups of products of separable metric groups.

Theorem 4.1. *Let G be a dense subgroup in the product $H = \prod_{i\in I} H_i$ of separable metric groups H_i. Then the following conditions are equivalent:*

(i) *G is weakly α-favourable;*
(ii) *all groups H_i are Polishable and $\pi_J(G) = \prod_{i\in J} H_i$ for every at most countable subset J of I, where $\pi_J : H \to \prod_{i\in J} H_i$ is the projection.*

In [4, Theorem 4.6], the authors give the following list of conditions equivalent to item (ii) of this theorem:

(iii) G is Sánchez-Okunev complete;
(iv) G is Oxtoby complete;
(v) G is Telgársky complete;
(vi) G is strongly Sánchez-Okunev complete;
(vii) G is strongly Oxtoby complete;
(viii) G is Sánchez-Okunev countably compact;
(ix) G is Oxtoby countably compact;
(x) G is Polish factorizable.

Since Todd complete spaces are weakly α-favourable [3, Theorem 6.10], now we can add another equivalent condition to this list, thereby answering positively [4, Question 11.9]:

(xi) G is Todd complete.

5. Two characterizations of pseudocompactness in topological groups

Definition 5.1. A topological group X is called *precompact* provided that for every open neighbourhood U of the identity of X one can find an a finite set S such that $X = SU$.

It is obvious from Definitions 3.3 and 5.1 that precompact groups are ω-bounded.

Theorem 5.2 (Weil)**.** *A topological group is precompact if and only if it is both topologically and algebraically isomorphic to a subgroup of some compact group.*

Fact 5.3. [2] Pseudocompact groups are precompact.

It follows from Facts 3.2 and 5.3 that pseudocompact groups are precompact and Polish factorizable. As it turns out, these two properties combined together *characterize* pseudocompact groups:

Theorem 5.4 (first characterization of pseudocompact groups)**.** *A topological group is pseudocompact if and only if it is both precompact and Polish factorizable.*

Fact 5.5. Pseudocompact (regular) spaces are weakly α-favourable.

Proof. Indeed, let X be a pseudocompact space. Presented with the nth move A_n by Player A, Player B can simply use regularity of X to select a non-empty open subset B_n of X which is contained in A_n together with its closure. This defines a (stationary) winning strategy β for Player B. Indeed, let (1) be the game obtained by following some strategy α of Player A and the strategy β for Player B defined above. Then $\{B_n : n \in \mathbb{N}\}$ is a sequence of non-empty open subsets of X such that the closure of each B_{n+1} is contained in B_n. Since X is pseudocompact, one can easily see that (2) holds. □

It follows from Facts 5.3 and 5.5 that pseudocompact groups are precompact and weakly α-favourable. As it turns out, these two properties combined together *characterize* pseudocompact groups:

Theorem 5.6 (second characterization of pseudocompact groups)**.** *A topological group is pseudocompact if and only if it is both precompact and weakly α-favourable.*

Proof. The "only if" implication follows from Facts 5.3 and 5.5. To show the "if" implication, suppose that G is a precompact weakly α-favourable group. Then G is also ω-bounded. Applying Theorem 3.5, we conclude that G is Polish factorizable. Now pseudocompactness of G follows from Theorem 5.4. □

Theorem 5.6 is "best possible" in a sense that weak α-favourability is the weakest of all known "completeness" properties which imply the Baire property, and a precompact Baire group need not be pseudocompact.

6. Answering questions of García-Ferreira, Rojas-Hernández and Tamariz-Mascarúa

In [5, Problem 7.5], the following problem was proposed:

Question 6.1. Is a precompact weakly α-favourable topological group Oxtoby complete?

Since pseudocompact spaces are Oxtoby complete, Theorem 5.6 gives a strong positive answer to this question.

In [5, Problem 7.7], the following problem was proposed:

Question 6.2. For which class of topological groups G it is true that $C_p(X, G)$ being weakly pseudocompact and dense in the Tychonoff product G^X implies $C_p(X, G)$ is Oxtoby complete, for every Tychonoff space X?

It follows from our results that the answer is positive for separable metric groups G and precompact groups G.

7. Two open questions about Baire spaces

Question 7.1. Is Baire property factorizable? That is, given a continuous map $h : X \to Z$ from a Baire space X to a separable metric space Z, can we find a separable metric Baire space Y and continuous maps $g : X \to Y$, $f : Y \to Z$ such that $h = f \circ g$ and $Y = g(X)$?

Question 7.2. Is Baire property factorizable in the category of topological groups? That is, given a continuous homomorphism $h : X \to Z$ from a Baire group X to a separable metric group Z, can we find a separable metric Baire group Y and continuous homomorphisms $g : X \to Y$, $f : Y \to Z$ such that $h = f \circ g$ and $Y = g(X)$?

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