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A Note on Impulse Control with Outside Jumps*

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1 Introduction

Reversible investment or capacity choice have been studied extensively as singular control or impulse control problems. Especially, capacity choice is one of the biggest interest of companies and to optimize their decision makings about capacity choice based on product prices or something that affect their production planning brings economic profits into them. However, capacity choice is usually represented by impulse control problem with outside jumps and any effective methods that give solutions of the problem have not been found yet. If those methods are found, more general and complicated capacity choice problems can be solved explicitly and the theory of capacity choice or even stochastic control as a whole dramatically makes progress.

One of the most epoch-making study of capacity choice in these days is Guo and Tomecek (2008). In Guo and Tomecek (2008), the theoretical connection between singular control and optimal switching control in two-regime case that was discussed in Vath and Pham (2007) was established and they were successful in solving a multidimensional capacity choice problem which is defined as a singular control by applying the connection. It means that they obtained solutions of singular control problem by solving corresponding optimal switching problem and at the same time, the conversion from the outside control in singular control to the inside control in optimal switching control is observed. Moreover, they expanded their theory in Guo (2009). This approach is more informative than other previous approaches because it does not require any special forms of utility functions which are required by previous researches and, furthermore, it allows non-smooth utility functions. However, many capacity choice problems include not only proportional cost but also fixed cost, so impulse control is more suitable for representing those kind of problems. And it is difficult to solve the problems through usual approaches as reported in Goto et al. (2006).

In this thesis, we consider a novel approach that gives solutions of impulse control problem with outside jumps and that is inspired by Guo and Tomecek (2008). If we can connect impulse control with optimal switching control, we can solve the problems through converting outside jumps of impulse control into inside jumps of optimal switching control. In our case, controls are accompanied by impulse, so we should change the form of optimal switching control to make it correspond to impulse control and we call the changed optimal switching control transformed switching control. For solving the problem, we have to clear three things. First, it

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is necessary that the theoretical connection between impulse control and transformed switching exists. The transformed switching that we now consider is not derived from the impulse control mathematically but assumed for convenience, so it might be possible that there is no connection between them in practice. Second, we need to reveal whether transformed switching has mathematical explicit solutions or not. It means that the transformed switching has to meet HJB equation but we currently do not know wether it meets the equation. Third, we have to verify that transformed switching can be solved practically and analyze the parameters' effects on the solutions. However, sophisticated mathematical discussion is unavoidable to prove the first and second problems, so we establish that the purpose of present study is to obtain the solutions of transformed switching and give examples of them under some different parameters.

2 Hypothesis for Correspondence between Impulse Control with Transformed Switching Control

2.1 Problem

Let (Ω, F, F, P) be a filtered probability space, and assume a given bounded interval $[a, b] \in (-\infty, \infty)$. Consider the following problem.

$$V(x, y) = \sup_{\omega \ni W} J(x, y, \xi_i), \tag{1}$$

with

$$J(x, y, \xi_i) := \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\rho t} H(Y_t) X_t^x \mathrm{d}t - \sum_{i=1}^\infty \mathrm{e}^{-\rho \tau_i} C(\xi_i)\right], \tag{2}$$

$$C(\xi) = \begin{cases} K_1 + E_1\xi, & \xi > 0, \\ 0, & \xi = 0, \ C(0) = K_1, \\ K_0 + E_0\xi, & \xi < 0, \end{cases} \tag{3}$$

subject to

$$\begin{aligned}
&\mathrm{d}X_t^x = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \ X_0 = x > 0, \\
&\omega = (\tau_1, \tau_2, ...; \xi_1, \xi_2, ...), \\
&N(t) = \sup[i \geq 0 : \tau_i \leq t], \\
&Yt := y + \sum_{i=1}^{N(t)} \xi_i, \quad y \in [a, b], \\
&H : [a, b], \quad H(y) = H(a) + \int_a^y h(z) \mathrm{d}z, \\
&K_1, K_0, E_1 > 0.
\end{aligned}$$

where the market price X is modeled by a geometric Brownian motion; an impulse control ω is represented by stopping time τ and impulse ξ ; the last stopping time $N(t)$ is defined by terminal time t; the capacity level Y is controlled process represented by ξ_i; the resource extraction rate is modeled by a function $H(Y)$; $K_1 > 0$ is the fixed cost per investment and $K_0 > 0$ is the fixed cost per disinvestment; $E_1 > 0$ is the cost of capacity increase and E_0 is the cost of capacity reduction. The aim of the firm is to maximize J by controlling Y.

2.2 Preliminary

Throughout the thesis, we define $m < 0 < 1 < n$ to be the roots of $\sigma^2 x^2 + (\mu - \sigma^2)x - \rho = 0$, so that

$$m, n = \frac{-(\mu - \sigma^2) \pm \sqrt{(\mu - \sigma^2)^2 + 4\sigma^2\rho}}{2\sigma^2}. \tag{4}$$

We also observe the identity $\rho = -\sigma^2 mn$ and define the useful quantity $\eta > 0$:

$$\eta := \frac{1}{\rho - \mu} = \frac{-mn}{(n-1)(1-m)\rho} = \frac{1}{\sigma^2(n-1)(1-m)}. \tag{5}$$

Next, let $R(x, y) := J(x, y, 0)$ be the no-action expected payoff. Then,

$$R(x, y) := \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\rho t} H(y) X_t^x \mathrm{d}t\right] = \eta H(y)x, \tag{6}$$

$$r(x, y) := R_y(x, y) = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\rho t} h(y) X_t^x \mathrm{d}t\right] = \eta h(y)x. \tag{7}$$

2.3 Corresponding transformed switching and the value function

Optimal switching control corresponding with singular control in Guo and Tomecek (2008) is as follows.

$$v_k(x, z) := \sup_{\substack{\alpha \in B \\ \kappa_0 = k}} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\rho t}[h(z) X_t^x] I_t \mathrm{d}t - \sum_{n=1}^\infty \mathrm{e}^{-\rho\tau_n} K_{\kappa_n}\right], \tag{8}$$

where $\alpha = (\tau_i, \kappa_n)_{n\geq 0}$ is an admissible switching control, B is the subset of admissible switching controls $\alpha = (\tau_i, \kappa_n)_{n\geq 0}$ such that $\mathbb{E}[\sum_{n=1}^\infty \mathrm{e}^{-\rho\tau_n}] < 0$, and I_t is the regime indicator function defined as follows.

$$I_t := \sum_{n=0}^\infty \kappa_n 1_{\{\tau_n < t \leq \tau_{n+1}\}}, \quad I_0 = \kappa_0. \tag{9}$$

Here, we consider the two-regime switching between the stoping status and the operating status, $k \in [0, 1]$. Finally, we have the value function represented by the payoff of optimal switching control:

$$V(x, y) = \eta H(a)x + \int_a^y v_1(x, z)\mathrm{d}z + \int_y^b v_0(x, z)\mathrm{d}z. \tag{10}$$

According to the theoretical connection between singular control and optimal switching control established in Guo and Tomecek (2008), the value function represents that of singular control problem.

