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# Remark on skew $m$-complex symmetric operators

by

Muneo Chō, Eungil Ko, and Ji Eun Lee

Abstract

In this paper we study skew $m$-complex symmetric operators. In particular, we prove that if $T \in \mathcal{L}(\mathcal{H})$ is a skew $m$-complex symmetric operator with a conjugation $C$, then $e^{itT}$, $e^{-itT}$, and $e^{-itT^*}$ are $(m, C)$-isometric for every $t \in \mathbb{R}$. Moreover, we investigate some conditions for skew $m$-complex symmetric operators to be skew $(m-1)$-complex symmetric.

## 1 Introduction

*The results in this paper will be appeared in other journals.* Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$.

**Definition 1.1** *An operator $C$ is said to be a conjugation on $\mathcal{H}$ if the following conditions hold:*
(i) *$C$ is antilinear; $C(ax+by) = \overline{a}Cx + \overline{b}Cy$ for all $a, b \in \mathbb{C}$ and $x, y \in \mathcal{H}$,*
(ii) *$C$ is isometric; $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$, and*
(iii) *$C$ is involutive; $C^2 = I$.*

Moreover, if $C$ is a conjugation on $\mathcal{H}$, then $\|C\| = 1$, $\big(CTC\big)^* = CT^*C$ and $\big(CTC\big)^k = CT^kC$ for every positive integer $k$. For any conjugation $C$, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all $n$ (see [11] for more details). We first consider the following examples for conjugations.

**Example 1.2** Let's define an operator $C$ as follows:
(i) $C(x_1, x_2, x_3, \cdots, x_n) = (\overline{x_1}, \overline{x_2}, \overline{x_3}, \cdots, \overline{x_n})$ on $\mathbb{C}^n$.
(ii) $C(x_1, x_2, x_3, \cdots, x_n) = (\overline{x_n}, \overline{x_{n-1}}, \overline{x_{n-2}}, \cdots, \overline{x_1})$ on $\mathbb{C}^n$.
(iii) $[Cf](x) = \overline{f(x)}$ on $\mathcal{L}^2(\mathcal{X}, \mu)$.
(iv) $[Cf](x) = \overline{f(1-x)}$ on $L^2([0,1])$.
(v) $[Cf](x) = \overline{f(-x)}$ on $L^2(\mathbb{R}^n)$.
(vi) $Cf(z) = \overline{zf(z)}u(z) \in \mathcal{K}_u^2$ for all $f \in \mathcal{K}_u^2$ where $u$ is inner function and $\mathcal{K}_u^2 = H^2 \ominus uH^2$ is Model space.
Then each $C$ in (i)-(vi) is a conjugation.

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In 1970, J. W. Helton [15] initiated the study of operators $T \in \mathcal{L}(\mathcal{H})$ which satisfy an identity of the form;

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}T^{m-j}=0. \tag{1}$$

Using the identity (1) and a conjugation operator, we define skew $m$-complex symmetric operators as follows; an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a *skew $m$-complex symmetric operator* if there exists some conjugation $C$ such that

$$\sum_{j=0}^{m}\binom{m}{j}T^{*j}CT^{m-j}C=0$$

for some positive integer $m$. In this case, we say that $T$ is skew $m$-complex symmetric with conjugation $C$. In particular, if $m=1$, then $T$ is said to be *skew complex symmetric*, i.e., $T=-CT^*C$. Set $\Gamma_m(T;C) := \sum_{j=0}^{m}\binom{m}{j}T^{*j}CT^{m-j}C$. Then $T$ is a skew $m$-complex symmetric operator with conjugation $C$ if and only if $\Gamma_m(T;C)=0$. Note that

$$T^*\Gamma_m(T;C)+\Gamma_m(T;C)(CTC)=\Gamma_{m+1}(T;C). \tag{2}$$

From (2), if $T$ is skew $m$-complex symmetric with conjugation $C$, then $T$ is skew $n$-complex symmetric with conjugation $C$ for $n \geq m$. In general, skew $m$-complex symmetric operators are not skew $(m-1)$-complex symmetric.

**Example 1.3** Let $Cx=\binom{\overline{x_2}}{\overline{x_1}}$ for $x=\binom{x_1}{x_2}$ and $T=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ on $\mathbb{C}^2$. Then $T^*=CTC=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ and so $CT^2C+2T^*CTC+T^{*2}=0$. But, $CTC+T^*=2\begin{pmatrix}0&0\\1&0\end{pmatrix}\neq 0$. Hence $T$ is a skew 2-complex symmetric operator which is not skew complex symmetric (see [3]).

