NOTES ON $\kappa$-SUBSETS OF COMPACT-LIKE SPACES

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In 1980 Gryzlov ([3], see also Hodel [4]) proved that every compact $T_1$ space has cardinality $\leq 2^{\psi(x)}$, where $\psi(X)$ is the pseudo-character of $X$, and later Stephenson generalized Gryzlov's result as follows:

**Theorem 1** (Stephenson [5]). Let $X$ be a $2^\kappa$-total $T_1$ space with $\psi(X) \leq \kappa$. Then $|X| \leq 2^\kappa$ and $X$ is compact.

A topological space $X$ is $\kappa$-total if for every subset $H$ of $X$ with $|H| \leq \kappa$, every filter base on $H$ has an adherent point in $X$.

On the other hand, Gryzlov obtained a similar result for $H$-closed spaces. Recall that, a Hausdorff space $X$ is $H$-closed if $X$ is closed in every Hausdorff space containing $X$ as a subspace. A subset $H \subseteq X$ is an $H$-set if for every family $\mathcal{V}$ of open sets which covers $H$, there are finitely many $V_0, \ldots, V_n \in \mathcal{V}$ with $H \subseteq \overline{V_0} \cup \cdots \cup \overline{V_n}$. It is known that a Hausdorff space $X$ is $H$-closed if and only if $X$ is an $H$-set in $X$.

For a space $X$, $\psi_c(X)$ denotes the closed pseudo-character of $X$, that is, $\psi_c(X)$ is the minimum infinite cardinal $\kappa$ such that for every $x \in X$ there is a family $\mathcal{V}$ of open neighborhood of $x$ with $|\mathcal{V}| \leq \kappa$ and $\{x\} = \cap \{\overline{V} | V \in \mathcal{V}\}$. Note that closed pseudo-character can be defined only for Hausdorff spaces.

**Theorem 2** (Gryzlov [3]). Let $X$ be an $H$-closed set with $\psi_c(X) = \omega$, then $|X| \leq 2^\omega$.

Dow and Porter [2] extended this result as that $|X| \leq 2^{\psi_c(X)}$ for every $H$-closed space $X$.

In this note we prove slightly general and strong results in term of $G_\kappa$-subsets. Recall that, for a topological space $X$ and an infinite cardinal $\kappa$, a $G_\kappa$-subset is the intersection of $\leq \kappa$ many open subsets in $X$.

**Proposition 3.** Let $\kappa$ be an infinite cardinal. Let $X$ be a $2^\kappa$-total space (no separation axiom of assumed), and $\mathcal{G}$ a cover of $X$ by $G_\kappa$-subsets. If for every $x \in X$, the set $\{G \in \mathcal{G} | x \in G\}$ has cardinality $\leq 2^\kappa$, then $\mathcal{G}$ has a subcover of size $\leq 2^\kappa$.

**Proof.** Suppose to the contrary that $\mathcal{G}$ has no subcover of size $\leq 2^\kappa$. Let $\lambda = |\mathcal{G}|$, and $\{G_\alpha | \alpha < \lambda\}$ be an enumeration of $\mathcal{G}$. Let $[\kappa]^{<\omega}$ denote the set of all finite subsets of $\kappa$. For $\alpha < \lambda$, we can take open sets $W_\alpha^a (a \in [\kappa]^{<\omega})$ such that $G_\alpha = \bigcap_{a \in [\kappa]^{<\omega}} W_\alpha^a$ and whenever $b \supseteq a$ we have $W_b^a \subseteq W_\alpha^a$.

Take a sufficiently large regular cardinal $\chi$, and take $M < H(\chi)$ containing all relevant objects such that $|M| = 2^\kappa \subseteq M$ and $[M]^\kappa \subseteq M$. Since $|M \cap \lambda| = 2^\kappa$,
we have \( \exists \alpha \in [\kappa]^{<\omega} \) with \( x^* \notin W_{a_{\alpha}}^\beta \).

