数理解析研究所講究録 第2112巻 2019年 148-153



# FIXED POINT THEOREMS FOR MIXED MONOTONE MAPPINGS IN ORDERED METRIC SPACES

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## 1. Introduction

A coupled fixed point theorem is a combination between fixed point results for contractive type mappings and the monotone iterative method proposed by Bhaskar and Lakshmikantham [4]. Several authors investigated it, see [2, 12, 15] and the reference there in. In this paper we consider product of metric spaces with the partial order and give a fixed point theorem for them. As an application, we give a solution for the following cantilever beam equations with fully nonlinear terms in Section 4:

$$\begin{cases} u''''(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \\ u(0) = A, u'(0) = B, u''(1) = C, u'''(1) = D, \end{cases} \tag{1.1}$$

where $f$ is a continuous mapping of $[0, 1] \times \mathbb{R}^4$ into $\mathbb{R}$. The solution of this problem was already given by the method of contraction principle, method of order reduction [9], and the theory of the fixed point index in cones [7]. We give a solution using a fixed point theorem in metric spaces with order. As the merit of this method, under the existence of lower solution or upper solution, we have a fixed point and also we can fixed point when the space satisfies the regularity. In this case the mapping is not necessary continuous. Moreover, under the setting we can give a unique fixed point, and multiple fixed point, see [12] and the examples there in.

## 2. Fixed point theorem

First of all, we cited the following definitions and preliminary results will be useful later. Let $(X, d)$ be a metric space endowed with a partial order $\preceq$. We say that a mapping $F : X \to X$ is nondecreasing if for any $x, y \in X$,

$$x \preceq y \Rightarrow Fx \preceq Fy.$$

Let $\Phi$ denote the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying

(a) $\varphi$ is continuous and nondecreasing;
(b) $\varphi^{-1}(\{0\}) = \{0\}$.

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2010 *Mathematics Subject Classification.* Primary 34A08, 34B15, 47H10.
*Key words and phrases.* Fixed point theorem, fractional order, differential equations, boundary value problem.

Let $\Psi$ denote the set of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying

(c) $\lim_{t\to r+} \psi(t) > 0$ (and finite) for all $r > 0$;
(d) $\lim_{t\to 0+} \psi(t) = 0$.

Let $\Theta$ denote the set of all functions $\theta : [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfying

(e) $\theta$ is continuous;
(f) $\theta(s1, s2, s3, s4) = 0$ if and only if $s1s2s3s4 = 0$.

Examples of functions $\psi$ of $\Psi$ are given in [10]; see also [2, 13]. Examples of functions $\theta$ in $\Theta$ are given in [5].

In [5, Theorem 3.1, 3.2], the following fixed point theorem is obtained. We require an additional assumption to the metric space $X$ with a partial order $\preceq$: We say that $(X, d, \preceq)$ is regular if $\{a_n\}$ is a nondecreasing sequence in $X$ with respect to $\preceq$ such that $a_n \to a \in X$ as $n \to \infty$, then $a_n \preceq a$ for all $n$.

**Theorem 2.1.** *Let $(X, d)$ be a complete metric space endowed with a partial order $\preceq$ and $F : X \to X$ a nondecreasing mapping such that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that for any $x, y \in X$ with $x \succeq y$,*

$$\begin{aligned}\varphi(d(Fx, Fy)) \le \varphi(d(x, y)) - \psi(d(x, y))\\ + \theta(d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)).\end{aligned} \tag{2.1}$$

*Suppose also that the following (i) or (ii) hold.*

(i) *$F$ is continuous*
(ii) *$(X, d, \le)$ is regular.*

*Also supose that there exists $x_0 \in X$ such that $x_0 \preceq Fx_0$ (or $x_0 \succeq Fx_0$). Then $F$ admits a fixed point, that is, there exists $\overline{x} \in X$ such that $\overline{x} = F\overline{x}$.*