In 1995, Agler and Stankus ([1]) studied the following operator. For a fixed $m \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *$m$-isometric* operator if it satisfies an identity;

$$\sum_{j=0}^{m}(-1)^{j}\binom{m}{j}T^{*m-j}T^{m-j}=0. \tag{3}$$

Using the identity (3) and a conjugation $C$, the authors of [9] define the following operator; An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *$(m,C)$-isometric operator* if there exists some conjugation $C$ such that

$$\sum_{j=0}^{m}(-1)^{j}\binom{m}{j}T^{*m-j}CT^{m-j}C=0 \tag{4}$$

for some $m \in \mathbb{N}$. In particular, if $T=CTC$, then $T$ is an $m$-isometric operator. Put $\Lambda_m(T) := \sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}CT^{m-j}C$. Thus $T$ is an $(m,C)$-isometric operator if and only if $\Lambda_m(T)=0$. Note that

$$T^*\Lambda_m(T)(CTC)-\Lambda_m(T)=\Lambda_{m+1}(T). \tag{5}$$

From (5), if $\Lambda_m(T) = 0$, then $\Lambda_n(T) = 0$ for all $n \geq m$. Moreover, $T$ is an $(m, C)$-isometry if and only if $CTC$ is an $(m, C)$-isometry (see [9]).

Next, we provide several examples of $(m, C)$-isometric operators with a conjugation $C$.

**Example 1.4** ([9]) Let $C$ be the canonical conjugation on $\mathcal{H}$ given by

$$C(\sum_{n=0}^{\infty} x_n e_n) = \sum_{n=0}^{\infty} \overline{x_n} e_n$$

where $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$ with $Ce_n = e_n$ for all $n$. Assume that $W$ is the weighted shift given by $We_n = \alpha_n e_{n+1}$ where $\alpha_n = \sqrt{\frac{n+\alpha}{n+1}}$ for $\alpha > 0$. If $\alpha = 1$, then $W = S$ is the unilateral shift. Hence $S$ is $(1, C)$-isometry. If $\alpha = 2$, then, since $W = CWC$, it holds that

$$I - 2W^* CWC + W^{*2} CW^2 C = 0.$$

Therefore, $W$ is an $(2, C)$-isometric operator which is called the Dirichlet shift. On the other hand, if $\alpha = m$, then, since $W = CWC$, it holds that

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} W^{*m-j} CW^{m-j} C = 0.$$

So, $W$ is an $(m, C)$-isometric operator.

**Example 1.5** ([9]) Let $C$ be a conjugation defined by $Cf(z) = \overline{f(\overline{z})}$ and let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of $H^2$. Set $\mathcal{C} = C \oplus C$. Then $\mathcal{C}$ is clearly a conjugation on $H^2 \oplus H^2$. Assume that

$$T = \begin{pmatrix} S & e_0 \otimes e_0 \\ 0 & I \end{pmatrix} \in \mathcal{L}(H^2 \oplus H^2)$$

where $S$ is the unilateral shift on $H^2$. Then

$$\begin{aligned} \Lambda_2(T) &= T^*(T^*\mathcal{C}T\mathcal{C} - I)\mathcal{C}T\mathcal{C} - (T^*\mathcal{C}T\mathcal{C} - I) \\ &= \begin{pmatrix} 0 & 0 \\ 0 & e_0 \otimes e_0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & e_0 \otimes e_0 \end{pmatrix} = 0. \end{aligned}$$

Hence $T$ is an $(2, \mathcal{C})$-isometric operator. If $R = S + e_0 \otimes e_0$, then

$$CRC = CSC + C(e_0 \otimes e_0)C = S + e_0 \otimes e_0.$$

Since $S^* e_0 = 0$, it follows that $R^*CRC = (S^* + e_0 \otimes e_0)(S + e_0 \otimes e_0) = I + e_0 \otimes e_0$ and so

$$\begin{aligned} \Lambda_2(R) &= R^*(R^*CRC - I)CRC - (R^*CRC - I) \\ &= (S^* + e_0 \otimes e_0)(e_0 \otimes e_0)(S + e_0 \otimes e_0) - e_0 \otimes e_0 = 0. \end{aligned}$$

Therefore, $R$ is an $(2, C)$-isometric operator.