Now we claim that there are finitely many \( \alpha_0, \ldots, \alpha_k \in M \cap \lambda \) such that \( M \cap X \subseteq \bigcup_{\alpha \leq \alpha_k} W_{a_{\alpha}}^\beta \). We can derive a contradiction by this claim; If \( M \cap X \subseteq \bigcup_{\alpha \leq \alpha_k} W_{a_{\alpha}}^\beta \), by the elementarity of \( M \) we have that \( \{W_{a_{\alpha}}^\alpha | \alpha \leq \alpha_k\} \) is a cover of \( X \). Hence there is \( \alpha \in M \cap \lambda \) with \( x^* \notin W_{a_{\alpha}}^\beta \), this is a contradiction.

Suppose that \( M \cap X \nsubseteq \bigcap_{\alpha \in M \cap \lambda} W_{a_{\alpha}}^\beta \) for every finitely many \( \alpha_{0}, \ldots, \alpha_k \in N \cap \lambda \).

Let \( \mathcal{F} = \{ \bigcap_{\alpha \leq \alpha_k} W_{a_{\alpha}}^\beta | \alpha \in [\kappa]^{<\omega} \} \). By the assumption, \( \mathcal{F} \) has the finite intersection property. In addition if \( x \in \bigcap_{F \in \mathcal{F}} \overline{F} \) then \( x \notin W_{a_{\alpha}}^\beta \) for every \( \alpha \in M \cap \lambda \).

Let \( \mathcal{F}' \subseteq \mathcal{P}(M \cap X) \) such that:

1. \( \mathcal{F} \subseteq \mathcal{F}' \) and every element of \( \mathcal{F}' \) is closed in \( M \cap X \).
2. \( \mathcal{F}' \) is a filter on \( M \cap X \), hence has the finite intersection property.
3. \( \mathcal{F}' \) is a maximal family satisfying (1) and (2).

Since \( X \) is \( 2^\kappa \)-total and \( |M \cap X| \leq 2^\kappa \), we can fix \( y \in \bigcap_{F \in \mathcal{F}'} \overline{F} \) and take \( \beta < \lambda \) with \( y \in G_{\beta} \). Then we have \( \beta \notin M \cap \lambda \).

For every \( a \in [\kappa]^{<\omega} \), we know that the family \( \{W_{a_{\alpha}}^\beta | \alpha \in [\kappa]^{<\omega} \} \) cannot have the finite intersection property; Otherwise, by the maximality of \( \mathcal{F}' \), we have \( (M \cap X) \setminus W_{a_{\alpha}}^\beta \in \mathcal{F}' \). This contradicts to the choice of \( y \). Hence there is \( C_{\alpha} \subseteq W_{a_{\alpha}}^\beta \). We may assume \( C_{\alpha} \subseteq C_{\alpha_0} \) for every \( \alpha \supseteq \alpha_0 \). Fix \( z_{a} \in C_{a} \) for each \( a \in [\kappa]^{<\omega} \).

Let \( H = \{ z_{a} | a \in [\kappa]^{<\omega} \} \). Then \( H \subseteq M \cap X \) with \( |H| \leq \kappa \), so \( H \in M \). Put \( B_{a} = \{ z_{b} | b \supseteq a \} \) for \( a \in [\kappa]^{<\omega} \). We know that \( \{B_{a} | \alpha \in [\kappa]^{<\omega} \} \) is a filter base on \( H \). By the \( 2^\kappa \)-totality of \( X \), we can pick \( z \in \bigcap_{a \in [\kappa]^{<\omega}} B_{a} \). Since \( H \in M \), we may assume \( z \in M \cap X \). Then we have \( z \in \bigcap_{a \in [\kappa]^{<\omega}} C_{a} \); If \( z \notin C_{a} \) for some \( a \), since \( C_{a} \) is closed in \( M \cap X \), pick an open neighborhood \( O \) of \( z \) with \( O \cap C_{a} = \emptyset \). Because \( z \in B_{a} \), there is \( b \supseteq a \) with \( z_{b} \in O \). However \( z_{b} \in C_{b} \subseteq C_{a} \), this is a contradiction.

