COMPOSITE ITERATIVE METHODS FOR A GENERAL SYSTEM OF VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

JONG SOO JUNG
DEPARTMENT OF MATHEMATICS, DONG-A UNIVERSITY

ABSTRACT. In this paper, we introduce two composite iterative methods (one implicit method and one explicit method) for finding a common element of the solution set of a general system of variational inequalities for continuous monotone mappings and the fixed point set of a continuous pseudocontractive mapping in a Hilbert space. First, this system of variational inequalities is proven to be equivalent to a fixed point problem of nonexpansive mapping. Then we establish strong convergence of the sequence generated by the proposed iterative methods to a common element of the solution set and the fixed point set, which is the unique solution of a certain variational inequality.

1. INTRODUCTION

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $C$ be a nonempty closed convex subset of $H$ and let $S : C \rightarrow C$ be a self-mapping on $C$. We denote by $Fix(S)$ the set of fixed points of $S$.

A mapping $F : C \rightarrow H$ is called monotone if

$$\langle x - y, Fx - Fy \rangle \geq 0, \quad \forall x, y \in C,$$

and $F$ is called $\alpha$-inverse-strongly monotone (see [5, 11]) if there exists a positive real number $\alpha$ such that

$$\langle x - y, Fx - Fy \rangle \geq \alpha \|F x - F y\|^2, \quad \forall x, y \in C.$$

The class of monotone mappings includes the class of $\alpha$-inverse-strongly monotone mappings.

A mapping $T : C \rightarrow H$ is said to be pseudocontractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C,$$

and $T$ is said to be $k$-strictly pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C,$$

where $I$ is the identity mapping. Note that the class of $k$-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, $T$ is nonexpansive (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$) if and only if $T$ is 0-strictly pseudocontractive. Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings as a subclass.

Let $F$ be a nonlinear mapping of $C$ into $H$. The variational inequality problem (VIP) is to find a $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

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The results presented in this lecture are collected mainly from the work [8] by the author of this report.
We denote the set of solutions of VIP(1.1) by \( VI(C, F) \). The variational inequality problem has been extensively studied in the literature; see [3, 5, 7, 10, 11, 14, 15, 17, 19] and the references therein.

In 2008, Ceng et al. [2] considered the following general system of variational inequalities:

\[
\begin{cases}
\langle \lambda F_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C \\
\langle \nu F_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C,
\end{cases}
\]

where \( F_1 \) and \( F_2 \) are an \( \alpha \)-inverse-strongly monotone mapping and a \( \beta \)-inverse-strongly monotone mapping, respectively; and \( \lambda \in (0, 2\alpha) \) and \( \nu \in (0, 2\beta) \) are two constants. For finding an element \( \text{Fix}(S) \cap \Gamma \), where \( S : C \rightarrow C \) is a nonexpansive mapping and \( \Gamma \) is the solution set of the problem (1.2), they introduced a relaxed extragradient method ([9]) and proved strong convergence to a common element of \( \text{Fix}(S) \cap \Gamma \).

In 2016, Alofi et al. [1] also considered the problem (1.2) coupled with the fixed point problem, and introduced two composite iterative algorithms (one implicit algorithm and one explicit algorithm) based on Jung’s composite iterative method [6] to find an element \( \text{Fix}(T) \cap \Gamma \), where \( T : C \rightarrow C \) is a \( k \)-strictly pseudocontractive mapping and \( \Gamma \) is the solution set of the problem (1.2), and showed strong convergence to a common element of \( \text{Fix}(T) \cap \Gamma \). The following problems arise:

Question 1. Can we extend the class of inverse-strongly monotone mappings in [1, 2] to the more general class of continuous monotone mappings?

Question 2. Can we extend the class of nonexpansive mappings in [2] or the class of strictly pseudocontractive mappings in [1] to the more general class of pseudocontractive mappings?