As for impulse control with outside jumps, we assume that the representation of the value function of impulse control is same as that of singular control. However, we define transformed switching may corresponds with impulse control as follows.

$$v_k(x, z) := \sup_{\substack{\alpha \in B \\ \kappa_0 = k}} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\rho t}[h(z) X_t^x] I_t \mathrm{d}t - \sum_{n=1}^\infty \mathrm{e}^{-\rho\tau_n}(K_{\kappa_n} + E_{\kappa_n}\xi_n)\right]. \tag{11}$$

Transformed switching deals with the switching cost of capacity increase or reduction in addition to the fixed switching cost. Moreover, we consider that the two-regime switching between two different operating statuses in this case. Therefore, we know the value function of impulse control problem is (10) and the payoff ν_0 and ν_1 are the solutions of transformed switching problem (11) are as follows:

$$v_0(x,z) = \sup_{\substack{\alpha\in B \\ \kappa_0=0}} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\rho t} h(z) X_t^x I_t \mathrm{d}t - \sum_{n=1}^{\infty} \mathrm{e}^{-\rho\tau_n}(K_1 + E_1\xi_n)\right], \tag{12}$$

$$v_1(x,z) = \sup_{\substack{\alpha\in B \\ \kappa_0=1}} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\rho t} (-h(z) X_t^x)(1-I_t) \mathrm{d}t - \sum_{n=1}^{\infty} \mathrm{e}^{-\rho\tau_n}(K_0 + E_0\xi_n)\right]. \tag{13}$$

3 Solutions of Transformed Switching Control

We assume that ν_0 and ν_1 are the unique solutions with linear growth condition to the following system of variational inequalities.

$$\min\{-\mathcal{L}v_0(x,z) - h(z)x, v_0(x,z) - v_1(x,z+\xi) + K_1 + E_1\xi\} = 0, \tag{14}$$

$$\min\{-\mathcal{L}v_1(x,z) - h(z)x, v_1(x,z) - v_0(x,z+\xi) + K_0 + E_0\xi\} = 0, \tag{15}$$

where $\mathcal{L}$ is the generator of the diffusion X^x, killed at rate ρ, given by $\mathcal{L}u(x,z) = \sigma^2 u_{xx}(x,z) + \mu u_x(x,z) - \rho u(x,z)$. Based on Vath and Pham (2007), we solve ν_0 and ν_1 in the following two cases.

3.1 Case 1: $E_0 \geq 0$

For each $z \in (a,b)$, we describe the switching regions as $d(z)$ and $u(z)$ which take values $(0,\infty]$.

Firstly, for each $z \in (a,b)$ such that $h(z) = 0$, it is never optimal to switch since $E_0 \geq 0, E_1 > 0, K_0 > 0, K_1 > 0, \xi_n > 0$ in ν_0 and $\xi_n < 0$ in ν_1, and so we have $d(z) = \infty = u(z)$. For this case, $v_0(x,z) = 0 = v_1(x,z)$.

Secondly, for each $z \in (a,b)$ such that $h(z) > 0$, it is optimal to switch from regime 0 to regime 1 when $X_t^x \in [u(z),\infty)$, and to switch from regime 1 to regime 0 when $X_t^x \in (0,d(z)]$, where $0 < d(z) < u(z) < \infty$. Furthermore, we have

$$v_0(x,z) = \begin{cases} A(z)x^n + \eta h(z)x, & x < u(z), \\ B(z)x^m + \eta h(z+\xi_1)x - K_1 - E_1\xi_1, & x \geq u(z), \end{cases} \tag{16}$$

$$v_1(x,z) = \begin{cases} A(z)x^n + \eta h(z+\xi_0)x - K_0 - E_0\xi_0, & x < d(z), \\ B(z)x^m + \eta h(z)x, & x \geq d(z). \end{cases} \tag{17}$$

where $\xi_1 \geq 0$ is accompanied by investment and $\xi_0 < 0$ is accompanied by disinvestment. The value matching condition, the smooth pasting condition and the condition of a differentiation with respect to an impulse lead to the following boundary conditions:

$$\begin{cases} A(z)u^n(z) + \eta h(z)u(z) = B(z)u^m(z) + \eta h(z+\xi)u(z) - K_1 - E_1\xi_1, \\ nA(z)u^{n-1}(z) + \eta h(z) = mB(z)u^{m-1}(z) + \eta h(z+\xi_1), \\ \eta\frac{\partial h(z+\xi_1)}{\partial \xi_1}u(z) - E_1 = 0, \end{cases} \tag{18}$$

$$\begin{cases} A(z)d^n(z) + \eta h(z+\xi_0)d(z) - K_0 - E_0\xi_0 = B(z)d^m(z) + \eta h(z)d(z), \\ nA(z)d^{n-1}(z) + \eta h(z+\xi_0) = mB(z)d^{m-1}(z) + \eta h(z), \\ \eta\frac{\partial h(z+\xi_0)}{\partial \xi_0}d(z) - E_0 = 0. \end{cases} \tag{19}$$

Here, we assume $h(z) = z^\gamma$ (γ is an arbitrary constant value), so given $d(z), u(z), A(z), B(z), \xi_0$ and ξ_1 are solved from (16),

$$\begin{cases} A(z) = \frac{\eta\left(1-\frac{1}{m}\right)(h(z+\xi_1)-h(z))u(z)-K_1-E_1\xi_1}{u^n(z)\left(1-\frac{n}{m}\right)} \\ \quad = \frac{\eta\left(1-\frac{1}{m}\right)(h(z+\xi_0)-h(z))d(z)-K_0-E_0\xi_0}{d^n(z)\left(\frac{n}{m}-1\right)}, \\ B(z) = \frac{\eta\left(1-\frac{1}{n}\right)(h(z+\xi_1)-h(z))u(z)-K_1-E_1\xi_1}{u^m(z)\left(\frac{m}{n}-1\right)} \\ \quad = \frac{\eta\left(1-\frac{1}{n}\right)(h(z+\xi_0)-h(z))d(z)-K_0-E_0\xi_0}{d^m(z)\left(1-\frac{m}{n}\right)}, \\ (K_1+E_1\xi_1)u^{-n}(z) + (K_0+E_0\xi_0)d^{-n}(z) \\ \quad = \eta\left(1-\frac{1}{m}\right)\left(h(z+\xi_1)u^{1-n}(z) + h(z+\xi_0)d^{1-n}(z) - h(z)(u^{1-n}(z)+d^{1-n}(z))\right), \\ (K_1+E_1\xi_1)u^{-m}(z) + (K_0+E_0\xi_0)d^{-m}(z) \\ \quad = \eta\left(1-\frac{1}{n}\right)\left(h(z+\xi_1)u^{1-m}(z) + h(z+\xi_0)d^{1-m}(z) - h(z)(u^{1-m}(z)+d^{1-m}(z))\right), \\ 0 = \eta\gamma(z+\xi_1)^{\gamma-1}u(z) - E_1, \\ 0 = \eta\gamma(z+\xi_0)^{\gamma-1}d(z) - E_0. \end{cases} \tag{20}$$

Lastly, for each $z \in (a, b)$ such that $h(z) < 0$, it is optimal to switch from regime 1 to regime 0 when $X_t^x \in (0, d(z)]$ and it is never optimal to switch from regime 0 to regime 1, so we have $d(z) < \infty$ and $u(z) = \infty$. Moreover, ν_0 and ν_1 are given by

$$v_0(x,z) = \eta h(z)x, \tag{21}$$

$$v_1(x,z) = \begin{cases} \eta h(z+\xi)x - K_0 - E_0\xi, & x \le d(z), \\ B(z)x^m + \eta h(z)x, & x > d(z), \end{cases} \tag{22}$$

where $\xi < 0$. We obtain the boundary condition as follows.