# 2 $(m,C)$-isometric operators

In this section, we state properties of $(m,C)$-isometric operators which are the known results in [9].

**Theorem 2.1** *Let $T \in \mathcal{L}(\mathcal{H})$ and let $C$ be a conjugation on $\mathcal{H}$. Then the following statements hold.*
(i) *If $T$ is an invertible, then $T$ is an $(m,C)$-isometric operator if and only if $T^{-1}$ is an $(m,C)$-isometry.*
(ii) *If $T$ is an $(m,C)$-isometric operator with the conjugation $C$ and $T$ is complex symmetric, i.e., $T = CT^*C$, then $T$ is an algebraic operator of order at most $2m$.* (iii) *If $\{T_k\}$ is a sequence of $(m,C)$-isometric operators with conjugation $C$ such that $\lim_{k\to\infty} \|T_k - T\| = 0$, then $T$ is also an $(m,C)$-isometric operator.*
(iv) *If $T$ is an $(m,C)$-isometric operator, then $T^n$ is also an $(m,C)$-isometric operator for any $n \in \mathbb{N}$.*

If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ for the spectrum, the point spectrum and the approximate point spectrum of $T$, respectively.

**Lemma 2.2** *Let $T \in \mathcal{L}(\mathcal{H})$ be an $(m,C)$-isometric operator where $C$ is a conjugation on $\mathcal{H}$. Then $0 \notin \sigma_a(T)$.*

We observe from Lemma that both $ran(T)$ and $\ker(T)$ are closed complemented subspaces. If $ran(T) = \mathcal{H}$, then $T$ is invertible. Otherwise, $ran(T)$ is a nontrivial invariant subspace of $T$. Hence the representation of $T$ with respect to the Hilbert space decomposition $ran(T) \oplus \ker(T^*) = \mathcal{H}$ is the upper triangular matrices

$$\begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} : ran(T) \oplus \ker(T^*) \to ran(T) \oplus \ker(T^*)$$

where $T_1 = T|_{ran(T)}$, and $T_2$ is an operator mapping $\ker(T^*)$ into $ran(T)$ and $\ker(T^*)$, respectively.

**Theorem 2.3** *Let $T \in \mathcal{L}(\mathcal{H})$ be an $(m,C)$-isometric operator where $C$ is a conjugation on $\mathcal{H}$. If $\lambda \in \sigma_a(T)$, then $\frac{1}{\overline{\lambda}} \in \sigma_a(T^*)$. In particular, if $\lambda$ is an eigenvalue of $T$, then $\frac{1}{\overline{\lambda}}$ is an eigenvalue of $T^*$.*

**Theorem 2.4** *Let $T \in \mathcal{L}(\mathcal{H})$ be an $(m,C)$-isometric operator where $C$ is a conjugation on $\mathcal{H}$. Let $\lambda, \mu \in \mathbb{C}$ with $\lambda\mu \neq 1$. If $\{x_n\}$ and $\{y_n\}$ are sequences of unit vectors such that $\lim_{n\to\infty} (T-\lambda)x_n = 0$ and $\lim_{n\to\infty} (T-\mu)y_n = 0$, then $\lim_{n\to\infty} \langle Cx_n, y_n\rangle = 0$. In particular, if $(T-\lambda)x = 0$ and $(T-\mu)y = 0$, then $\langle Cx, y\rangle = 0$.*

**Corollary 2.5** *Let $C$ be a conjugation on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$ is an $(m,C)$-isometric operator with a conjugation $C$, then $\ker(T-\lambda) \subseteq C\ker((T^* - \frac{1}{\overline{\lambda}})^m)$.*

## 3 Skew $m$-complex symmetric operators

In this section, we study properties of skew $m$-complex symmetric operators. In [7], if $T$ is an $m$-complex symmetric operator, then $T^n$ is also $m$-complex symmetric for some $n$. Unlike an $m$-complex symmetric operator (see [7] and [9]), the power of a skew $m$-complex symmetric operator is not skew $m$-complex symmetric.

**Example 3.1** If $T = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & -1 \end{pmatrix}$ for $a \in \mathbb{C}$, then $T$ is skew complex symmetry with the conjugation $C(z_1, z_2, z_3) = (-\overline{z_3}, \overline{z_2}, -\overline{z_1})$ from [18]. A simple calculation shows that

$$T^2 = \begin{pmatrix} 1 & a & a^2 \\ 0 & 0 & -a \\ 0 & 0 & 1 \end{pmatrix} \text{ and } -CT^2C = \begin{pmatrix} -1 & 0 & 0 \\ -a & 0 & 0 \\ -a^2 & a & -1 \end{pmatrix}.$$

Hence $T^2$ is not skew complex symmetric with the conjugation $C$.