In this paper, in order to give the affirmative answers to the above two questions, we consider a general system of variational inequalities slightly different from the problem (1.2). More precisely, we introduce the following general system of variational inequalities (GSVI) for two continuous monotone mappings \( F_1 \) and \( F_2 \) of finding \((x^*, y^*) \in C \times C\) such that

\[
\begin{cases}
\langle \lambda F_1 x^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C \\
\langle \nu F_2 y^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C,
\end{cases}
\]

where \( \lambda > 0 \) and \( \nu \) are two constants. The solution set of GSVI(1.3) is denoted by \( \Omega \). First, we prove that the problem (1.3) is equivalent to a fixed point problem of nonexpansive mapping. Second, by using Jung’s composite iterative algorithms [6], we introduce a composite implicit iterative algorithm and a composite explicit iterative algorithm for finding a common element of \( \Omega \cap \text{Fix}(T) \), where \( T \) is a continuous pseudocontractive mapping. Then we establish strong convergence of these two composite iterative algorithms to a common element of \( \Omega \cap \text{Fix}(T) \), which is the unique solution of a certain variational inequality related to a minimization problem. As a direct consequence, we obtain strong convergence to a common element of \( VI(C, F) \cap \text{Fix}(T) \), where \( F \) is a continuous monotone mapping.

2. Preliminaries and Lemmas

Let \( H \) be a real Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). We write \( x_n \rightharpoonup x \) to indicate that the sequence \( \{x_n\} \) converges weakly to \( x \). \( x_n \rightarrow x \) implies that \( \{x_n\} \) converges strongly to \( x \).

For every point \( x \in H \), there exists a unique nearest point in \( C \), denoted by \( P_C(x) \), such that

\[ ||x - P_C(x)|| \leq ||x - y||, \quad \forall y \in C. \]
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$P_C$ is called the metric projection of $H$ onto $C$. It is well known that $P_C(x)$ is characterized by the property:

(2.1) $u = P_C(x) \iff (x - u, u - y) \geq 0, \forall x \in H, y \in C.$

In a Hilbert space $H$, we have

(2.2) $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle, \forall x, y \in H.$

We recall that

(i) an operator $A$ is said to be strongly positive on $H$ if there exists a constant $\overline{\gamma} > 0$ such that

$\langle Ax, x \rangle \geq \overline{\gamma}\|x\|^2, \forall x \in H;$

(ii) a mapping $V : C \to H$ is said to be $l$-Lipschitzian if there exists a constant $l \geq 0$ such that

$\|Vx - Vy\| \leq l\|x - y\|, \forall x, y \in C;$

(iii) a mapping $G : C \to H$ is said to be $\rho$-strongly monotone if there exists a constant $\rho > 0$ such that

$\langle Gx - Gy, x - y \rangle \geq \rho\|x - y\|^2, \forall x, y \in C.$

The following lemma is an immediate consequence of an inner product.

**Lemma 2.1.** In a real Hilbert space $H$, there holds the following inequality

$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \forall x, y \in H.$

We need the following lemmas for the proof of our main results.

**Lemma 2.2 ([16]).** Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying

$s_{n+1} \leq (1 - \omega_n)s_n + \omega_n\delta_n + \nu_n, \forall n \geq 1,$

where $\{\omega_n\}$, $\{\delta_n\}$, and $\{\nu_n\}$ satisfy the following conditions:

(i) $\{\omega_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty}\omega_n = \infty$ or, equivalently, $\prod_{n=1}^{\infty}(1 - \omega_n) = 0$;

(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ or $\sum_{n=1}^{\infty}\omega_n|\delta_n| < \infty$;

(iii) $\nu_n \geq 0 (n \geq 1)$, $\sum_{n=1}^{\infty}\nu_n < \infty$.

Then $\lim_{n\to\infty}s_n = 0.$

**Lemma 2.3 ([4]).** (Demiclosedness principle) Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, and let $S : C \to C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in $C$ such that $x_n \rightharpoonup x^*$ and $(I - S)x_n \to y$, then $(I - S)x^* = y.$

**Lemma 2.4 ([12]).** Let $H$ be a real Hilbert space. Let $A : H \to H$ be a strongly positive bounded linear operator with a constant $\overline{\gamma} > 1$. Then

$\langle (A - I)x - (A - I)y, x - y \rangle \geq (\overline{\gamma} - 1)\|x - y\|^2, \forall x, y \in C.$

That is, $A - I$ is strongly monotone with a constant $\overline{\gamma} - 1$.