We have known \( z \in \bigcap_{a \in [\kappa]^{<\omega}} C_{a} \subseteq \bigcap_{a \in [\kappa]^{<\omega}} W_{a_{\alpha}}^\beta = G_{\beta} \). The set \( \{ \alpha < \lambda | z \in G_{\alpha} \} \) is definable in \( M \) and has cardinality \( \leq 2^\kappa \), hence \( \beta \in \{ \alpha < \lambda | z \in G_{\alpha} \} \subseteq M \cap \lambda \) and \( \beta \in M \cap \lambda \). This is a contradiction. \( \square \)

For a topological space \( X \) and an infinite cardinal \( \kappa \), let \( X_{\kappa} \) be the space \( X \) with topology generated by all \( G_{\kappa} \)-subsets. Let \( L(X) \) denote the Lindelöf degree of \( X \).

**Corollary 4.** Let \( \kappa \) be an infinite cardinal, and \( X \) a \( 2^\kappa \)-total space. Then the following are equivalent:

1. \( L(X_{\kappa}) \leq 2^\kappa \).
2. For every cover \( \mathcal{G} \) of \( X \) by \( G_{\kappa} \)-subsets, there is a subcover \( \mathcal{G}' \) of \( \mathcal{G} \) such that \( |\{G \in \mathcal{G}' | x \in G\}| \leq 2^\kappa \) for every \( x \in X \).
3. For every cover \( \mathcal{G} \) of \( X \) by \( G_{\kappa} \)-subsets, there is a refinement cover \( \mathcal{G}' \) of \( \mathcal{G} \) by \( G_{\kappa} \)-subsets such that \( |\{G \in \mathcal{G}' | x \in G\}| \leq 2^\kappa \) for every \( x \in X \).
Note that there is a compact \( T_2 \) space \( X \) such that \( L(X_\omega) \) is much greater than \( 2^\omega \), e.g., see Usuba [6].

**Corollary 5.** If \( X \) is a \( 2^\kappa \)-total space and \( \mathcal{G} \) is a partition of \( X \) by \( G_\kappa \)-subsets, then \( |\mathcal{G}| \leq 2^\kappa \).

**Note 6.** Arhangel'skii [1] proved that if \( X \) is a compact Hausdorff space, then \( X \) cannot be partitioned into more than \( 2^\omega \)-many closed \( G_\delta \)-subsets. The above corollary is a generalization of this result.

Now Stephenson’s theorem is immediate from this corollary.

**Corollary 7.** If \( X \) is a \( 2^{\psi(X)} \)-total \( T_1 \) space, then \( |X| \leq 2^{\psi(X)} \) and \( X \) is compact.

For \( H \)-closed spaces, we use the following easy observation:

**Lemma 8.** For a Hausdorff space \( X \), the following are equivalent:

1. \( X \) is \( H \)-closed.
2. For every upward directed set \( D = \langle D, \leq \rangle \) and net \( \{ x_a \mid a \in D \} \subseteq X \), there is \( x \in X \) such that for every open neighborhood \( V \) of \( x \) and every \( a \in D \), there is \( b \geq a \) with \( x_b \in \overline{V} \).

For a space \( X \) and \( A \subseteq X \), the \( \theta \)-closure of \( A \), \( \overline{A}^\theta \), is the set \( \{ x \in X \mid A \cap \overline{V} \neq \emptyset \} \) for every open neighborhood \( V \) of \( x \). A subset \( A \subseteq X \) is \( \theta \)-closed if \( \overline{A}^\theta = A \). Note that the following:

1. For every \( A \subseteq X \), \( \overline{A}^\theta \) is \( \theta \)-closed.
2. Every \( \theta \)-closed set is closed in \( X \), and if \( X \) is regular then the converse holds.
3. If \( O \subseteq X \) is open, then \( \overline{O}^\theta \) is \( \theta \)-closed.
4. Even if \( X \) is \( H \)-closed, every closed subset of \( X \) needs not be an \( H \)-set, but every \( \theta \)-closed subset of \( X \) is an \( H \)-set.