$$\begin{cases} B(z)d(z)^m + \eta h(z)d(z) = \eta h(z+\xi)d(z) - K_0 - E_0\xi, \\ mB(z)d^{m-1}(z) + \eta h(z) = \eta h(z+\xi), \\ \eta\frac{\partial h(z+\xi)}{\partial \xi}d(z) - E_0 = 0. \end{cases} \tag{23}$$

Therefore, assuming $h(z) = z^\gamma - \alpha$ (γ is an arbitrary constant value and α is a positive number) for the existence of $z \in (a, b)$ such that $h < (0), B(z), d(z)$ and ξ are given by

$$\begin{cases} B(z) = \frac{\eta[(z+\xi)^\gamma - z^\gamma]d(z) - K_0 - E_0\xi}{d(z)^m}, \\ d(z) = \frac{m}{m-1}\frac{K_0+E_0\xi}{\eta[(z+\xi)^\gamma - z^\gamma]}, \\ 0 = \gamma(z+\xi)^{\gamma-1}\frac{K_0/E_0+\xi}{(z+\xi)^\gamma - z^\gamma} - \frac{m-1}{m}. \end{cases} \tag{24}$$

3.2 Case 2: $E_0 < 0$

First, for each $z \in (a, b)$ such that $h(z) = 0$, it is never optimal to switch since $E_0 < 0$. Therefore, $d(z) = \infty = u(z)$ and $v_0(x, z) = 0 = v_1(x, z)$ in this case.

Then, for each $z \in (a, b)$ such that $h(z) > 0$, it is optimal to switch from regime 0 to regime 1 when $X_t^x \in [u(z), \infty)$ and it is never optimal to switch from regime 1 to regime 0. That is, $d(z) = \infty$ and $u(z) < \infty$ and we have

$$v_0(x, z) = \begin{cases} A(z)x^n + \eta h(z)x, & x < u(z), \\ h(z+\xi)x - K_1 - E_1\xi, & x \geq u(z), \end{cases} \tag{25}$$

$$v_1(x, z) = \eta h(z)x. \tag{26}$$

According to the conditions, the boundary condition is as follows.

$$\begin{cases} A(z)u(z)^n + \eta h(z)u(z) = \eta h(z+\xi)u(z) - K_1 - E_1\xi, \\ nA(z)u^{n-1}(z) + \eta h(z) = \eta h(z+\xi), \\ \eta\frac{\partial h(z+\xi)}{\partial \xi}u(z) - E_1 = 0. \end{cases} \tag{27}$$

Here, $h(z) = z^\gamma$ (γ is an arbitrary constant value) is assumed. That is,

$$\begin{cases} A(z) = \frac{\eta[(z+\xi)^\gamma - z^\gamma]u(z) - K_1 - E_1\xi}{u(z)^n}, \\ u(z) = \frac{n}{n-1}\frac{K_1+E_1\xi}{\eta[(z+\xi)^\gamma - z^\gamma]}, \\ 0 = \gamma(z+\xi)^{\gamma-1}\frac{K_1/E_1+\xi}{(z+\xi)^\gamma - z^\gamma} - \frac{n-1}{n}. \end{cases} \tag{28}$$

Finally, for each $z \in (a, b)$ such that $h(z) > 0$, it is optimal to switch from regime 1 to regime 0 when $X_t^x \in (0, d(z)]$ and it is never optimal to switch from regime 0 to regime 1, so we have $d(z) < \infty$ and $u(z) = \infty$. Therefore, we obtain clearly same conclusion as Case 1 with each $z \in (a, b)$ such that $h(z) < 0$ in this case.

In short, we have 4 patterns of transformed switching, no switching, downside switching (only to switch from regime 1 to regime 0), upside switching(only to switch from regime 0 to regime 1), bilateral switching (to switch between regime 0 and regime 1). They are summarized as follows.

1. No switching: $h(z) = 0$

$$v_0(x, z) = v_1(x, z) = 0, d(z) = u(z) = \infty.$$

2. Downside switching: $h(z) = z^\gamma - \alpha < 0, \xi < 0$

$$\begin{cases} v_0(x,z) = \eta h(z) x \\ v_1(x,z) = \begin{cases} \eta h(z+\xi)x - K_0 - E_0\xi, & x \le d(z), \\ B(z)x^m + \eta h(z)x, & x > d(z), \end{cases} \end{cases}$$

$$\begin{cases} B(z) = \frac{\eta[(z+\xi)^\gamma - z^\gamma]d(z) - K_0 - E_0\xi}{d(z)^m}, \\ d(z) = \frac{m}{m-1}\frac{K_0+E_0\xi}{\eta[(z+\xi)^\gamma - z^\gamma]}, \\ 0 = \gamma(z+\xi)^{\gamma-1}\frac{K_0/E_0+\xi}{(z+\xi)^\gamma - z^\gamma} - \frac{m-1}{m}. \end{cases}$$

3. Upside switching: $h(z) = z^\gamma > 0, \xi > 0, E_0 < 0$

$$\begin{cases} v_0(x,z) = \begin{cases} A(z)x^n + \eta h(z)x, & x < u(z), \\ h(z+\xi)x - K_1 - E_1\xi, & x \ge u(z), \end{cases} \\ v_1(x,z) = \eta h(z)x. \end{cases}$$

$$\begin{cases} A(z) = \frac{\eta[(z+\xi)^\gamma - z^\gamma]u(z) - K_1 - E_1\xi}{u(z)^n}, \\ u(z) = \frac{n}{n-1}\frac{K_1+E_1\xi}{\eta[(z+\xi)^\gamma - z^\gamma]}, \\ 0 = \gamma(z+\xi)^{\gamma-1}\frac{K_1/E_1+\xi}{(z+\xi)^\gamma - z^\gamma} - \frac{n-1}{n}. \end{cases}$$

4. Bilateral switching: $h(z) = z^\gamma > 0, \xi_1 > 0, \xi_0 < 0, E_0 \ge 0$

$$\begin{cases} v_0(x,z) = \begin{cases} A(z)x^n + \eta h(z)x, & x < u(z), \\ B(z)x^m + \eta h(z+\xi_1)x - K_1 - E_1\xi_1, & x \ge u(z), \end{cases} \\ v_1(x,z) = \begin{cases} A(z)x^n + \eta h(z+\xi_0)x - K_0 - E_0\xi_0, & x < d(z), \\ B(z)x^m + \eta h(z)x, & x \ge d(z), \end{cases} \end{cases}$$

$$\begin{cases} A(z) = \frac{\eta\left(1-\frac{1}{m}\right)(h(z+\xi_1)-h(z))u(z) - K_1 - E_1\xi_1}{u^n(z)\left(1-\frac{n}{m}\right)} \\ \qquad = \frac{\eta\left(1-\frac{1}{m}\right)(h(z+\xi_0)-h(z))d(z) - K_0 - E_0\xi_0}{d^n(z)\left(\frac{n}{m}-1\right)}, \\ B(z) = \frac{\eta\left(1-\frac{1}{n}\right)(h(z+\xi_1)-h(z))u(z) - K_1 - E_1\xi_1}{u^m(z)\left(\frac{m}{n}-1\right)} \\ \qquad = \frac{\eta\left(1-\frac{1}{n}\right)(h(z+\xi_0)-h(z))d(z) - K_0 - E_0\xi_0}{d^m(z)\left(1-\frac{m}{n}\right)}, \\ (K_1 + E_1\xi_1)u^{-n}(z) + (K_0 + E_0\xi_0)d^{-n}(z) \\ \qquad = \eta\left(1 - \frac{1}{m}\right)\left(h(z+\xi_1)u^{1-n}(z) + h(z+\xi_0)d^{1-n}(z) - h(z)(u^{1-n}(z) + d^{1-n}(z))\right), \\ (K_1 + E_1\xi_1)u^{-m}(z) + (K_0 + E_0\xi_0)d^{-m}(z) \\ \qquad = \eta\left(1 - \frac{1}{n}\right)\left(h(z+\xi_1)u^{1-m}(z) + h(z+\xi_0)d^{1-m}(z) - h(z)(u^{1-m}(z) + d^{1-m}(z))\right), \\ 0 = \eta\gamma(z+\xi_1)^{\gamma-1}u(z) - E_1, \\ 0 = \eta\gamma(z+\xi_0)^{\gamma-1}d(z) - E_0. \end{cases}$$

