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Applications of Convex-valued KKM maps

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Abstract

This is to introduce the contents of two articles of Granas and Lassonde [2, 3] on applications of some elementary principles of convex analysis. In the first article, they presented a geometric approach in the theory of minimax inequalities, which has numerous applications in different areas of mathematics. In the second article, they complement and elucidate the preceding approach within the context of complete metric spaces. In this paper, we give abstract convex space versions of the basic results of [2, 3], and, as the supplements of overviews on recently developed KKM theory in [5, 8], we introduce applications appeared in [2, 3]. Consequently, many of known results in the traditional convex analysis can be deduced from the KKM theory.

1. Introduction

This survey article concerns with applications of convex-valued KKM maps as shown in the works of Granas and Lassonde [2, 3]. We obtain abstract versions of basic results in [2, 3] and introduce applications of some elementary principles of convex analysis given there. Consequently, many of known results in the traditional convex analysis now belong to the KKM theory.

In 1991, Granas and Lassonde [2] presented a new geometric approach in the theory of minimax inequalities, which has numerous applications in different areas of mathematics. Actually they are based on a particular form of the well-known KKM lemma due to Ky Fan in 1961. Their proof of the form is very simple and depends only on the geometric structure induced by convexity. Many applications to known results are systemically given on systems of inequalities, variational inequalities, minimax equalities, theorems

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of Markhoff-Kakutani, Mazur-Orlicz and Hahn-Banach, variational problems, maximal monotone operators, and others in convex analysis.

Moreover, in 1995, Granas and Lassonde [3] complement and elucidate the preceding approach within the context of complete metric spaces. Their aim of [3] is to provide simple proofs of several known results, stated in super-reflexive Banach spaces, concerning minimization of quasi-convex functions, variational inequalities, game theory, systems of inequalities, and maximal monotone operators, by using their intersection principle which is elementary.

Recently, the present author has tried to give overviews on currently developing KKM theory of abstract convex spaces in [5, 8]. Moreover, we studied the contributions of Granas to the KKM theory in [7], where we introduced the contents of most of works of Granas and his coworkers on the KKM theory and gave some comments to compare them with current results in the theory. Motivated by such works, we found that, as their supplements, the contents of [2, 3] seem to be essential and worth to be examined.

This article is organized as follows: Section 2 is a brief introduction on some basic facts on our abstract convex space theory. In Section 3, basic results of [2] are extended to our abstract convex spaces. Section 4 devotes to introduce the applications of the geometric principle in [2]. In Section 5, basic results of [3] are compared with corresponding results in our abstract convex space theory. Section 6 devotes to introduce the applications of elementary general principles in [3]. Finally, in Sections 7 and 8, we add-up the contents of later works of Horvath [4] and Ben-El-Mechaiekh [1], respectively, which are closely related to [2, 3].

2. Abstract convex spaces

Let $\langle D \rangle$ denote the collection of all nonempty finite subsets of a set D. Recall the following; see [5, 6] and the references therein.

Definition. Let E be a topological space, D a nonempty set, and $\Gamma : \langle D \rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$; that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$*-map* [resp., a $\mathfrak{KO}$*-map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp. A (partial) KKM space $(E, D; \Gamma)$ is said to be *compact* whenever E is compact.

Now we have the following well-known diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y \in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y \in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y \in D} \overline{G(y)} = \overline{\bigcap_{y \in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y \in D} \overline{G(y)} = \bigcap_{y \in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

The following is one of the most general KKM type theorems in [6]:

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $G : D \multimap Z$ a map such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*

(2) *there exists a nonempty compact subset K of Z such that either*

(i) $K = Z$;

(ii) $\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ *for some* $M \in \langle D\rangle$; *or*

(iii) *for each* $N \in \langle D\rangle$, *there exists a* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$, $\overline{F(L_N)}$ *is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if* G *is transfer closed-valued, then* $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$; *and*

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.

Our KKM theory concerns with the study of partial KKM spaces and their applications.

3. Abstractions of basic results in [2]

Recall that each of two articles [2] and [3] consist of basic results and their applications. In this section, we present some abstract space versions of basic results mainly given in [2].

