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Equilibrium problems on geodesic spaces and approximation to their solutions
測地距離空間上の均衡問題と解近似

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Abstract

In this paper, we consider an equilibrium problem for a bifunction defined on a Hadamard space. Using the notion of resolvent for this bifunction, we generate an iterative sequence of Halpern type and prove its convergence to a solution of the problem.

1 Introduction

Let K be a metric space and F a bifunction of $K \times K$ to $\mathbb{R}$. An equilibrium problem for F is defined as to find $z \in X$ such that $F(z, y) \geq 0$ for all $y \in K$. This problem has been widely investigated by a large number of researchers in the case that K is a nonempty closed convex subset of a Banach space. In this theory, it is known that the notion of resolvents for F plays an important role.

Recently, Kimura and Kishi [7] proposed the notion of resolvent for F in the setting of Hadamard spaces. They also study several basic properties of this operator.

In this paper, we apply the resolvent for F to generate an iterative sequence approximating a solution to the equilibrium problem for F defined on a Hadamard space. We adopt the Halpern type scheme for an approximating sequence and prove its convergence.

2 Preliminaries

Let X be a metric space with a metric d. For $x, y \in X$, a mapping $c : [0, l] \to X$ is called a geodesic with endpoints x, y if $c(0) = x$, $c(l) = y$, and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, l]$, where $l = d(x, y)$. We say that X is a geodesic space if there exists a geodesic for any two endpoints in X. In what follows, we assume that a geodesic is uniquely determined for every $x, y \in X$. The image of the geodesic with endpoints $x, y \in X$ is denoted by $[x, y]$. A subset C of X is said to be convex if $[x, y] \subset C$ for any $x, y \in C$. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = (1-t)d(x, y)$ and $d(z, y) = td(x, y)$. We denote it by $tx \oplus (1-t)y$. Therefore, the following hold by definition.

$$\begin{aligned} d(x, tx \oplus (1-t)y) &= (1-t)d(x, y), \\ d(tx \oplus (1-t)y, y) &= td(x, y). \end{aligned}$$

A Hadamard space X can be characterized by a complete geodesic space satisfying the following inequality for every $x, y, z \in X$ and every $t \in [0, 1]$:

$$d(z, tx \oplus (1-t)y)^2 \leq td(z, x)^2 + (1-t)d(z, y)^2 - t(1-t)d(x, y)^2.$$

For the exact definition of a Hadamard space and more details, see [1, 3] for instance.

A mapping $T : X \to X$ is said to be firmly nonspreading if

$$2d(Tx, Ty)^2 \leq d(x, Ty)^2 + d(Tx, y)^2 - d(x, Tx)^2 - d(y, Ty)^2$$

for every $x, y \in X$. We know that the inequality

$$d(x, u)^2 + d(v, y)^2 - d(x, v)^2 - d(y, u)^2 \leq 2d(x, y)d(v, u)$$

holds for any $x, y, u, v \in X$ in a Hadamard space X. Therefore, if T is firmly nonspreading, then we have

$$\begin{aligned} 2d(Tx, Ty)^2 &\leq d(x, Ty)^2 + d(Tx, y)^2 - d(x, Tx)^2 - d(y, Ty)^2 \\ &\leq 2d(x, y)d(Tx, Ty) \end{aligned}$$

and thus T is nonexpansive; $d(Tx, Ty) \leq d(x, y)$ for any $x, y \in X$. We say that T is nonspreading if

$$2d(Tx, Ty)^2 \leq d(x, Ty)^2 + d(Tx, y)^2$$

for every $x, y \in X$. We denote by $\operatorname{Fix} T$ the set of all fixed points of T. A mapping T is said to be quasinonexpansive if $\operatorname{Fix} T \neq \emptyset$ and

$$d(Tx, z) \leq d(x, z)$$

for every $x \in X$ and $z \in \operatorname{Fix} T$. It is obvious that every firmly nonspreading mapping is nonspreading, and every nonspreading mapping and every nonexpansive mapping is quasinonexpansive if $\operatorname{Fix} T \neq \emptyset$.

Let C be a nonempty closed convex subset of a Hadamard space X. Then for each $x \in X$, there exists a unique point $y_x \in C$ such that $d(x, y_x) = \inf_{y \in C} d(x, y)$. Using this fact, we define a mapping $P_C : X \to C$ by $P_C x = y_x$ for every $x \in X$. This mapping P_C is called a metric projection onto C. We know that P_C is firmly nonspreading and therefore it is nonexpansive and nonspreading.