4 Illustrations of Solutions

4.1 Constraint for calculation

In all cases, it is clear that $A(z) > 0$ and $B(z) > 0$ since $A(z)$ and $B(z)$ represent the option values for waiting to switch from current regime to another regime. We also clearly have $d(z) > 0$ and $u(z) > 0$ because $X_t^x \in (0, \infty)$. In addition, $\xi > 0$ in upside switching, $\xi < 0$ in downside switching, $\xi_1 > 0$ and $\xi_0 < 0$ in bilateral switching.

As for γ, first, $0 < \gamma < (n-1)/n$ for the convergence in upside switching since $E_1 > 0$ requires $\gamma > 0$ and taking the limit of (28) proves $\gamma < (n-1)/n$. Then, in downside switching, it is possible that $\gamma \geq 0$ and $\gamma < 0$ since $E_0 < 0$ requires $\gamma < 0$ and $E_0 \geq 0$ requires $\gamma \geq 0$. Lastly, $\gamma > 0$ because $E_1 > 0$ requires $\gamma > 0$.

4.2 Examples of numerical calculations

4.2.1 Downside switching

In this thesis, we use R programming codes to solve the problem. First of all, we set the parameter values such that $\rho = 0.1, \mu = 0.01, \sigma = 0.2, \gamma = 0.3, K_0 = 1$ and $E_0 = 1$. Next, we calculate $B(z), d(z)$ and ξ with those parameters by the Newton method. Consequently, we obtain the graph as in Fig. 1. $D^{-1}(z) = z + \xi > 0$ is the new z level after we add an impulse ξ to the current z level. This graph illustrates the switching region $(0, d(z)]$ and the continuous region $(d(z), \infty)$, and when X_t^x reaches the switching region, the current capacity level jumps to the new level $D^{-1}(z)$. As for the option values, $B(z) > 0$ for each $z \geq 2$ (we start to calculate from $z = 2$) as in Fig. 2. So downside switching is satisfied with the all constraints under these settings.

Moreover, we also calculate a $E_0 < 0$ and $\gamma < 0$ case and obtain the graphs as in Fig. 3 (we set $E_0 = -1$ and $\gamma = -0.3$ in this case). However, in this case, we have $B(z) < 0$ for each $z \geq 2$ as in Fig. 4. It proves that $E_0 < 0$ and $\gamma < 0$ don't match downside switching.

4.2.2 Upside switching

Firstly, we set the parameter values such that $\rho = 0.1, \mu = 0.01, \sigma = 0.2, \gamma = 0.3, K_1 = 1$ and $E_1 = 1$. Secondly, we determine $A(z), u(z)$ and ξ based on the parameters. Finally, we have the graph as in Fig. 5. $U^{-1}(z) = z + \xi > 0$ is the new z level after we add an impulse ξ to the current z level. In this figure, the switching region and the continuous region are described as $[u(z), \infty)$ and $[0, u(z))$, respectively, and when X_t^x reaches the switching region, the current capacity level jumps to the new level $U^{-1}(z)$. Furthermore, $A(z) > 0$ for each $z \geq 0$ (calculation starts from $z = 0$) as in Fig. 6. So upside switching is satisfied with the all constraints under these settings.

4.2.3 Bilateral switching

First, we set the parameter values such that $\rho = 0.1, \mu = 0.01, \sigma = 0.2, \gamma = 0.3, K_1 = 1, K_0 = 1, E_1 = 2$ and $E_0 = 1$. Then, we calculate $A(z), B(z), u(z), d(z), \xi_1$ and ξ_0. Lastly, the graph as in Fig. 7 is given below. We have the switching region $[0, d(z))$ and $[u(z), \infty)$ and the continuous region $(d(z), u(z))$ in bilateral switching, so when X_t^x reaches the downside and upside switching regions, the current capacity level jumps to the new level $D^{-1}(z)$ and $U^{-1}(z)$, respectively. Moreover, for each $z \geq 5$ (we start to calculate from $z = 5$), $A(z) > 0$ and $B(z) > 0$ as in Figs. 8 and 9. So bilateral switching is satisfied with the all constraints under these settings.

In bilateral switching, it is a critical problem that the calculation occasionally diverge at higher z level. For the problem, we find rather high value of σ or ρ suppress the divergence of the calculation.

4.3 Sensitivity analysis

4.3.1 Downside switching

We analyze the parameters' effects to the solutions. Firstly, as for downside switching, we establish the standard condition such that $\rho = 0.1, \mu = 0.01, \sigma = 0.2, \gamma = 0.3, K_0 = 1$ and $E_0 = 1$ and the solutions under the condition are as in Fig. 10. Next, we see the following cases under the different parameters. Fig. 11 is given by $\gamma = 0.4$, and the downside threshold $d(z)$ and the jumping destinations $D(z)$ decrease at the same capacity level compared with the standard condition. Fig. 12 is given by $\sigma = 0.3$, and $d(z)$ decrease and $D(z)$ is scarcely affected at the same capacity level. Fig. 13 is given by $\rho = 0.2$, and $d(z)$ and $D(z)$ increase at the same capacity level. Fig. 14 is given by $\mu = 0.04$, and $d(z)$ and $D(z)$ increase at the same capacity level. We summarize the above findings from Figs. 11–14 as follows: the increase of γ, σ, μ and the reduction of ρ reduce the downside threshold $d(z)$ at the same capacity level; the increase of γ, μ and the reduction of ρ reduce the jumping destinations $D(z)$ and the values of σ scarcely effect $D(z)$.

Fig. 15 is given by the downside fixed cost $K_0 = 2.4$, and we recognize that K_0 that is over around 2.4 leads to a reversal between $d(z)$ and $D(z)$. Moreover, the bigger K_0 becomes, the more expanded the reversal is as in Fig. 16, which is given by $K_0 = 100$, so it is supposed that the reversal is not dissolved at any z level in this case. However, it is not revealed why the threshold of the reversal is around 2.4 and what affects it. Fig. 17 is given by $E_0 = 10$, and it proves that the downside proportional cost E_0 becomes bigger, $d(z)$ and $D(z)$ increase with the same magnification of E_0 at the same capacity level.