**Proposition 9.** Let \( \kappa \) be an infinite cardinal. Let \( X \) be an \( H \)-closed space, and \( \mathcal{G} \) a cover of \( X \) by \( G_\kappa \)-sets such that for every \( G \in \mathcal{G} \), there is a family \( \{ W_\xi \mid \xi < \kappa \} \) of open sets with \( G = \bigcap_{\xi < \kappa} W_\xi = \bigcap_{\xi < \kappa} \overline{W_\xi} \). If for every \( x \in X \), the set \( \{ G \in \mathcal{G} \mid x \in G \} \) has cardinality \( \leq 2^\kappa \), then \( \mathcal{G} \) has a subcover of size \( \leq 2^\kappa \).

**Proof.** Suppose to the contrary that \( \mathcal{G} \) has no such a subcover, and let \( \{ G_\alpha \mid \alpha < \lambda \} \) be an enumeration of \( \mathcal{G} \). For \( \alpha < \lambda \), take open sets \( \{ W_\xi^\alpha \mid \xi < \kappa \} \) with \( G_\alpha = \bigcap_{\xi < \kappa} W_\xi^\alpha = \bigcap_{\xi < \kappa} \overline{W_\xi^\alpha} \).

Take a sufficiently large regular cardinal \( \chi \), and take \( M \prec H(\chi) \) containing all relevant objects such that \( |M| = 2^\kappa \subseteq M \) and \( |M|^\kappa \subseteq M \). We have \( X \neq \bigcup_{\alpha \in M \cap \lambda} G_\alpha \). Fix \( x^* \in X \setminus \bigcup_{\alpha \in M \cap \lambda} G_\alpha \). For \( \alpha \in M \cap \lambda \), fix \( \xi_\alpha < \kappa \) with \( x^* \notin \overline{W_{\xi_\alpha}^\alpha} \).

Now we claim that there are finitely many \( \alpha_0, \ldots, \alpha_k \in M \cap \lambda \) such that \( M \cap X \subseteq \bigcup_{i \leq k} \overline{W_{\xi_{\alpha_i}}^{\alpha_i}} \). As before, however, this is impossible.
Suppose that \( M \cap X \not\subseteq \bigcup_{i \leq k} \overline{W_{\xi_{\alpha_i}}^{\alpha_i}} \) for every finitely many \( \alpha_0, \ldots, \alpha_k \in M \cap \lambda \). Let \( \mathcal{F} = \{(M \cap X) \setminus \overline{W_{\xi_{\alpha}^{\alpha}}}, \alpha \in M \cap \lambda \} \). By the assumption, \( \mathcal{F} \) has the finite intersection property. Take a family \( \mathcal{F}' \subseteq \mathcal{P}(M \cap X) \) such that:

1. \( \mathcal{F} \subseteq \mathcal{F}' \).
2. \( \mathcal{F}' \) is a filter over \( M \cap X \), hence has the finite intersection property.
3. \( \mathcal{F}' \) is a maximal family satisfying (1) and (2).

For \( C \in \mathcal{F}' \), take \( y_C \in C \). Let \( D = \{\mathcal{F}' \supseteq \langle \rangle \} \), this is an upward directed set. Hence by Lemma 8, we can find \( y \in X \) such that for every open neighborhood \( V \) of \( y \) and \( C \in \mathcal{F} \), there is \( C' \in \mathcal{F} \) with \( C' \subseteq C \) and \( y_C \in \overline{V} \).

Choose \( \beta < \lambda \) with \( y \in G_{\beta} \). As before, we have \( \beta \notin M \cap \lambda \); If \( \beta \in M \cap \lambda \), then \( (M \cap X) \setminus \overline{W_{\xi_{\beta}}^{\beta}} \in \mathcal{F}' \), but \( y \in W_{\xi_{\beta}}^{\beta} \) and \( \overline{W_{\xi_{\beta}}^{\beta}} \cap ((M \cap X) \setminus \overline{W_{\xi_{\beta}}^{\beta}}) = \emptyset \). This is impossible.