The paper [2] concerns with many known applications of the convex-valued KKM maps. The main result (called the geometric principle) is as follows: Let E be a t.v.s., let $\emptyset \neq D \subset E$, and let $G : D \multimap E$ be a multimap satisfying (1) $G(x)$ is a closed convex set for all $x \in D$, and (2) the convex hull of A is contained in $\bigcup\{G(x) \mid x \in A\}$ for all finite subsets A of D (that is, G is a KKM map); then the family $\{G(x) \mid x \in D\}$ has the finite intersection property.

Note that the proof of Ky Fan's 1961 KKM lemma contains the geometric principle without assuming convexity of $G(x)$ for all $x \in D$ based on the original KKM theorem in 1929. Since the authors' $(E, D; \mathrm{co})$ is a particular partial KKM space, it satisfies a large number of equivalent results in [5]. Some of them appear in [3] in particular forms.

In fact, the geometric principle in [3] is a particular form of the following in [5]:

The KKM principle. *For an abstract convex space* $(E, D; \Gamma)$, *any closed-valued [resp., open-valued] KKM map* $G : D \multimap E$, *the family* $\{G(z)\}_{z \in D}$ *has the finite intersection property.*

The closed-valued case is called the *partial KKM principle*; and any abstract convex space satisfying the (partial) KKM principle is called a *(partial) KKM space*, resp.

For an abstract convex space $(E, D; \Gamma)$, at first we did not assumed any topology on E. Under such circumstance, we define

Definition. For $(E \supset D; \Gamma)$, a map $G : D \multimap E$ is said to be *strongly KKM* provided (i) $x \in G(x)$ for each $x \in D$, and (ii) the cofibers $G^*(y) := D \setminus G^-(y), y \in E$, of G are convex.

Proposition 3.1. *In $(E \supset D; \Gamma)$, if D is Γ-convex and $G : D \multimap E$ is strongly KKM, then G is a KKM map.*

Proof. Let $A \in \langle D \rangle$ and $y_0 \in \Gamma_A$. We have to show that $y_0 \in G(A)$. Since $y_0 \in G(y_0)$, we see that $y_0 \notin G^*(y_0)$ and therefore Γ_A is not contained in $G^*(y_0)$. Since the set $G^*(y_0)$ is Γ-convex, at least one point $x \in A$ does not belong to $G^*(y_0)$, which means that $y_0 \in G(x)$. $\square$

When E is a vector space, Proposition 3.1 reduces to [2, Proposition 4.2], where examples of three strongly KKM maps and one KKM map were given.

Corollary 3.2. *Let $(X; \Gamma)$ be a compact partial KKM space and $F, G : X \multimap X$ two multimap satisfying*

(i) *$F(x) \subset G(x)$ for each $x \in X$,*
(ii) *G has closed values,*
(iii) *F has Γ-convex cofibers.*

If $x \in F(x)$ for each $x \in X$, then $\bigcap\{G(x) \mid x \in X\} \neq \emptyset$.

Proof. Since $x \in F(x)$ for each $x \in X$, by Proposition 3.1, F is a KKM map and so is G by (i). Since $(X; \Gamma)$ is a compact partial KKM space, the conclusion follows immediately. $\square$

Note that Corollary 3.2 reduces to [3, Corollaire 1.1] when X is a nonempty compact convex subset of a t.v.s.

The following is a particular form of [5, (XXIV)]:

Theorem 3.3. (The Fan type analytic alternative) *Let $(X; \Gamma)$ be a compact partial KKM space and $f, g : X \times X \to \mathbb{R}$ be real functions such that*

(i) *$g(x, y) \leq f(x, y)$ for each $x, y \in X$,*
(ii) *g is l.s.c. on each y [that is, $\{x \in X \mid g(x, y) \leq 0\}$ is closed in X].*
(iii) *f is quasi-concave on each x [that is, $\{y \in X \mid g(x, y) > 0\}$ is Γ-convex in X].*

Then either

(a) *there exists $y_0 \in X$ such that $g(x, y_0) \leq 0$ for all $x \in X$; or*

(b) *there exists* $x_0 \in X$ *such that* $f(x_0, x_0) > 0$.