4.3.2 Upside switching

First, we set the standard condition such that $\rho = 0.1, \mu = 0.01, \sigma = 0.2, \gamma = 0.3, K_1 = 1$ and $E_1 = 1$ and the solutions under the condition are as in Fig. 18. Next, we see the following cases under the different parameters and $\gamma = 0.4, \sigma = 0.3, \rho = 0.2, \mu = 0.04, K_1 = 1000$ and $E_1 = 10$. Fig. 19 is given by $\gamma = 0.4$, and the upside threshold $u(z)$ and the jumping destinations $U(z)$ decrease at the same capacity level compared with the standard condition. Fig. 20 is given by $\sigma = 0.3$, and $u(z)$ increase and $U(z)$ is scarcely affected at the same capacity level. Fig. 21 is given by $\rho = 0.2$, and $u(z)$ and $U(z)$ increase at the same capacity level. Fig. 22 is given by

$\mu = 0.04$, and $u(z)$ increase and $U(z)$ decrease at the same capacity level. We summarize the above findings from Figs. 19–22 as follows: the increase of γ, μ and the reduction of σ, ρ reduce the upside threshold $u(z)$ at the same capacity level; the increase of γ and the reduction of ρ, μ reduce the jumping destinations $U(z)$ and the values of σ scarcely effect $U(z)$.

Fig. 23 is given by the upside fixed cost $K_1 = 1000$, and threshold $u(z)$ and the amount of impulse ξ also increase at the same capacity level. Fig. 24 is given by the upside proportional cost $E_1 = 10$, and it proves that $u(z)$ and $U(z)$ increase with the same magnification of E_1 at the same capacity level.

5 Conclusion

The main contribution of this thesis is giving a note on a new method to solve impulse control with outside jumps. It is required to clear three things in order to solve the impulse control problem with outside jumps in terms of transformed switching that we consider in this thesis. Firstly, we have to prove that the theoretical connection between impulse control and transformed switching really exists. Secondly, we need to reveal that transformed switching meets HJB equation. Finally, it is necessary to verify that transformed switching can be solved practically and how the parameters effect on the solutions. In this thesis, we only study the third problem due to some mathematical difficulties of the first and second problems.

We assume transformed switching that is considered to corresponds to impulse control problem and solve it. We are successful in obtaining the solutions, but the solutions of impulse ξ are not given as analytical solutions in all cases and neither are the solutions of the thresholds $d(z)$ and $u(z)$ in bilateral switching. So we calculate the solutions with some different parameters and get the examples of the solutions using R programing. Consequently, although something to be revealed still remain such as the convergence condition of the calculation, it is shown that we can solve transformed switching by numerical calculations under some conditions and how the effects are brought into the solutions by each parameter.

However, we also find the problems of the solutions of transformed switching. First, it is not allowed to set $\gamma < 0$ in all cases. It means that we have to consider that the utility function H is convex and it may contradicts usual diminishing utilities. In Guo (2009), the authors say that H is concave, so the problem may be our original. Second, we can not consider the bigger K_0 in downside switching. This method can not deal with the problem with the bigger downside proportional cost. Third, the bigger K_0 is not allowed unless we consider the higher capacity level in bilateral switching. Only in the z range in which the reversal between $d(z)$ and $D(z)$ is dissolved the solutions are valid. Lastly, especially in bilateral switching, strict parameter setting is required to converge the calculation because the divergence of the calculation is easily observed. Due to these problems, it is possible that the solutions are useless. In addition, as the above statement, we still leave the two more essential steps to complete our method. If these assumptions are not valid, this method may be useless for solving impulse control problem with outside jumps. Thus, there are a lot of things to be studied in order to prove our method's effectiveness, but our attempt must give some useful knowledge to the theory of capacity choice.

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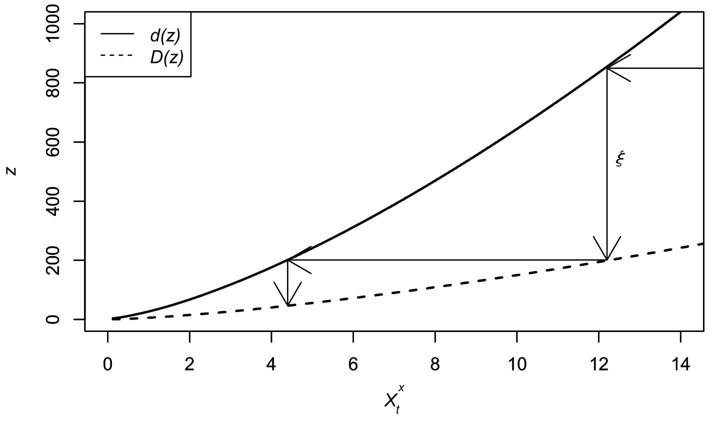

Figure 1: The downside thresholds and the jumping destinations from the current z level ($E_0 \geq 0$).

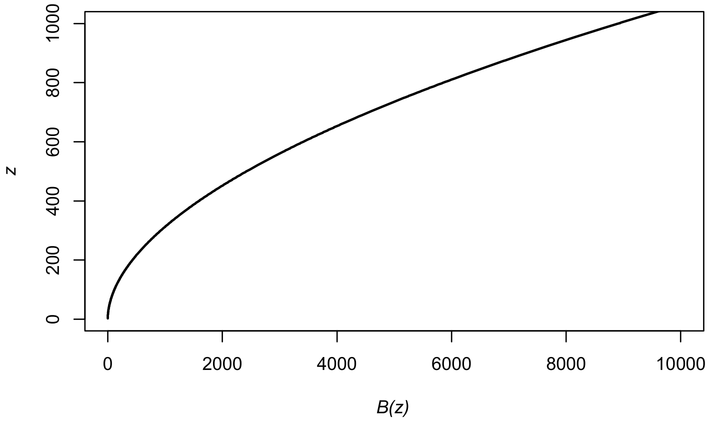

Figure 2: The option values in downside switching ($E_0 \geq 0$).

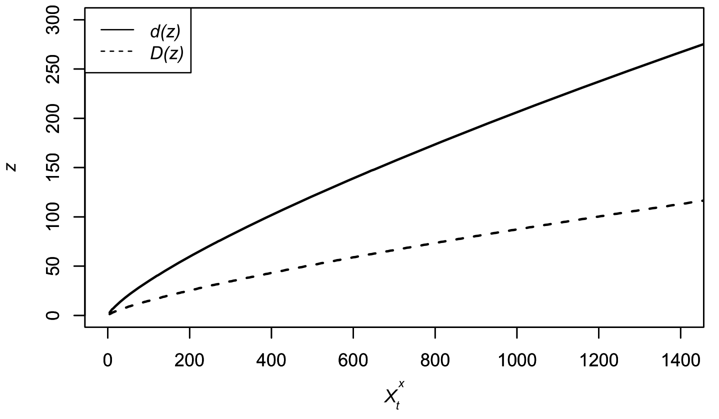

Figure 3: The downside thresholds and the jumping destinations from the current z level ($E_0 < 0$).

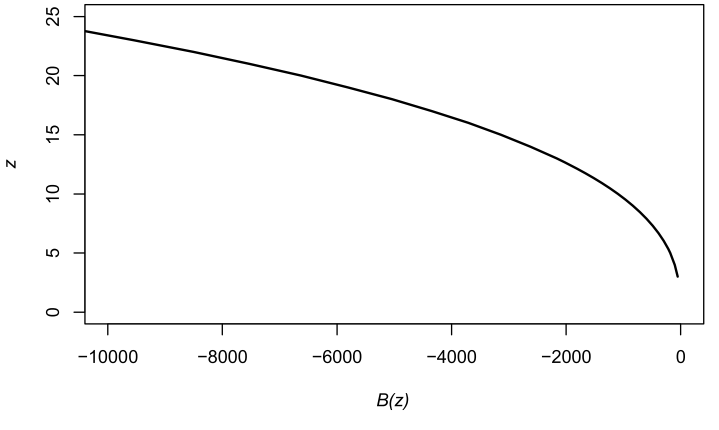

Figure 4: The option values in downside switching ($E_0 < 0$).