## 3. Fixed point theorem for monotone mapping

**Definition 3.1.** We say that a mapping $F$ of $X^4$ into $X$ has mixed monotone property, if it satisfies the following, see [6]: for any $x, y, z, w \in X$,

$$\begin{cases} x_1, x_2 \in X, x_1 \succeq x_2, \Rightarrow F(x_1, y, z, w) \succeq F(x_2, y, z, w),\\ y_1, y_2 \in X, y_1 \succeq y_2, \Rightarrow F(x, y_1, z, w) \succeq F(x, y_2, z, w),\\ z_1, z_2 \in X, z_1 \succeq z_2, \Rightarrow F(x, y, z_1, w) \succeq F(x, y, z_2, w),\\ w_1, w_2 \in X, w_1 \preceq w_2, \Rightarrow F(x, y, z, w_1) \preceq F(x, y, z, w_2).\end{cases}$$

Let $(X, d)$ be a metric space. Let $F_1$, $F_2$, $F_3$ and $F_4$ be mappings of $X^4$ into $X$. We also consider the mapping $A$ of $X^4$ into $[0,\infty)$ defined by

$$\begin{aligned} AU = \frac{d(x, F_1U) + d(y, F_2U)}{4} + \frac{d(z, F_3U) + d(w, F_4U)}{4},\\ U = (x, y, z, w) \in X^4,\end{aligned}$$

and the mapping $B$ of $X^8$ into $[0,\infty)$ defined by

$$\begin{aligned}&B(U,V) = \frac{d(x_1,F_1V)+d(y_1,F_2V)}{4} + \frac{d(z_1,F_3V)+d(w_1,F_4V)}{4},\\ &U=(x_1,y_1,z_1,w_1), V=(x_2,y_2,z_2,w_2)\in X^4.\end{aligned}$$

We consider the mapping $T$ from $X^4$ into $X^4$ defined by

$$TU = (F_1U, F_2U, F_3U, F_4U), U=(x,y,z,w)\in Y. \tag{3.1}$$

In this case, we consider the metric $\eta$ for the product set $X^4$ defined by

$$\begin{aligned}&\eta(U,V) = \frac{d(x_1,x_2)+d(y_1,y_2)+d(z_1,z_2)+d(w_1,w_2)}{4},\\ &U=(x_1,y_1,z_1,w_1), V=(x_2,y_2,z_2,w_2)\in Y.\end{aligned}$$

Note that if $(X,d)$ is complete, then clearly $(X^4,\eta)$ is also complete. Also if $F_1$, $F_2$, $F_3$ and $F_4$ are continuous, then $T$ is also continuous in $(X^4,\eta)$.

Next we consider the partial order $\ll$ in $X^4$ defined by

$$(x_2,y_2,z_2,w_2)\ll(x_1,y_1,z_1,w_1)\Leftrightarrow x_1\succeq x_2, y_1\succeq y_2, z_1\succeq z_2, w_1\preceq w_2$$

for any $(x_1,y_1,z_1,w_1),(x_2,y_2,z_2,w_2)\in Y$.

Under the above settings, we consider the following inequality ; there exist $\varphi\in\Phi$, $\psi\in\Psi$ and $\theta\in\Theta$ such that for any $x_1,y_1,z_1,w_1,x_2,y_2,z_2,w_2\in X$ with $x_1\succeq x_2$, $y_1\succeq y_2$, $z_1\succeq z_2$ and $w_1\preceq w_2$, the following holds:

$$\begin{aligned}&\varphi\left(\frac{d(F_1U_1,F_1U_2)+d(F_2U_1,F_2U_2)}{4}+\frac{d(F_3U_1,F_3U_2)+d(F_4U_1,F_4U_2)}{4}\right)\\ &\le\varphi\left(\frac{d(x_1,x_2)+d(y_1,y_2)+d(z_1,z_2)+d(w_1,w_2)}{4}\right)\\ &-\psi\left(\frac{d(x_1,x_2)+d(y_1,y_2)+d(z_1,z_2)+d(w_1,w_2)}{4}\right)\\ &+\theta\left(A_1U_1, A_1U_2, B_1(U_1,U_2), B_1(U_2,U_1)\right).\end{aligned} \tag{3.2}$$