**Example 3.2** Let $C$ be a conjugation given by $C(z_1, z_2, z_3) = (\overline{z_3}, \overline{z_2}, \overline{z_1})$ on $\mathbb{C}^3$. If $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$, then $T^* \neq CTC = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $T^{*2} = CT^2C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$. Hence $T^2$ is a 1-complex symmetric operator but $T$ is not a 1-complex symmetric operator with conjugation $C$.

Now we will introduce exponential operators $T := e^{-iA}$ which act on a wave function to move it in time and space (see [1]). Note that $T$ is a function of an operator $f(A)$ which is defined its expansion in a Taylor series

$$T = exp(-iA) = \sum_{n=0}^{\infty} \frac{(-iA)^n}{n!} = 1 - iA + \frac{(-iA)^2}{2!} + \cdots .$$

The most common one is the time-propagator or time-evolution operator $U$ which is the Hamiltonian function and propagates the wave function forward in time;

$$U = exp(\frac{-iHt}{h}) = 1 + \frac{-iHt}{h} + \frac{1}{2!}(\frac{-iHt}{h})^2 + \cdots .$$

For an operator $T \in \mathcal{L}(\mathcal{H})$, if $t \in \mathbb{R}$, then

$$e^{itT} = I + itT + \frac{(it)^2}{2!}T^2 + \frac{(it)^3}{3!}T^3 + \cdots . \tag{6}$$

**Theorem 3.3** *If $T \in \mathcal{L}(\mathcal{H})$ is a skew $m$-complex symmetric operator with a conjugation $C$, then $e^{itT}$, $e^{-itT}$, and $e^{-itT^*}$ are $(m, C)$-isometric for every $t \in \mathbb{R}$.*

In general, the converse of the previous theorem may not be hold. But, if $e^{itT}$ is $(1, C)$-isometric operator and $T$ is a skew 2-complex symmetric operator with the conjugation $C$, then $T$ is a skew complex symmetric operator.

**Corollary 3.4** *Let $T \in \mathcal{L}(\mathcal{H})$. Then the following statements hold:*
(i) *Assume that $T$ is skew $m$-complex symmetric with a conjugation $C$. If $\lambda \in \sigma_a(e^{itT})$, then $\dfrac{1}{\overline{\lambda}} \in \sigma_a(e^{-itT^*})$. In particular, if $\lambda \in \sigma_p(e^{itT})$, then $\dfrac{1}{\overline{\lambda}} \in \sigma_p(e^{-itT^*})$.*
(ii) *If $T$ is skew $m$-complex symmetric with a conjugation $C$, then $e^{itnT}$ is an $(m, C)$-isometric operator for any $n \in \mathbb{N}$.*
(iii) *Let $\{T_k\}$ be a sequence of skew $m$-complex symmetric operators with a conjugation $C$ such that $\lim_{k\to\infty} \|e^{itT_k} - e^{itT}\| = 0$. Then $e^{itT}$ is an $(m, C)$-isometric operator.*

Recall that
$$\cos(tT) = \frac{e^{itT} + e^{-itT}}{2} \text{ and } \sin(tT) = \frac{e^{itT} - e^{-itT}}{2i}$$
for every $t \in \mathbb{R}$.

**Corollary 3.5** *Let $T \in \mathcal{L}(\mathcal{H})$ be skew complex symmetric with a conjugation $C$ and let $t \in \mathbb{R}$. Then the following statements hold.*
(i) *$\cos(tT)$ is a $(1, C)$-isometric operator if and only if $\cos(2tT^*) = I$.*
(ii) *$\sin(tT)$ is a $(1, C)$-isometric operator if and only if $\cos(2tT^*) = -I$.*

A closed subspace $\mathcal{M} \subset \mathcal{H}$ is *invariant* for $T$ if $T\mathcal{M} \subset \mathcal{M}$.