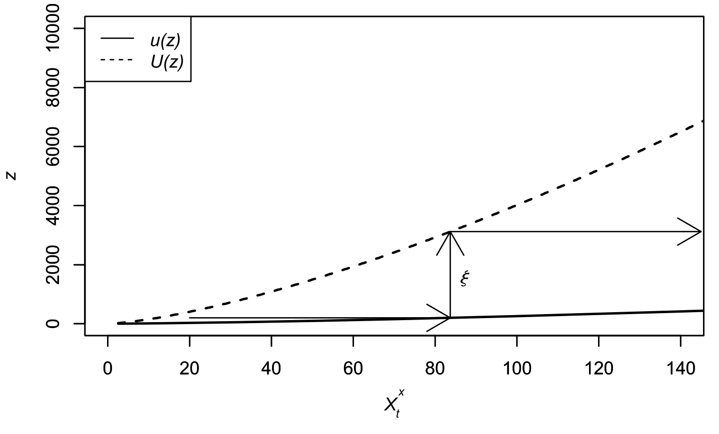

Figure 5: The upside thresholds and the jumping destinations from the current z level.

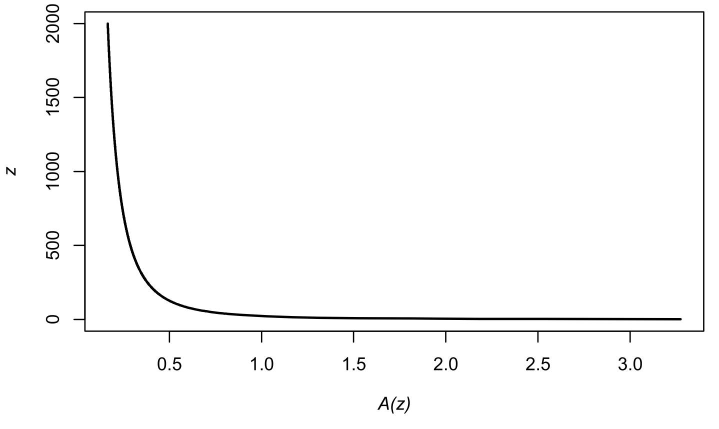

Figure 6: The option values in upside switching.

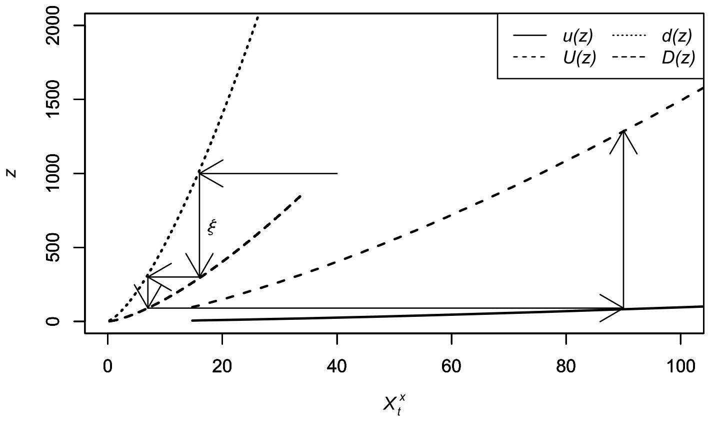

Figure 7: The bilateral thresholds and the jumping destinations from the current z level.

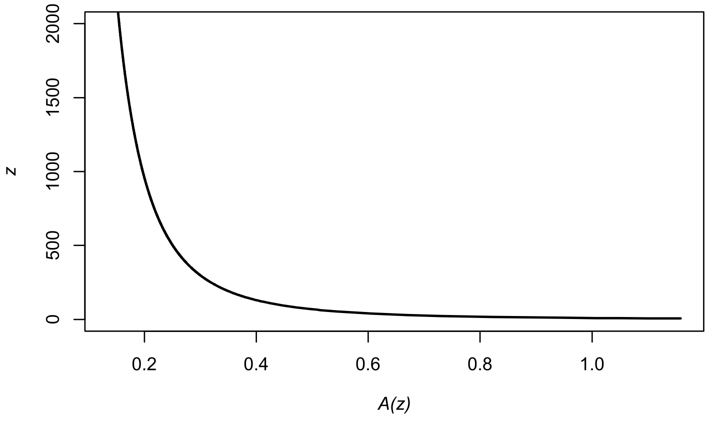

Figure 8: The upside option values in bilateral switching.

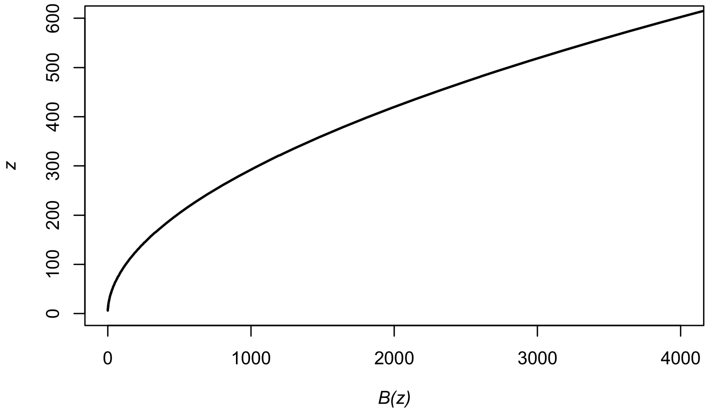

Figure 9: The downside option values in bilateral switching.

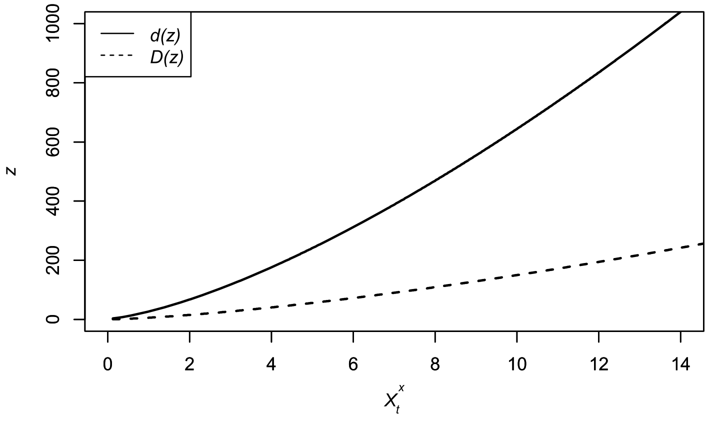

Figure 10: The downside solutions under the standard condition.