Proof. Consider multimaps $F : x \mapsto \{y \in X \mid f(x, y) \leq 0\}$ and $G : x \mapsto \{y \in X \mid g(x, y) \leq 0\}$ from X to X. From the hypotheses, all the conditions of Corollary 3.2 are satisfied. Hence, if $x \in F(x)$ for every $x \in X$, there exists $y_0 \in X$ such that $y_0 \in G(x)$ for all $x \in C$, and hence (a) holds; otherwise, if there exists $x_0 \in X$ such that $x_0 \notin F(x_0)$, then the case (b) holds. □.

Note that Theorem 3.3 reduces to [2, Theorem 2] when X is a nonempty compact convex subset of a t.v.s. Moreover, three results given in this section are "mutually equivalent":

In the above proofs, we have seen:

The partial KKM principle $\Longrightarrow$ *Corollary 3.2.* and

Corollary 3.2 $\Longrightarrow$ *Theorem 3.3.*

Moreover, we can show

Theorem 3.3 $\Longrightarrow$ *Corollary 3.2.* We take f and g are functions indicating the graphs of F and G, resp.

Corollary 3.2 $\Longrightarrow$ *The partial KKM principle for the case* $D \subset E$. If $G : D \multimap E$ is a KKM map and A is a finite subset of D, by letting $G(x) := E$ when $x \in \Gamma_A \setminus D$, we can construct an $F : \Gamma_A \multimap \Gamma_A$ having convex cofibers and satisfying $x \in F(x) \subset G(x)$ for every $x \in \Gamma_A$.

In our previous work [5], we gave a large number of equivalent formulations of the (partial) KKM principle. Recall that [5] contains incorrect statements such as (V), (VI), Theorem 4, (XVI), and (XVII). These can be easily corrected.

In [2], it is noted that the geometric principle is given by Valentine (1964) and Asakawa (1986), and the geometric lemma in [2](which is a basis of the geometric principle) is a reformulation of a lemma of Klee (1951).

Abstract forms of many equivalent forms of these statements also can be also established as in our work [5].

4. Applications of the geometric principle

In [2], it is shown that many known results can be proved by the aid of the geometric principle. Therefore, those results are contained in the realm of the KKM theory. Actually they listed as follows:

4.1. Systems of inequalities

Theorem 3. [M. Neumann 1977]

Corollary 3.1. [Fan-Glicksberg-Hoffman 1957]

Theorem 4. (Generalization of [Fan 1957])

Corollary 4.1. [Bohnenblust-Karlin-Shapley 1950, Fan 1957]

4.2. Minimax equalities

Theorem 5. [König 1968]

Theorem 6. (Reformulation)

Corollary 6.1. [Kneser 1952, Fan 1953]

It is also noted that minimax theorems of [Nikaido 1954] and [Sion 1958] can be equally obtained by the geometric principle.

4.3. Markov-Kakutani Theorem

In this subsection, E stands for a t.v.s. having sufficiently many continuous linear functionals.

Theorem 7. (A fixed point theorem)

Theorem 8. [Markov-Kakutani]

4.4. Theorems of Mazur-Orlicz and Hahn-Banach

Theorem 9. [Mazur-Orlicz]

Corollary 9.1.

Theorem 10. [Hahn-Banach]

4.5. Variational Problems

Theorem 11. [Mazur-Schauder]

Theorem 12. [Stampacchia]

Theorem 13. [Hartmann-Stampacchia]

Corollary 13.1. [Browder-Minty]

Corollary 13.2. [Browder-Goehde-Kirk] Hilbert space case.

4.6. Maximal Monotone Multivalued Operators

Theorem 14. (Particular case of [Debrunner-Flor 1964])

Corollary 14.1. [Minty]

Corollary 14.2. [Minty]

5. Abstractions of basic results in [3]

The aim of [3] is to provide simple proofs of several results, stated in super-reflexive Banach spaces, concerning minimization of quasi-convex functions, variational inequalities, game theory, systems of inequalities, and maximal monotone operators, by using the following "intersection principle": Let $(E, ||\cdot||)$ be super-reflexive and let $\{C_i \mid i \in I\}$ be a family of closed convex sets in E with the finite intersection property. If C_{i_0} is bounded for some $i_0 \in I$, then the intersection $\bigcap\{C_i \mid i \in I\}$ is not empty. As the authors point out, most of the results are valid for arbitrary reflexive spaces.