**Corollary 3.6** *If $T \in \mathcal{L}(\mathcal{H})$ is skew $m$-complex symmetric and complex symmetric with a conjugation $C$, i.e., $T^* = CTC$, then the following statements hold:*
(i) *$e^{itT}$ is an algebraic operator of order at most $2m$.*
(ii) *$C \ker(\Gamma_{m-1}(e^{itT}; C))$ is invariant for $e^{itT}$.*

**Corollary 3.7** *If $T \in \mathcal{L}(\mathcal{H})$ is skew $m$-complex symmetric and complex symmetric with a conjugation $C$, then the following statements hold.*
(i) *$e^{itT}$ is unitarily equivalent to a finite operator matrix of the form:*
$$\begin{pmatrix} \alpha_1 & A_{12} & \cdots & \cdots & \cdots & A_{1,2m} \\ 0 & \alpha_2 & A_{23} & \cdots & \cdots & A_{2,2m} \\ 0 & 0 & \alpha_3 & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \ddots & A_{2m-1,2m} \\ 0 & 0 & \cdots & \cdots & \cdots & \alpha_{2m} \end{pmatrix}$$
*where $\alpha_j$ are the roots of the polynomial $p(z)$ of degree at most $2m$.*
(ii) *The dimension of $\bigvee_{k=0}^{\infty}\{(e^{itT})^k x\}$ is less than or equals to $2m$.*

It is known from [15] that if $T$ is $m$-symmetric and $m$ is even, then $T$ is $(m-1)$-symmetric. In 2012, M. Chō, S. Ôta, K. Tanahashi, and A. Uchiyama proved that if $T$ is an invertible $m$-isometric operator and $m$ is even, then $T$ is an $(m-1)$-isometric operator (see [6] for more details). In view of these results, we will consider the following question; *if $T \in \mathcal{L}(\mathcal{H})$ is skew $m$-complex symmetric with a conjugation $C$ and $m$ is even, is it skew $(m-1)$-complex symmetric?* In the next theorem, we give a partial solution for the previous question.

**Theorem 3.8** *Let $T \in \mathcal{L}(\mathcal{H})$ and let $C$ be a conjugation on $\mathcal{H}$. Suppose that $\Lambda_{m-1}(e^{itT};C)$ and $((e^{itT})^*)^{m-1}\Lambda_{m-1}(e^{-itT};C)C(e^{itT})^{m-1}C$ are nonnegative. If $T$ is a skew $m$-complex symmetric operator with the conjugation $C$ where $m$ is even, then $T$ is skew $(m-1)$-complex symmetric and $e^{itT}$ is an $(m-1,C)$-isometric operator for all $t \in \mathbb{R}$.*

**Corollary 3.9** *If $T \in \mathcal{L}(\mathcal{H})$ is skew $m$-complex symmetric with a conjugation $C$, $m$ is even, and $[T,C]=0$, then $T$ is skew $(m-1)$-complex symmetric.*

# 4 On an operator $T$ commuting with $CTC$

In this section, we focus on an operator $T$ commuting with $CTC$. Given $T \in \mathcal{L}(\mathcal{H})$ and a conjugation $C$ on $\mathcal{H}$, let

$$\mathcal{C}_C(T) := \{S \in \mathcal{L}(\mathcal{H}) \mid [CTC,S]=0\}$$

where $[R,S] := RS - SR$. *In this section, we study the case when*

$$T \in \mathcal{C}_C(T), \quad \textit{that is,} \quad [CTC,T]=0.$$

We observe that $\mathcal{C}_C(T)$ need not contain complex symmetric operators.

**Example 4.1** Let $\mathcal{H} = \ell^2$, let $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$ and let $C:\mathcal{H}\to\mathcal{H}$ be the conjugation given by $C(\sum_{n=0}^{\infty} x_n\, e_n) = \sum_{n=0}^{\infty} \overline{x_n}\, e_n$ where $\{x_n\}$ is a sequence in $\mathbb{C}$ with $\sum_{n=0}^{\infty}|x_n|^2 < \infty$ and $Ce_n = e_n$ for all $n$. If $W \in \mathcal{L}(\mathcal{H})$ is the weighted shift given by $We_n = \alpha_n e_{n+1}$ for all $n \geq 1$, then it is easy to compute $WCWCe_n = CWCWe_n$ for all $n$. Hence $W \in \mathcal{C}_C(W)$. In particular, if $\alpha_n = 1$ for all $n$, then $W = S$ is the unilateral shift and so $S \in \mathcal{C}_C(S)$. However, $S$ is not complex symmetric.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$ and *binormal* if $T^*T$ and $TT^*$ commute where $T^*$ is the adjoint of $T$. Note that every normal operator is binormal.