For a bounded sequence $\{x_n\}$ in X, we say that $\{x_n\}$ is Δ-convergent to $x_0 \in X$ if

$$\limsup_{i\to\infty} d(x_{n_i}, x_0) = \inf_{y\in X} \limsup_{i\to\infty} d(x_{n_i}, y)$$

for every subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

It is proved in [6] that if $\{x_n\}$ is Δ-convergent to x_0, it follows that $d(u, x_0) \leq \liminf_{n\to\infty} d(u, x_n)$ for any $u \in X$. We also know that every bounded sequence $\{x_n\}$ in a Hadamard space has a Δ-convergent subsequence; see [5, 9].

A mapping $T : X \to X$ is said to be Δ-demiclosed if $x_0 \in \operatorname{Fix} T$ whenever a sequence $\{x_n\}$ in X is Δ-convergent to $x_0 \in X$ and $\{d(x_n, Tx_n)\}$ converges to 0.

For a subset C of X, we denote by $\operatorname{clco} C$ a closed convex hull of C, that is, $\operatorname{clco} C$ is the intersection of all closed convex subsets of X including C. A Hadamard space X has the convex hull finite property [12] if for every finite subset E of X, every continuous selfmapping on $\operatorname{clco} E$ has a fixed point.

In the end of this section, we show the following lemma proved in [11]; see also [8, 10].

Lemma 1 (Saejung and Yotkaew [11]). *Let $\{s_n\}$, $\{t_n\}$, and $\{\alpha_n\}$ be real sequences such that $s_n \geq 0$ and $0 < \alpha_n \leq 1$ for all $n \in \mathbb{N}$. Suppose that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and that $\limsup_{i\to\infty} t_{n_i} \leq 0$ whenever $\{n_i\}$ is a subsequence of $\mathbb{N}$ satisfying $\liminf_{i\to\infty}(s_{n_i+1} - s_{n_i}) \leq 0$. If*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n$$

for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} s_n = 0$.

3 Equilibrium problems and a convergence theorem

Let X be a Hadamard space and K a nonempty closed convex subset of X. Suppose that a bifunction $F : K \times K \to \mathbb{R}$ satisfies the following conditions:

(E1) $F(x, x) = 0$ for all $x \in K$;
(E2) $F(x, y) + F(y, x) \leq 0$ for all $x, y \in K$;
(E3) for every $x \in K$, $F(x, \cdot) : K \to \mathbb{R}$ is lower semicontinuous and convex;
(E4) for every $y \in K$, $F(\cdot, y) : K \to \mathbb{R}$ is upper hemicontinuous; for any $x, y, z \in X$, the inequality $F(x, y) \geq \limsup_{t\to 0+} F((1-t)x \oplus tz, y)$ holds.

We denote the set of solutions to the equilibrium problem for F by S;

$$S = \{z \in K : F(z, y) \geq 0 \text{ for all } y \in K\}.$$

A typical example satisfying the conditions above is $F(x, y) = f(y) - f(x)$ for $x, y \in K$, where $f : K \to \mathbb{R}$ is a lower semicontinuous convex function on K. In this case, the solutions to the equilibrium problem for F coincides with the minimizers of f. For other examples, see [2].

We define a resolvent J_F of bifunction $F : K \times K \to \mathbb{R}$ by

$$J_F x = \left\{ z \in K : F(z, y) + \frac{1}{2}(d(x, y)^2 - d(x, z)^2 - d(y, z)^2) \geq 0 \text{ for all } y \in K \right\}$$

for $x \in X$. As an analogous result of Combettes and Hirstoaga [4], Kimura and Kishi [7] obtain the following fundamental properties of J_F.

Theorem 1 (Kimura and Kishi [7]). *Let X be a Hadamard space satisfying the convex hull finite property, and K a nonempty closed convex subset of X. Let $F : K \times K \to \mathbb{R}$ and S the set of solutions of the equilibrium problem for F. Suppose that F satisfies the conditions* (E1)–(E4). *Then,*

(i) *J_F is single-valued mapping of X into itself;*
(ii) *J_F is firmly nonspreading and Δ-demiclosed;*
(iii) $\operatorname{Fix} J_F = S$;
(iv) *S is closed and convex.*

Now we show a convergence theorem of an iterative sequence approximating a solution to the equilibrium problem on a Hadamard space.

Theorem 2. *Let X be a Hadamard space satisfying the convex hull finite property, and K a nonempty closed convex subset of X. Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the conditions* (E1)–(E4) *and suppose that the set S of the solutions to the equilibrium problem for F is nonempty. Let $\{\alpha_n\}$ and $\{\lambda_n\}$ be positive real sequences such that $0 < \alpha_n < 1$ for all $n \in \mathbb{N}$, $\alpha_n \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} = \infty$, and $\inf_{n \in \mathbb{N}} \lambda_n > 0$. For $n \in \mathbb{N}$, let $J_{\lambda_n F}$ be a resolvent of $\lambda_n F$. Let $\{u_n\}$ be a sequence in X converging to $u \in X$. For given $x_1 \in X$, generate an iterative sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n u_n \oplus (1 - \alpha_n) J_{\lambda_n F} x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges to $P_S u \in S$, where P_S is the metric projection of X onto S.