where $U_1=(x_1,y_1,z_1,w_1)$, $U_2=(x_2,y_2,z_2,w_2)$ If each mapping $F_1$, $F_2$, $F_3$ and $F_4$ satisfies that there exist $a,b,c,d\in X$ such that $a=F_1(a,b,c,d)$, $b=F_2(a,b,c,d)$, $c=F_3(a,b,c,d)$ and $d=F_4(a,b,c,d)$, then $(a,b,c,d)\in X^4$ is a fixed point of the mapping $T$.

Motivated by [5, Theorem 3.4], we have the following theorems for the mapping $T$.

**Theorem 3.2.** *Let $(X,d)$ be a complete metric space endowed with a partial order $\preceq$, mappings $F_1$, $F_2$, $F_3$ and $F_4$ of $X^4$ into $X$ continuous mixed monotone mappings. We assume that there exist $\varphi\in\Phi$, $\psi\in\Psi$ and $\theta\in\Theta$ such that for any $x_1,y_1,z_1,w_1,x_2,y_2,z_2,w_2\in X$ with $x_1\succeq x_2$, $y_1\succeq y_2$, $z_1\succeq z_2$ and*

$w_1 \preceq w_2$, *the inequality (3.2) holds: If there exist* $x_0, y_0, z_0, w_0 \in X$ *such that*

$$x_0 \preceq F_1(x_0, y_0, z_0, w_0), y_0 \preceq F_2(x_0, y_0, z_0, w_0),$$
$$z_0 \preceq F_3(x_0, y_0, z_0, w_0), w_0 \succeq F_4(x_0, y_0, z_0, w_0), \tag{3.3}$$

*or*

$$x_0 \succeq F_1(x_0, y_0, z_0, w_0), y_0 \succeq F_2(x_0, y_0, z_0, w_0),$$
$$z_0 \succeq F_3(x_0, y_0, z_0, w_0), w_0 \preceq F_4(x_0, y_0, z_0, w_0), \tag{3.4}$$

*then the mapping* $T$ *defined by (3.1) has fixed point, that is, there exists* $(a, b, c, d) \in X^4$ *such that* $(a, b, c, d) = T(a, b, c, d)$.

The previous results, Theorem 3.2 is still valid for mixed monotone mappings $F_1, F_2, F_3$ and $F_4$, and $F_1, F_2$ and $F_3$, which are not necessarily continuous, respectively. Instead, we require additional assumptions to the metric space $X$ with a partial order $\preceq$:

**Definition 3.3.** Let $(X, d)$ be a complete metric space endowed with a partial order $\preceq$. We say that
(i) $(X, d, \preceq)$ is nondecreasing-regular ($\uparrow$-regular) if a nondecreasing sequence $\{x_n\} \subset X$ converges to $x$, then $x_n \preceq x$ for all $n$;
(ii) $(X, d, \preceq)$ is nonincreasing-regular ($\downarrow$-regular) if a nonincreasing sequence $\{x_n\} \subset X$ converges to $x$, then $x_n \succeq x$ for all $n$.

Motivated by [5, Theorem 3.5], we have the following result.

**Theorem 3.4.** *Let* $(X, d)$ *be a complete metric space endowed with a partial order* $\preceq$, *and mappings* $F_1$, $F_2$, $F_3$ *and* $F_4$ *of* $X^4$ *into* $X$ *mixed monotone mappings. We assume that there exist* $\varphi \in \Phi$, $\psi \in \Psi$ *and* $\theta \in \Theta$ *such that for any* $x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2 \in X$ *with* $x_1 \succeq x_2$, $y_1 \succeq y_2$, $z_1 \succeq z_2$ *and* $w_1 \preceq w_2$, *the inequality (3.2) holds. We also assume that* $(X, d, \preceq)$ *is nondecreasing-regular and nonincreasing-regular* ($\uparrow\downarrow$*-regular), and there exist* $x_0, y_0, z_0, w_0 \in X$ *such that (3.3) or (3.4) hold, then the mapping* $T$ *defined by (3.1) has fixed point, that is, there exists* $(a, b, c, d) \in X^4$ *such that* $(a, b, c, d) = T(a, b, c, d)$.