**Example 4.2** Let $\mathcal{H} = \mathbb{C}^2$ and let $C$ be a conjugation on $\mathcal{H}$ given by $C(x,y) = (\overline{y},\overline{x})$. Assume that $R = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$ on $\mathcal{H}$. Then $CRC = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} = R$. Hence $R \in \mathcal{C}_C(R)$. However, $R$ is not normal, but binormal.

**Example 4.3** Let $C$ and $J$ be conjugations on $\mathcal{H}$. Assume that $T = \begin{pmatrix} 0 & CJ \\ I & 0 \end{pmatrix}$ and $\mathcal{J} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$. Then $\mathcal{J}T\mathcal{J}T = T\mathcal{J}T\mathcal{J} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Hence $T \in \mathcal{C}_{\mathcal{J}}(T)$ is normal.

In the next example, we know that there exists $T$ such that $T \notin \mathcal{C}_C(T)$, in general.

**Example 4.4** Let $\mathcal{H} = \mathbb{C}^n$ and $C(z_1, z_2, z_3, \cdots, z_n) = (\overline{z_n}, \cdots, \overline{z_3}, \overline{z_2}, \overline{z_1})$. If

$$T = \begin{pmatrix} 0 & \lambda_1 & 0 & & \dots & 0 \\ 0 & 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \dots & 0 \\ \vdots & \vdots & . & 0 & \ddots & 0 \\ . & . & . & . & 0 & \lambda_{n-1} \\ 0 & 0 & . & . & \dots & 0 \end{pmatrix} \quad \text{and} \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

for all $\lambda_j \neq 0$, then $0 = (CTC)T\, e_1 \neq T(CTC)\, e_1 = \lambda_1 \cdot \overline{\lambda_{n-1}} \cdot e_1$. Hence $T \notin \mathcal{C}_C(T)$. But, it is clear that $T$ is binormal.

**Theorem 4.5** *If $T \in \mathcal{L}(\mathcal{H})$ is a normal operator, then $T \in \mathcal{C}_C(T)$ for some conjugation $C$.*

Note that every normal operator is complex symmetric (see [11]).

**Proposition 4.6** *Let $T \in \mathcal{C}_C(T)$ for some conjugation $C$. Then the following statements hold.*
(i) $T^* \in \mathcal{C}_C(T^*)$.
(ii) $p(T) \in \mathcal{C}_C(p(T))$ *for every polynomial* $p$.
(iii) *If $T$ is invertible, then $T^{-1} \in \mathcal{C}_C(T^{-1})$.*
(iv) *If $X \in \mathcal{L}(\mathcal{H})$ is invertible with $[X, C] = 0$, then $X^{-1}TX \in \mathcal{C}_C(X^{-1}TX)$.*
(v) *If $R \in \mathcal{L}(\mathcal{H})$ is unitarily equivalent to $T$, i.e., $R = UTU^*$, then $R \in \mathcal{C}_D(R)$ for a conjugation $D = UCU^*$.*
(vi) $[T^m, CT^nC] = 0$ *for all* $n, m \in \mathbb{N}$.
(vii) *The class of operators which satisfy $T \in \mathcal{C}_C(T)$ is norm closed.*

**Proposition 4.7** *Let $C, C_1, C_2$ be conjugations on $\mathcal{H}$. Then the following statements hold.*
(i) *If $T_i \in \mathcal{L}(\mathcal{H}_i)$ be such that $T_i \in \mathcal{C}(T_i)$ for conjugations $C_i$ with $i = 1, 2$, respectively, then $T_1 \oplus T_2 \in \mathcal{C}_{C_1 \oplus C_2}(T_1 \oplus T_2)$ for a conjugation $C_1 \oplus C_2$.*
(ii) *Let $T \in \mathcal{C}_C(T)$ and $S \in \mathcal{C}_C(S)$. If $[T, S] = 0$ and $[CTC, S] = 0$, then $T+S \in \mathcal{C}_C(T+S)$ and $TS \in \mathcal{C}_C(TS)$ for a conjugation $C$.*
(iii) *If $T \in \mathcal{C}_{C_1}(T)$ and $S \in \mathcal{C}_{C_2}(S)$ for conjugations $C_1$ and $C_2$, respectively, then $T \otimes S \in \mathcal{C}_{C_1 \otimes C_2}(T \otimes S)$ for a conjugation $C_1 \otimes C_2$.*

In [11], if $T$ is complex symmetric, then $ReT$ and $ImT$ are complex symmetric.