For \( \xi < \kappa \), we have that \( \{(M \cap X) \setminus \overline{W_{\xi}^{\beta}}\} \cup \mathcal{F}' \) cannot have the finite intersection property; If so, then \( (M \cap X) \setminus \overline{W_{\xi}^{\beta}} \in \mathcal{F}' \) by the maximality of \( \mathcal{F}' \). Put \( C = (M \cap X) \setminus \overline{W_{\xi}^{\beta}} \). By the choice of \( y \), we can find \( C' \in \mathcal{F} \) with \( z_C' \in C' \subseteq C \) and \( z_{C'} \in \overline{W_{\xi}^{\beta}} \), this is impossible. Hence there is \( C_{\xi} \in \mathcal{F}' \) with \( C_{\xi} \subseteq \overline{W_{\xi}^{\beta}} \). For \( a \in [\kappa]^{<\omega} \), let \( C_a = \bigcap_{\xi \in a} C_{\xi} \in \mathcal{F}' \). We have that \( C_b \subseteq C_a \) for every \( b \supseteq a \). Fix \( z_a \in O_a \) for each \( a \in [\kappa]^{<\omega} \).

Let \( H = \{z_a \mid a \in [\kappa]^{<\omega}\} \). We have \( H \in M \). Then \( H \) is a net associated with the directed set \( [\kappa]^{<\omega} \), hence we can find \( z \) such that for every open neighborhood \( V \) of \( z \) and \( a \in [\kappa]^{<\omega} \), there is \( b \supseteq a \) with \( z_b \in \overline{V} \). Since \( H \in M \), we may assume that \( z \in M \cap X \). Then we have \( z \in \bigcap_{\xi < \kappa} \overline{W_{\xi}^{\beta}} = G_{\beta} \); Suppose \( z \notin \overline{W_{\xi}^{\beta}} \) for some \( \xi < \kappa \). Since \( W_{\xi}^{\beta} \) is open, we have that \( \overline{W_{\xi}^{\beta}} \) is \( \theta \)-closed. Hence we can pick an open neighborhood \( V \) of \( z \) with \( \overline{V} \cap \overline{W_{\xi}^{\beta}} = \emptyset \). On the other hand we can choose \( b \supseteq \{\xi\} \) with \( z_b \in \overline{V} \). \( z_b \in O_b \subseteq \overline{W_{\xi}^{\beta}} \), this is impossible.

The set \( \{\alpha < \lambda \mid z \in G_{\alpha}\} \) is definable in \( M \) and has cardinality \( \leq 2^\kappa \), hence \( \beta \in \{\alpha < \lambda \mid z \in G_{\alpha}\} \subseteq M \cap \lambda \) and \( \beta \in M \cap \lambda \). This is a contradiction. \( \Box \)

For a \( \theta \)-closed set \( G \subseteq X \), let \( \psi_{c}(G, X) \) denote the minimum infinite cardinal \( \kappa \) such that there is an open sets \( \{V_\alpha \mid \alpha < \kappa\} \) with \( G = \bigcap_{\alpha < \kappa} V_\alpha = \bigcap_{\alpha < \kappa} \overline{V_\alpha} \). It is clear that \( \psi_{c}(G, X) \leq \chi(G, X) \).

Corollary 10. Let \( X \) be an \( H \)-closed space, and \( \kappa \) an infinite cardinal.

1. For every partition \( G \) of \( X \) by \( \theta \)-closed sets, if \( \psi_{c}(G, X) \leq \kappa \) for every \( G \in \mathcal{G} \) then \( |\mathcal{G}| \leq 2^\kappa \).
2. (Gryzlov [3], Dow-Porter [2]) Let \( X \) be an \( H \)-closed space. Then \( |X| \leq 2^{\psi_{c}(X)} \).
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REFERENCES


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