Proof. Let $q \in S$. Since $\{u_n\}$ is convergent, we have

$$M = \max \left\{ \sup_{n \in \mathbb{N}} d(u_n, q)^2, d(x_1, q)^2 \right\} < \infty.$$

We will show that $d(x_n, q)^2 \leq M$ for all $n \in \mathbb{N}$ by induction. It is trivial that $d(x_1, q)^2 \leq M$. Suppose that $d(x_n, q) \leq M$ for some $n \in \mathbb{N}$. Then, since $J_{\lambda_n F}$ is quasinonexpansive, it follows that

$$\begin{aligned} d(x_{n+1}, q)^2 &= d(\alpha_n u_n \oplus (1 - \alpha_n) J_{\lambda_n F} x_n, q)^2 \\ &\leq \alpha_n d(u_n, q)^2 + (1 - \alpha_n) d(J_{\lambda_n F} x_n, q)^2 \\ &\leq \alpha_n d(u_n, q)^2 + (1 - \alpha_n) d(x_n, q)^2 \\ &\leq \alpha_n M + (1 - \alpha_n) M \leq M. \end{aligned}$$

Therefore $d(x_n, q)^2 \leq M$ for all $n \in \mathbb{N}$ and thus $\{x_n\}$ is bounded. Then, since $q \in S = \operatorname{Fix} J_{\lambda_n F}$ and $d(J_{\lambda_n F}x_n, q) \leq d(x_n, q)$ for all $n \in \mathbb{N}$, $\{J_{\lambda_n F}x_n\}$ is also bounded.

Let $s_n = d(x_n, P_S u)^2$ and $t_n = d(u_n, P_S u)^2 - (1-\alpha_n)d(u_n, J_{\lambda_n F}x_n)^2$ for each $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
s_{n+1} &= d(x_{n+1}, P_S u)^2 \\
&= d(\alpha_n u_n \oplus (1-\alpha_n)J_{\lambda_n F}x_n, P_S u)^2 \\
&= \alpha_n d(u_n, P_S u)^2 + (1-\alpha_n)d(J_{\lambda_n F}x_n, P_S u)^2 - \alpha_n(1-\alpha_n)d(u_n, J_{\lambda_n F}x_n)^2 \\
&\leq (1-\alpha_n)d(x_n, P_S u)^2 + \alpha_n(d(u_n, P_S u)^2 - (1-\alpha_n)d(u_n, J_{\lambda_n F}x_n)^2) \\
&\leq (1-\alpha_n)s_n + \alpha_n t_n
\end{aligned}$$

for any $n \in \mathbb{N}$. We also have

$$\begin{aligned}
s_n - s_{n+1} &= d(x_n, P_S u)^2 - d(x_{n+1}, P_S u)^2 \\
&= d(x_n, P_S u)^2 - d(\alpha_n x_n \oplus (1-\alpha_n)J_{\lambda_n F}x_n, P_S u)^2 \\
&\geq d(x_n, P_S u)^2 - (\alpha_n d(x_n, P_S u)^2 + (1-\alpha_n)d(J_{\lambda_n F}x_n, P_S u)^2) \\
&= (1-\alpha_n)(d(x_n, P_S u)^2 - d(J_{\lambda_n F}x_n, P_S u)^2) \\
&\geq (1-\alpha_n)d(x_n, J_{\lambda_n F}x_n)^2
\end{aligned}$$

for $n \in \mathbb{N}$.

We will apply Lemma 1 to show that $s_n = d(x_n, P_S u)^2$ converges to 0 as $n \to \infty$. To verify the assumptions in this lemma, let $\{n_i\}$ be a subsequence of $\mathbb{N}$ such that

$$\liminf_{i\to\infty}(s_{n_i+1} - s_{n_i}) \geq 0.$$

Then, since $\alpha_{n_i} \to 0$, it follows that

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty} d(x_{n_i}, J_{\lambda_{n_i} F}x_{n_i})^2 \\
&\leq \limsup_{n\to\infty} d(x_{n_i}, J_{\lambda_{n_i} F}x_{n_i})^2 \\
&= \limsup_{n\to\infty}(1-\alpha_{n_i})d(x_{n_i}, J_{\lambda_{n_i} F}x_{n_i})^2 \\
&\leq \limsup_{n\to\infty}(s_{n_i} - s_{n_i+1}) \\
&= -\liminf_{n\to\infty}(s_{n_i+1} - s_{n_i}) \leq 0,
\end{aligned}$$