**Proposition 4.8** *Let $T \in \mathcal{C}_C(T)$. Then the following statements hold:*
(i) *Let $R = \dfrac{T + CTC}{2}$ and $S = \dfrac{T - CTC}{2i}$. Then $R$ and $S$ belong to $\mathcal{C}_C(T)$ such that $T = R + iS$ and $[R, S] = 0$, $[R, C] = 0$, and $[S, C] = 0$ hold.*
(ii) *If $T$ is normal, then $\mathrm{Re}\ T \in \mathcal{C}_C(\mathrm{Re}\ T)$ and $\mathrm{Im}\ T \in \mathcal{C}_C(\mathrm{Im}\ T)$.*

**Lemma 4.9** ([17]) *Let $T \in \mathcal{L}(\mathcal{H})$ and let $C$ be a conjugation on $\mathcal{H}$. Then $\sigma(CTC) = \sigma(T)^*$ and $\sigma_a(CTC) = \sigma_a(T)^*$.*

Therefore, if $T$ satisfies $[T, C] = 0$, then $\sigma(T) = \sigma(T)^*$, that is, $\sigma(T)$ is a symmetric set with the real line. For a commuting pair $(T, S) \in \mathcal{L}(\mathcal{H})^2$, $\sigma_T(T, S)$ and $\sigma_{ja}(T, S)$ denote the *Taylor spectrum* and the *joint approximate point spectrum* of $(T, S)$, respectively (see [2] and [19] for more details).

**Corollary 4.10** *Let $T \in \mathcal{C}_C(T)$. Then there exist commuting operators $R$ and $S$ such that the following statements hold:*
(i) *$T = R + iS$ and $(T, R, S)$ is a commuting 3-tuple.*
(ii) *$\sigma(R)$ and $\sigma(S)$ are symmetric sets with the real line.*
(iii) *If $\lambda \in \sigma(T)$, then there exist $\alpha \in \sigma(R)$ and $\beta \in \sigma(S)$ such that $\lambda = \alpha + i\beta$.*
(iv) *If $\alpha \in \sigma(R)$, then there exist $\lambda \in \sigma(T)$ and $\beta \in \sigma(S)$ such that $\lambda = \alpha + i\beta$.*
(v) *If $\beta \in \sigma(S)$, then there exist $\lambda \in \sigma(T)$ and $\alpha \in \sigma(R)$ such that $\lambda = \alpha + i\beta$.*

Remark that the statements (iii), (iv) and (v) hold for the approximate point spectra $\sigma_a(T), \sigma_a(R)$ and $\sigma_a(S)$. Please see [2] for the spectral mapping theorem for the joint approximate point spectrum.

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a conjugation $C$, we define the operator $\alpha_m(T; C)$ by

$$\alpha_m(T; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^j.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an $[m, C]$-symmetric operator if $\alpha_m(T; C) = 0$ (see [5]).

**Theorem 4.11** *If $T \in \mathcal{C}_C(T)$ is an $[m, C]$-symmetric operator, then $CTC - T$ is $m$-nilpotent, i.e., $(CTC - T)^m = 0$.*

**Corollary 4.12** *If $T \in \mathcal{C}_C(T)$ is an $[m, C]$-symmetric operator, then*

$$\sigma_T(CTC, T) = \{(\lambda, \lambda) : \lambda \in \sigma(T)\}.$$

*In this case, it holds $\sigma(CTC) = \sigma(T) = \sigma(T)^*$. Moreover, it holds $\sigma_{ja}(CTC, T) = \{(\lambda, \lambda) : \lambda \in \sigma_a(T)\}$.*

For an operator $T \in \mathcal{L}(\mathcal{H})$, $T$ is said to be *normaloid* if $r(T) = \|T\|$, where $r(T)$ is the spectral radius of $T$.

**Corollary 4.13** *Let $T \in \mathcal{C}_C(T)$ be an $[m, C]$-symmetric operator. If $CTC - T$ is normaloid, then $CTC - T = 0$.*

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a conjugation $C$, we define the operator $\lambda_m(T; C)$ by

$$\lambda_m(T; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^{m-j}.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an $[m, C]$-isometric operator if $\lambda_m(T; C) = 0$. See [4] for properties of $[m, C]$-isometric operators.