From [3], recall that a Banach space $(E, ||\cdot||)$ is *uniformly convex* provided its norm $||\cdot||$ has the following property: If (x_n), (y_n) are sequences in E such that the three sequences $||x_n||$, $||y_n||$, and $\frac{1}{2}||x_n + y_n||$ converge to 1, then $||x_n - y_n|| \to 0$. Recall that any Hilbert space is uniformly convex.

A Banach space $(E, ||\cdot||)$ is called *super-reflexive* provided it admits an equivalent uniformly convex norm. In weak topology, closed convex bounded subsets of a super-reflexive Banach spaces are compact.

Lemma 5.1. [3] *Let $(E, ||\cdot||)$ be super-reflexive and (C_n) be a decreasing sequence of nonempty closed convex subsets of E. Suppose that $d = \sup_n d(0, C_n)$ is finite. Then there exists a unique point $\bar{x} \in \bigcap_n C_n$ and $||\bar{x}|| = d$.*

Theorem 5.2. (Intersection Principle [3]) *Let $(E, ||\cdot||)$ be super-reflexive and $\{C_i \mid i \in I\}$ be a family of closed convex sets in E with the finite intersection property. If C_{i_0} is bounded for some $i_0 \in I$, then the intersection $\bigcap\{C_i \mid i \in I\}$ is not empty.*

In [3], this has an elegant proof using Lemma 5.1, but, by switching to the weak topology, this is clear since C_{i_0} becomes compact.

Lemma 5.3. *Let E be a t.v.s., X be a nonempty subset of E, and $G : X \multimap E$ be a closed valued KKM map. Then the $\{G(x)\}_{x\in X}$ has the finite intersection property.*

This simply tells that $(E, X; \mathrm{co})$ is a partial KKM space and was first proved by Ky Fan in the proof of his 1961 KKM lemma based on the original KKM theorem in 1929.

In [3, Theorem 5.1], Lemma 5.3 was proved for a super-reflexive Banach space E and for a KKM map with closed convex values in a elegant method.

From Theorem C in Section 2, we have the following immediately by considering the weak topology on E:

Theorem 5.4. (Elementary Principle of KKM maps [3]) *Let E be super-reflexive, X be a nonempty subset of E and $G : X \multimap E$ be a KKM map with convex closed values. Assume, furthermore, that one of the following conditions is satisfied:*

(i) *X is bounded,*
(ii) *all $G(x)$ are bounded,*
(iii) *$G(x_0)$ is bounded for some $x_0 \in X$.*

Then the intersection $\bigcap\{G(x) \mid x \in X\}$ is not empty.

6. Applications of elementary principles

In [3], the elementary principle is applied to obtain the following many known results:

6.1. Minimization of quasiconvex functions

Theorem 3.1. Existence of minimum.

Theorem 3.2. Specialized to quadratic forms in Hilbert spaces.

Corollary 3.3. (F. Riesz representation theorem)

Corollary 3.4. (Projection on closed convex sets)

Corollary 3.5. (Separation of closed convex sets)

6.2. Variational inequalities

Theorem 6.1. [Stampacchia]

Corollary 6.2. [Lax-Milgram] A generalization of Riesz representation theorem.

Theorem 6.3. [Hartman-Stampacchia] A generalization of theorems of Stampacchia and Lax-Milgram.

Corollary 6.4. [Minty-Browder]

Corollary 6.5. [Browder-Goehde-Kirk] For nonexpansive maps on Hilbert space.

6.3. Minimax theorem of von Neumann

Theorem 7.1. [von Neumann] This is for two super-reflexive spaces. For partial KKM spaces, a general form is given in [5, (XXVI)].

6.4. Systems of inequalities

Theorem 8.1. Existence of common solutions of a system of inequalities on a super-reflexive space.

Theorem 8.2. A variant for Ky Fan type family of functions.

Theorem 8.3. A minimax theorem which is a consequence of Theorem 8.2.

Corollary 8.4. [Kneser-Fan] A minimax equality.

6.5. Maximal monotone operators

Theorem 9.1. A basic result in the theory of maximal monotone operators in a Hilbert space H.