**Lemma 2.5 ([12]).** Assume that $A$ is a strongly positive bounded linear operator on $H$ with a coefficient $\overline{\gamma} > 0$ and $0 < \zeta \leq \|A\|^{-1}$. Then $\|I - \zeta A\| \leq 1 - \zeta\overline{\gamma}.$

**Lemma 2.6 ([17]).** Let $H$ be a real Hilbert space. Let $G : H \to H$ be a $\rho$-Lipschitzian and $\eta$-strongly monotone mapping with constants $\rho, \eta > 0$. Let $0 < \mu < \frac{\overline{\gamma}^2}{\rho^2}$ and $0 < t < \sigma \leq 1$. Then $S := \sigma I - t\mu G : H \to H$ is a contractive mapping with constant $\sigma - t\tau$, where

$\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho)}.$

The following lemmas are Lemma 2.3 and Lemma 2.4 of Zegeye [18], respectively.
Lemma 2.7 ([18]). Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $F : C \rightarrow H$ be a continuous monotone mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that
\[
\langle y - z, Fz \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.
\]
For $r > 0$ and $x \in H$, define $F_r : H \rightarrow C$ by
\[
F_r x = \left\{ z \in C : \langle y - z, Fz \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}.
\]
Then the following hold:
(i) $F_r$ is single-valued;
(ii) $F_r$ is firmly nonexpansive, that is,
\[
\|F_r x - F_r y\|^2 \leq \langle x - y, F_r x - F_r y \rangle, \quad \forall x, y \in H;
\]
(iii) $\text{Fix}(F_r) = \text{VI}(C, F)$;
(iv) $\text{VI}(C, F)$ is a closed convex subset of $C$.

Lemma 2.8 ([18]). Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $T : C \rightarrow H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that
\[
\langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \quad \forall y \in C.
\]
For $r > 0$ and $x \in H$, define $T_r : H \rightarrow C$ by
\[
T_r x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \quad \forall y \in C \right\}.
\]
Then the following hold:
(i) $T_r$ is single-valued;
(ii) $T_r$ is firmly nonexpansive, that is,
\[
\|T_r x - T_r y\|^2 \leq \langle x - y, T_r x - T_r y \rangle, \quad \forall x, y \in H;
\]
(iii) $\text{Fix}(T_r) = \text{Fix}(T)$;
(iv) $\text{Fix}(T)$ is a closed convex subset of $C$.

3. MAIN RESULTS

Throughout the rest of this paper, we always assume the following:
\begin{itemize}
    \item $H$ is a real Hilbert space;
    \item $C$ is a nonempty closed subspace subset of $H$;
    \item $A : C \rightarrow C$ is a strongly positive linear bounded self-adjoint operator with a constant $\overline{\gamma} \in (1, 2)$;
    \item $V : C \rightarrow C$ is $l$-Lipschitzian with constant $l \in [0, \infty)$;
    \item $G : C \rightarrow C$ is a $\rho$-Lipschitzian and $\eta$-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$;
    \item Constants $\mu$, $l$, $\tau$, and $\gamma$ satisfy $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$;
    \item $F_1$ and $F_2 : C \rightarrow H$ are continuous monotone mapping;
    \item $\Omega$ is the solution set of GSVI (1.3) for $F_1$ and $F_2$;
    \item $F_{1\lambda} : H \rightarrow C$ is a mapping defined by
    \[
    F_{1\lambda} x = \left\{ z \in C : \langle y - z, F_1 z \rangle + \frac{1}{\lambda} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}
    \]
    for $\lambda > 0$;
\end{itemize}
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- $F_{2\nu} : H \to C$ is a mapping defined by
  \[
  F_{2\nu}x = \left\{ z \in C : \langle y - z, F_2z \rangle + \frac{1}{\nu}(y - z, z - x) \geq 0, \quad \forall y \in C \right\}
  \]
  for $\nu > 0$;
- $R : H \to C$ is a mapping defined by $Rx = F_{1\lambda}F_{2\nu}x$ for each $x \in H$;
- $T : C \to C$ is a continuous pseudocontractive mapping such that $Fix(T) \neq \emptyset$;
- $T_{r_t} : H \to C$ is a mapping defined by
  \[
  T_{r_t}x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r_t}\langle y - z, (1+r_t)z - x \rangle \leq 0, \quad \forall y \in C \right\}
  \]
  for $r_t \in (0, \infty)$, $t \in (0,1)$, and $\lim \inf_{t \to 0} r_t > 0$;
- $T_{r_n} : C \to H$ is a mapping defined by
  \[
  T_{r_n}x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r_n}\langle y - z, (1+r_n)z - x \rangle \leq 0, \quad \forall y \in C \right\}
  \]
  for $r_n \in (0, \infty)$ and $\lim \inf_{n \to \infty} r_n > 0$;
- $\Omega \cap Fix(T) \neq \emptyset$.