## 4. Application

In this section, as applications of Theorem 3.4, we study the existence of solutions of two types fourth-order two-point boundary value problems. First of all, we study the existence of solutions of the following fourth-order two-point boundary value problem (1.1). Let $\Omega$ be a set of functions $\omega$ of $[0, \infty)$ into $[0, \infty)$ satisfying
(i) $\omega$ is nondecreasing;
(ii) there exists $\psi \in \Psi$ such that $\omega(r) = \frac{r}{2} - \psi(\frac{r}{2})$ for all $r \in [0, \infty)$.
For examples of such functions, see [10]. Next we consider the following assumptions $(A1)$ and $(A2)$.

(A1) There exists $\omega \in \Omega$ such that for all $t \in I$ and for all $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R}$, with $a_1 \geq b_1$, $a_2 \geq b_2$, $a_3 \geq b_3$ and $a_4 \leq b_4$,

$$\begin{aligned} 0 &\leq f(t, a_1, a_2, a_3, a_4) - f(t, b_1, b_2, b_3, b_4) \\ &\leq \omega(a_1 - b_1) + \omega(a_2 - b_2) + \omega(a_3 - b_3) + \omega(b_4 - a_4). \end{aligned} \tag{4.1}$$

(A2) There exist $\alpha, \beta, \gamma, \delta \in C(I, \mathbb{R})$ which are solutions of

$$\begin{aligned} \alpha(t) \leq\; & Bt + A - \int_0^1 H_2(t,s)(C - D + Ds)ds \\ & + \int_0^1 G(t,s) f(s, \alpha(s), \beta(s), \gamma(s), \delta(s))ds, t \in I, \\ \beta(t) \leq\; & B - \int_0^t (C - D + Ds)ds \\ & + \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s, \alpha(s), \beta(s), \gamma(s), \delta(s))ds, t \in I, \\ \gamma(t) \leq\; & -C + D - Dt + \int_0^1 H_1(t,s) f(s, \alpha(s), \beta(s), \gamma(s), \delta(s))ds, t \in I, \\ \delta(t) \geq\; & -D - \int_0^1 \frac{\partial H_1}{\partial t}(t,s) f(s, \alpha(s), \beta(s), \gamma(s), \delta(s))ds, t \in I, \end{aligned} \tag{4.2}$$

where the Green functions $G$ and $H_1$ are defined by

$$G(t,s) = \begin{cases} \frac{1}{6}s^2(3t - s), & (0 \leq s \leq t \leq 1), \\ \frac{1}{6}t^2(3s - t), & (0 \leq t \leq s \leq 1), \end{cases} \tag{4.3}$$

$$H_1(t,s) = \begin{cases} 0, & (0 \leq s \leq t \leq 1), \\ s - t, & (0 \leq t \leq s \leq 1), \end{cases} \tag{4.4}$$

It is easy to see that

$$0 \leq G(t,s) \leq \frac{1}{2}t^2 s \text{ for all } t, s \in I, \tag{4.5}$$

$$0 \leq \frac{\partial G}{\partial t}(t,s) \leq ts \text{ for all } t, s \in I, \tag{4.6}$$

$$0 \leq H_1(t,s) \leq \min\{s, t\} \text{ for all } t, s \in I. \tag{4.7}$$

Now we have the following theorem.

**Theorem 4.1.** *Under the assumptions (A1) and (A2), the fourth-order two-point boundary value problem (1.1) has a solution.*

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