Corollary 9.2. [Minty] Surjectivity of a maximal monotone operator with bounded domain.

Corollary 9.3. [Minty] If $T : H \to H$ is maximal monotone, then $1_H + T$ is onto.

7. Add-up of Horvath [4] in 2014

In 2014, Horvath [4] published a related article to [2] as follows:

Abstract : If one adds one extra assumption to the classical Knaster-Kuratowski-Mazurkiewicz (KKM) theorem, namely that the sets F_i are convex, one gets the Elementary KKM theorem; the name is due to A. Granas and M. Lassonde [2] who gave a simple proof of the Elementary KKM theorem and showed that despite being elementary, it is powerful and versatile. It is shown here that this Elementary KKM theorem is equivalent to Klee's theorem, the Elementary Alexandroff. Pasynkov theorem, the Elementary Ky Fan theorem and the Sion-von Neumann minimax theorem, as well as a few other classical results with an extra convexity assumption; hence the adjective elementary. The Sion.von Neumann minimax theorem itself can be proved by simple topological arguments using connectedness instead of convexity. This work answers a question of Professor Granas regarding the logical relationship between the Elementary KKM theorem and the Sion-von Neumann minimax theorem.

8. Add-up of Ben-El-Mechaiekh [1] in 2015

In 2015, Ben-El-Mechaiekh [1] published a related article to [2] as follows:

Abstract : A number of landmark existence theorems of nonlinear functional analysis follow in a simple and direct way from the basic separation of convex closed sets in finite dimension via elementary versions of the Knaster-Kuratowski-Mazurkiewicz principle - which we extend to arbitrary topological vector spaces - and a coincidence property for so-called von Neumann relations. The method avoids the use of deeper results of topological essence such as the Brouwer fixed point theorem or the Sperner's lemma and underlines the crucial role played by convexity. It turns out that the convex KKM principle is equivalent to the Hahn-Banach theorem, the Markov-Kakutani fixed point theorem, and the Sion-von Neumann minimax principle.

Comments: Note that the convex KKM principle is the geometric principle in [2]. This principle can be generalized to abstract convex spaces as follows:

Consider the case $E = Z$ and $F = \mathrm{id}_E$ in Theorem C. Then we have the following conclusion:

(α) *if* G *is* Γ*-convex transfer closed-valued, then* $K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$*; and*

(β) *if* G *is* Γ*-convex intersectionally closed-valued, then* $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.

Then the conclusion generalizes the geometric principle of Granas and Lassonde [2] and the convex KKM theorem (Theorem 6) of Ben-El-Mechaiekh [1].

Final Remark: Recall that some people complained against our use of triples for abstract convex spaces. Note that here also appears a large number of triples $(E, D; \Gamma)$.

References

[1] H. Ben-El-Mechaiekh, *Intersection theorems for closed convex sets and applications*, Missouri J. Math. Sc. **27**(1) (2015), 44–63. arXiv:1501.05813v1 [math.FA] 23Jan2015.

[2] A. Granas and M. Lassonde, *Sur un principe géométrique en analyse convexe*, (French. English summary) [*On a geometric principle in convex analysis*], Studia Math. **101**(1) (1991), 1–18.

[3] A. Granas and M. Lassonde, *Some elementary general principles of convex analysis. Contributions dedicated to Ky Fan on the occasion of his 80th birthday*, Topol. Methods Nonlinear Anal. **5**(1) (1995), 23–37.

[4] C. Horvath, *Some of Sion's heirs and relatives*, J. Fixed Point Theory Appl. **16** (2014), 385–409. DOI 10.1007/s11784-015-0225-4

[5] S. Park, *The KKM principle in abstract convex spaces: Equivalent formulations and applications*, Nonlinear Anal. **73** (2010), 1028–1042.

[6] S. Park, *A genesis of general KKM theorems for abstract convex spaces: Revisited*, J. Nonlinear Anal. Optim. **4**(1) (2013), 127–132.

[7] S. Park, *Contributions of A. Granas to the KKM theory*, Nonlinear Analysis Forum **23**(1) (2018) 20–35.

[8] S. Park, *A panoramic view of the KKM theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **57**(2) (2018) 1–46.