By Lemma 2.7 and Lemma 2.8, we note that $F_{1\lambda}$, $F_{2\nu}$, $T_{r_t}$, and $T_{r_n}$ are nonexpansive, and $Fix(T_{r_t}) = Fix(T) = Fix(T_{r_n})$.

First, we prove that the problem (1.3) is equivalent to a fixed point problem of nonexpansive mapping.

**Proposition 3.1.** Let $C$ be a closed convex subset of a real Hilbert space $H$. For given $x^* \in C$, $(x^*, y^*)$ is a solution of GSVI (1.3) for continuous monotone mappings $F_1$ and $F_2$ if and only if $x^*$ is a fixed point of the mapping $R : H \to C$ defined by
\[
Rx = F_{1\lambda}F_{2\nu}x, \quad \forall x \in H,
\]
where $y^* = F_{2\nu}x^*$.

First, we introduce the following composite algorithm that generates a net $\{x_t\}$ in an implicit way:
\[
(3.1) \quad x_t = (I - \theta_t A)T_{r_t}Rx_t + \theta_t[t\gamma Vx_t + (I - t\mu G)T_{r_t}Rx_t],
\]
where $t \in (0, \min\{1, \frac{2-\gamma}{\tau-\gamma}\})$ and $\theta_t \in (0, \|A\|^{-1})$.

We summarize the basic properties of $\{x_t\}$, which can be proved by the same method as in [6].

**Proposition 3.2.** Let $\{x_t\}$ be defined via (3.1). Then

(i) $\{x_t\}$ is bounded for $t \in (0, \min\{1, \frac{2-\gamma}{\tau-\gamma}\})$;
(ii) $\lim_{t \to 0} \|x_t - T_{r_t}Rx_t\| = 0$ provided $\lim_{t \to 0} \theta_t = 0$;
(iii) $\lim_{t \to 0} \|x_t - y_t\| = 0$, where $y_t = t\gamma Vx_t + (I - t\mu G)T_{r_t}Rx_t$;
(iv) $\lim_{t \to 0} \|x_t - Rx_t\| = 0$;
(v) $x_t$ defines a continuous path from $(0, \min\{1, \frac{2-\gamma}{\tau-\gamma}\})$ into $H$ provided $\theta_t : (0, \min\{1, \frac{2-\gamma}{\tau-\gamma}\}) \to (0, \|A\|^{-1})$ is continuous, and $r_t : (0, \min\{1, \frac{2-\gamma}{\tau-\gamma}\}) \to (0, \infty)$ is continuous.