**Theorem 4.14** *If $T \in \mathcal{C}_C(T)$ is an $[m, C]$-isometric operator, then $CTCT - I$ is $m$-nilpotent, i.e., $(CTCT - I)^m = 0$.*

**Corollary 4.15** *If $T \in \mathcal{C}_C(T)$ is an $[m, C]$-isometric operator, then $\sigma_T(CTC, T) = \{(\frac{1}{\lambda}, \lambda) : \lambda \in \sigma(T)\}$. In this case, it holds $\sigma(CTC) = \{\frac{1}{\lambda} : \lambda \in \sigma(T)\}$. Moreover, it holds $\sigma_{ja}(CTC, T) = \{(\frac{1}{\lambda}, \lambda) : \lambda \in \sigma_a(T)\}$.*

**Theorem 4.16** *Let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric with a conjugation $C$. Suppose that $T = U|T|$ is the polar decomposition of $T$ where $U = CJ$ and $J$ is a partial conjugation supported on $\overline{ran(|T|)}$, which commutes with $|T|$. Then the following statements are equivalent.*
(i) *$T$ is binormal.*
(ii) *$|T| \in \mathcal{C}_C(|T|)$.*
(iii) *$[|\widetilde{T}^D|, |T|] = 0$ where $\widetilde{T}^D := |T|U$ is the Duggal transform of $T$.*

**Corollary 4.17** *Let $T \in \mathcal{L}(\mathcal{H})$ be such that $T^2$ is normal. Then $|T| \in \mathcal{C}_C(|T|)$.*

**Example 4.18** Let $T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{C}^2$. Then $T$ is complex symmetric with the conjugation $C$ defined by $C(z_1, z_2) = (\overline{z_2}, \overline{z_1})$ for $z_1, z_2 \in \mathbb{C}$. Since $|T| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$, it follows that

$$C|T|C|T| = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \text{ and } |T|C|T|C = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

Hence $T$ is not binormal by Theorem 4.16.

**Example 4.19** Let $\mathcal{H} = \ell^2$ and let $C$ be the canonical conjugation given by $C(\sum_{n=0}^{\infty} x_n e_n) = \sum_{n=0}^{\infty} \overline{x_n} e_n$ with $Ce_n = e_n$ for all $n$. Assume that $T = \begin{pmatrix} S^* & I \\ 0 & S \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$, where $S \in \mathcal{L}(\mathcal{H})$ is the unilateral shift. Then $S$ and $S^*$ commute with the conjugation $C$. Denote the conjugation $\mathcal{C}$ given by $\mathcal{C} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$. Then we obtain that

$$\mathcal{C}T^* - T\mathcal{C} = \begin{pmatrix} C & CS^* \\ CS & 0 \end{pmatrix} - \begin{pmatrix} C & S^*C \\ SC & 0 \end{pmatrix} = 0.$$

Hence $T$ is a complex symmetric operator (cf.[14]). Moreover, since $T = \begin{pmatrix} S^* & I \\ 0 & S \end{pmatrix}$, it follows that $T^*T = \begin{pmatrix} SS^* & S \\ S^* & 2I \end{pmatrix}$ and $TT^* = \begin{pmatrix} 2I & S^* \\ S & SS^* \end{pmatrix}$. So, we have $TT^*T^*T = \begin{pmatrix} 2SS^* + S^{*2} & 2S + 2S^* \\ S^2S^* + SS^{*2} & S^2 + 2SS^* \end{pmatrix}$ and $T^*TTT^* = \begin{pmatrix} S^2 + 2SS^* & SS^{*2} + S^2S^* \\ 2S + 2S^* & S^{*2} + 2SS^* \end{pmatrix}$. Hence $T$ is not binormal. On the other hand, if $S$ is the unilateral shift on $\mathcal{H}$, then $T = S^* \oplus S$ is binormal and complex symmetric.

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Muneo Chō
Department of Mathematics, Kanagawa University, Hiratuka 259-1293, Japan
e-mail: chiyom01@kanagawa-u.ac.jp

Eungil Ko
Department of Mathematics, Ewha Womans University, Seoul 120-750, Korea
e-mail: eiko@ewha.ac.kr

Ji Eun Lee
Department of Mathematics and Statistics, Sejong University, Seoul 143-747, Korea
e-mail: jieunlee7@sejong.ac.kr; jieun7@ewhain.net