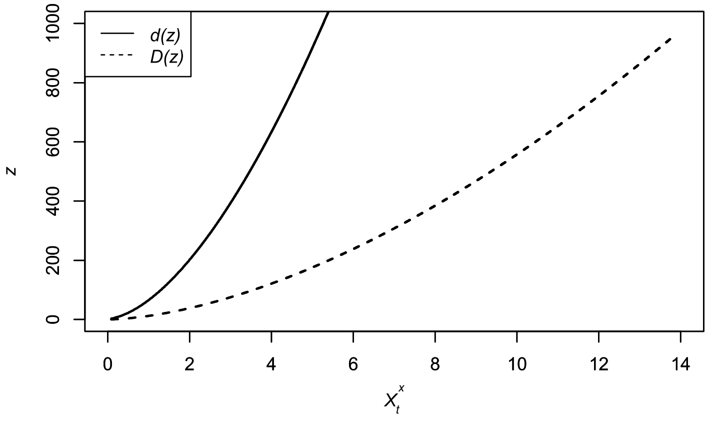

Figure 11: The downside solutions under $\gamma = 0.4$.

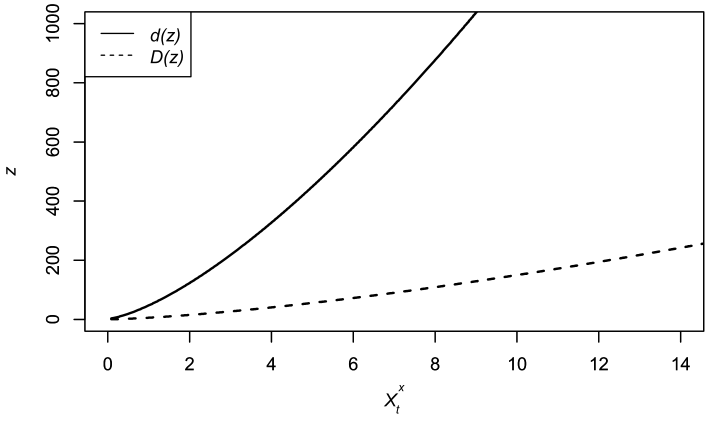

Figure 12: The downside solutions under $\sigma = 0.3$.

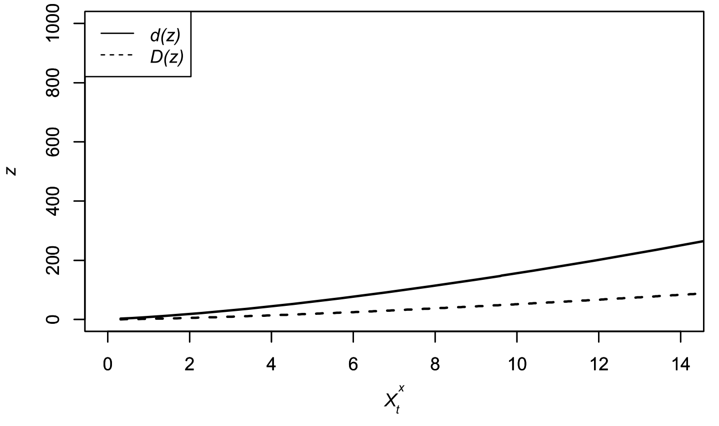

Figure 13: The downside solutions under $\rho = 0.2$.

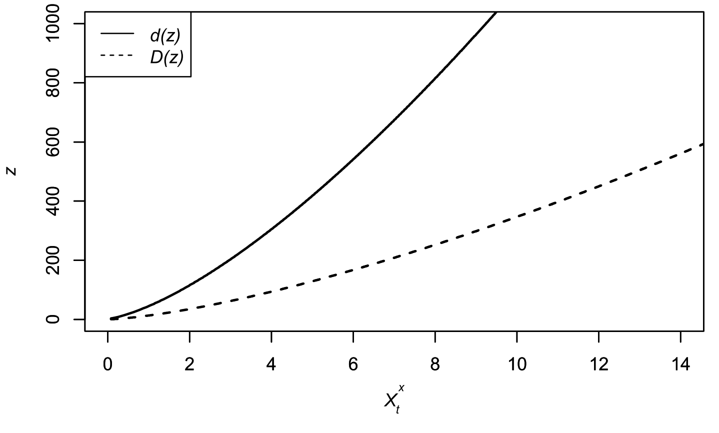

Figure 14: The downside solutions under $\mu = 0.04$.

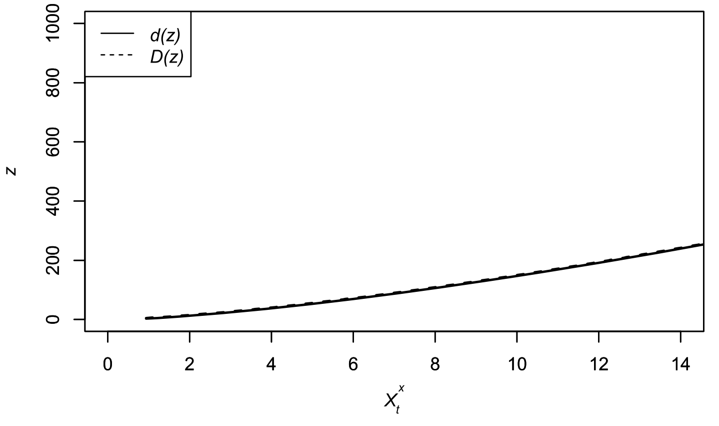

Figure 15: The downside solutions under $K_0 = 2.4$.

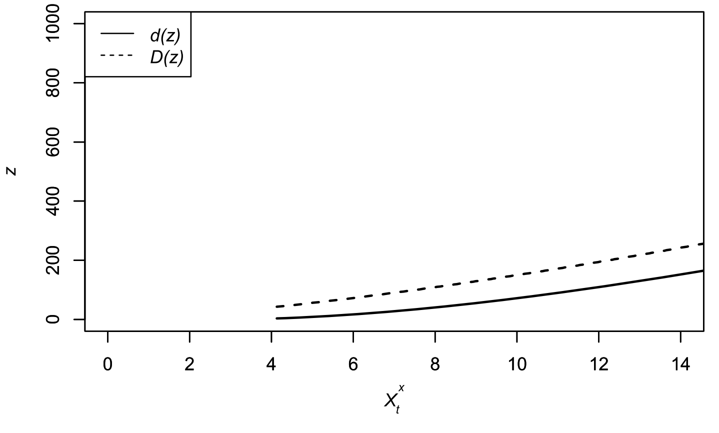

Figure 16: The downside solutions under $K_0 = 100$.

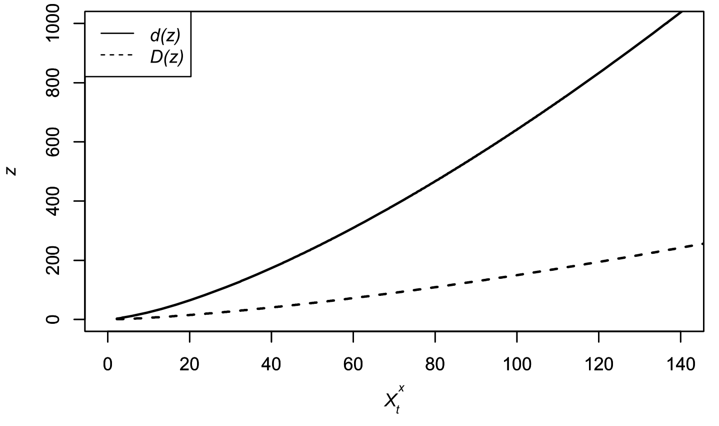

Figure 17: The downside solutions under $E_0 = 10$.