We obtain the following theorem for strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the variational inequality (3.2) below.
Theorem 3.3. Let the net \( \{x_t\} \) be defined via (3.1). If \( \lim_{t \to 0} \theta_t = 0 \), then \( x_t \) converges strongly to \( \bar{x} \) in \( \Omega \cap Fix(T) \) as \( t \to 0 \), which solves the variational inequality

\[(A - I)\bar{x}, \bar{x} - p \leq 0, \quad \forall p \in \Omega \cap Fix(T),\]

Equivalently, we have

\[P_{\Omega \cap Fix(T)}(2I - A)\bar{x} = \bar{x}.\]

Now, we propose the following composite algorithm which generates a sequence in an explicit way:

\[
\begin{aligned}
&\ y_n = \alpha_n \gamma Vx_n + (I - \alpha_n \mu G)T_{r_n}Rx_n, \\
&\ x_{n+1} = (I - \beta_n A)T_{r_n}Rx_n + \beta_n y_n, \quad \forall n \geq 0,
\end{aligned}
\]

where \( \{\alpha_n\} \subset [0,1] \); \( \{\beta_n\} \subset (0,1) \); \( \{r_n\} \subset (0, \infty) \); and \( x_0 \in C \) is an arbitrary initial guess, and establish strong convergence of this sequence to \( \bar{x} \in \Omega \cap Fix(T) \), which is the unique solution of the variational inequality (3.2).

Theorem 3.4. Let \( \{x_n\} \) be the sequence generated by the explicit algorithm (3.3). Let \( \{\alpha_n\}, \{\beta_n\}, \) and \( \{r_n\} \) satisfy the following conditions:

(C1) \( \{\alpha_n\} \subset [0,1] \) and \( \{\beta_n\} \subset (0,1) \), \( \alpha_n \to 0 \) and \( \beta_n \to 0 \) as \( n \to \infty \);

(C2) \( \sum_{n=0}^{\infty} \beta_n = \infty \);

(C3) \( \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \), and \( |\beta_{n+1} - \beta_n| \leq o(\beta_{n+1}) + \sigma_n, \sum_{n=0}^{\infty} \sigma_n < \infty \) (the perturbed control condition);

(C4) \( \{r_n\} \subset (0, \infty), \liminf_{n \to \infty} r_n > 0 \), and \( \sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty \).

Then \( \{x_n\} \) converges strongly to \( \bar{x} \in \Omega \cap Fix(T) \), which is the unique solution of the variational inequality (3.2).

Taking \( G \equiv I, \mu = 1 \), and \( \gamma = 1 \) in Theorem 3.5, we obtain the following corollary.

Corollary 3.5. Let \( \{x_n\} \) be generated by the following iterative algorithm:

\[
\begin{aligned}
&\ y_n = \alpha_n Vx_n + (1 - \alpha_n)T_{r_n}Rx_n, \\
&\ x_{n+1} = (I - \beta_n A)T_{r_n}Rx_n + \beta_n y_n, \quad \forall n \geq 0.
\end{aligned}
\]

Assume that the sequences \( \{\alpha_n\}, \{\beta_n\}, \) and \( \{r_n\} \) satisfy the conditions (C1) – (C4) in Theorem 3.5. Then \( \{x_n\} \) converges strongly to \( \bar{x} \in \Omega \cap Fix(T) \), which is the unique solution of the variational inequality (3.2).

Remark 3.6. 1) The \( \bar{x} \in \Omega \cap Fix(T) \) in our results is the unique solution of minimization problem

\[(3.4) \quad \min_{x \in D} \frac{1}{2} \langle (A - I)x, x \rangle,\]

where the constraint set \( D \) is \( \Omega \cap Fix(T) \). In fact, the variational inequality (3.2) is the optimality condition for the minimization problem (3.4). Thus, for finding an element of \( \Omega \cap Fix(T) \), where \( T \) is a continuous pseudocontractive mapping, and \( F_1 \) and \( F_2 \) are continuous monotone mappings, Theorem 3.4, Theorem 3.5 and Corollary 3.6 are new ones different from previous those introduced by some authors (for example, see [1, 2]).

2) Taking \( F_1 = F_2 = F, \lambda = \nu \) and \( x^* = y^* \) in GSVI(1.3) and replacing \( F_{\lambda} \) by \( F_{r_n} \) along with the condition (C4) on \( \{r_n\} \), we can obtain a new result, which improves, supplements and develops the corresponding results of [3, 5, 7, 14, 15, 19].
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DEPARTMENT OF MATHEMATICS, DONG-A UNIVERSITY, BUSAN 49315, KOREA
E-mail address: jungjs@dau.ac.kr