and thus $\lim_{i\to\infty} d(x_{n_i}, J_{\lambda_{n_i} F}x_{n_i})^2 = 0$. Since $\{J_{\lambda_{n_i} F}x_{n_i}\}$ is bounded, there exists a subsequence $\{y_j\}$ of $\{J_{\lambda_{n_i} F}x_{n_i}\}$ such that

$$\lim_{j\to\infty} d(u, y_j) = \liminf_{i\to\infty} d(u, J_{\lambda_{n_i} F}x_{n_i})$$

and that $\{y_j\}$ Δ-converges to some $p \in X$, where $y_j = J_{\lambda_{n_{i_j}} F} x_{n_{i_j}}$ for all $j \in \mathbb{N}$. From the definition of the resolvent $J_{\lambda_{n_{i_j}} F}$, we have

$$\lambda_{n_{i_j}} F(y_j, y) + \frac{1}{2}(d(x_{n_{i_j}}, y)^2 - d(x_{n_{i_j}}, y_j)^2 - d(y, y_j)^2) \geq 0$$

for all $y \in K$. In particular, letting $y = J_F p$, we have

$$d(x_{n_{i_j}}, J_F p)^2 - d(x_{n_{i_j}}, y_j)^2 - d(J_F p, y_j)^2 \geq -2\lambda_{n_{i_j}} F(y_j, J_F p).$$

Similarly, from the definition of J_F, we have

$$d(p, y_j)^2 - d(p, J_F p)^2 - d(y_j, J_F p)^2 \geq -2F(J_F p, y_j).$$

Since $F(y_j, J_F p) + F(J_F p, y_j) \leq 0$, we obtain

$$\begin{aligned} d(x_{n_{i_j}}, J_F p)^2 - d(x_{n_{i_j}}, y_j)^2 - d(J_F p, y_j)^2 \\ + \lambda_{n_{i_j}} d(p, y_j)^2 - \lambda_{n_{i_j}} d(p, J_F p)^2 - \lambda_{n_{i_j}} d(y_j, J_F p)^2 \geq 0, \end{aligned}$$

and hence

$$\begin{aligned} &(1 + \lambda_{n_{i_j}}) d(y_j, J_F p)^2 \\ &\leq d(x_{n_{i_j}}, J_F p)^2 - d(x_{n_{i_j}}, y_j)^2 + \lambda_{n_{i_j}} d(p, y_j)^2 - \lambda_{n_{i_j}} d(p, J_F p)^2 \\ &\leq d(x_{n_{i_j}}, J_F p)^2 + \lambda_{n_{i_j}} d(p, y_j)^2. \end{aligned}$$

It follows that

$$\begin{aligned} d(y_j, J_F p)^2 &\leq \frac{1}{\lambda_{n_{i_j}}}(d(x_{n_{i_j}}, J_F p)^2 - d(y_j, J_F p)^2) + d(y_j, p)^2 \\ &\leq \frac{1}{\lambda_{n_{i_j}}} d(x_{n_{i_j}}, y_j)(d(x_{n_{i_j}}, J_F p) - d(y_j, J_F p)) + d(y_j, p)^2 \end{aligned}$$

for all $j \in \mathbb{N}$ and consequently we obtain

$$\limsup_{j \to \infty} d(y_j, J_F p)^2 \leq \limsup_{j \to \infty} d(y_j, p)^2.$$

Since the asymptotic center of $\{y_j\}$ is a unique point p, we have $p = J_F p$, that is, $p \in S$. Using the assumption that $\lim_{n \to \infty} u_n = u$, we get

$$\begin{aligned} \limsup_{i \to \infty} t_{n_i} &= \limsup_{i \to \infty} (d(u_{n_i}, P_S u)^2 - (1 - \alpha_{n_i}) d(u_{n_i}, J_{\lambda_{n_i} F} x_{n_i})^2) \\ &= d(u, P_S u)^2 - \liminf_{i \to \infty} d(u, J_{\lambda_{n_i} F} x_{n_i})^2 \\ &= d(u, P_S u)^2 - \lim_{i \to \infty} d(u, y_j)^2 \end{aligned}$$

$$\leq d(u, P_S u)^2 - d(u, p)^2 \leq 0.$$

This shows that all the assumptions in Lemma 1 is verified. Thus we have

$$\lim_{n\to\infty} d(x_n, P_S u)^2 = \lim_{n\to\infty} s_n = 0$$

and hence $\{x_n\}$ converges to $P_S u$, which is the desired result. □

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