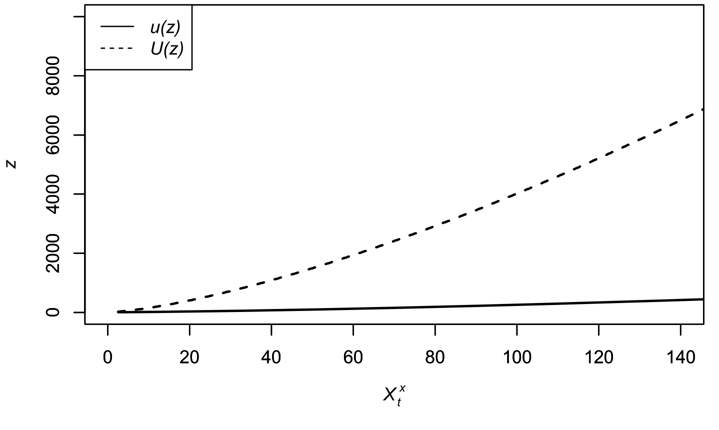

Figure 18: The upside solutions under the standard condition.

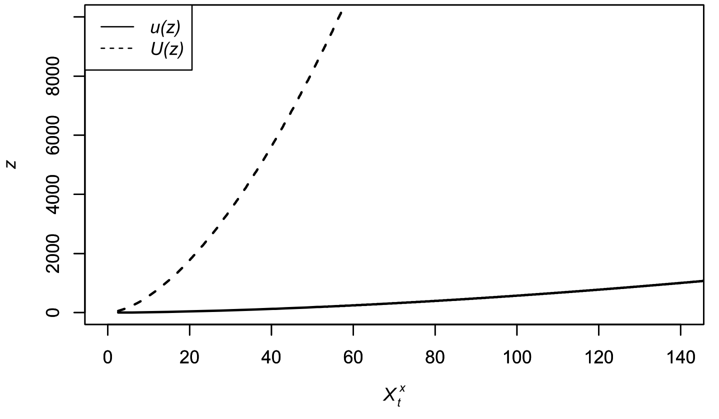

Figure 19: The upside solutions under $\gamma = 0.4$.

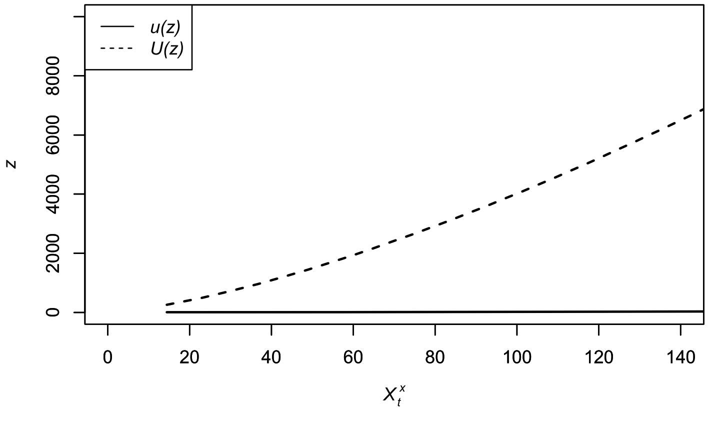

Figure 20: The upside solutions under $\sigma = 0.3$.

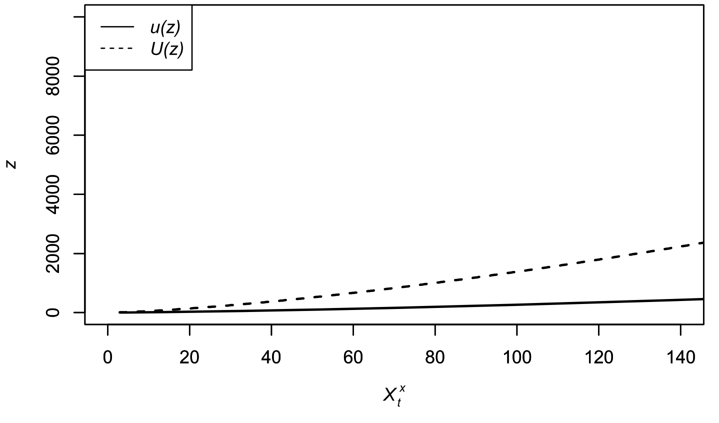

Figure 21: The upside solutions under $\rho = 0.2$.

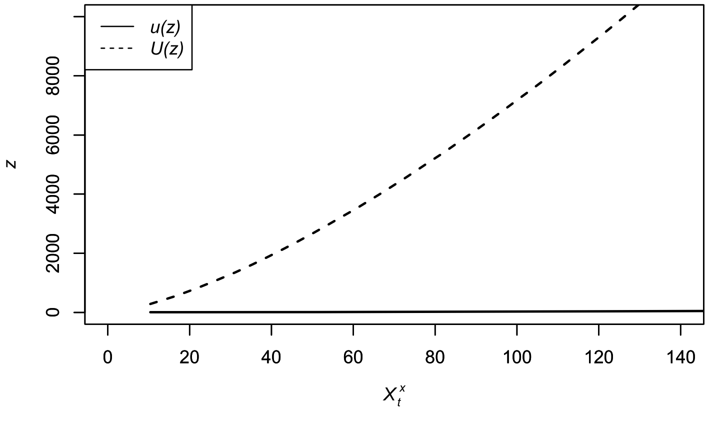

Figure 22: The upside solutions under $\mu = 0.04$.

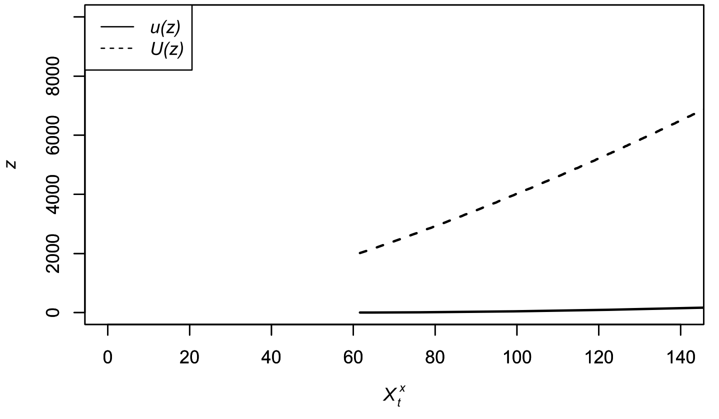

Figure 23: The downside solutions under $K_1 = 1000$.

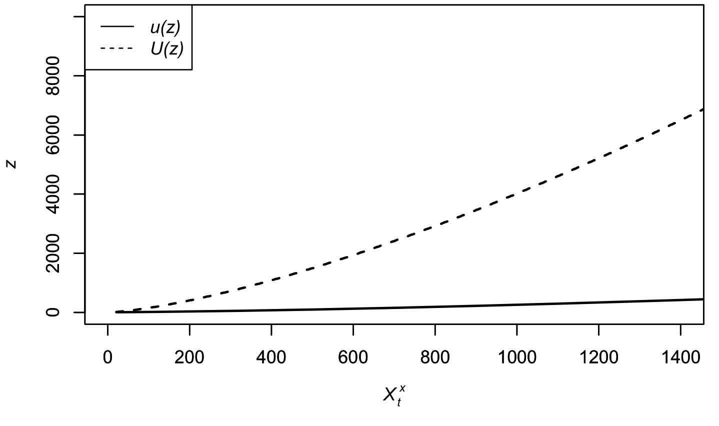

Figure 24: The downside solutions under $E_1 = 10$.