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A SURVEY OF CLASS p-$wA(s,t)$ OPERATORS

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1. Introduction

The aim of this paper is to survey recent study of class p-$wA(s,t)$ operators where $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. These results are proved in [7, 8, 21, 22, 23, 25].

Let $T \in B(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$ and let $T = U|T|$ be polar decomposition with $\ker U = \ker |T|$. T is called hyponormal if

$$TT^* \leq T^*T.$$

Aluthge [2] studied p-hyponormal operator

$$|T^*|^{2p} = (TT^*)^p \leq (T^*T)^p = |T|^{2p} \quad (0 < p \leq 1)$$

which is a generalization of hyponormal operator. Aluthge defined Aluthge transform

$$T(1/2, 1/2) = |T|^{1/2}U|T|^{1/2}$$

and proved that if T is p-hyponormal operator with $0 < p \leq 1/2$, then

$$|T(1/2,1/2)^*|^{2p+1} \leq |T|^{2p+1} \leq |T(1/2,1/2)|^{2p+1}$$

by using Furuta's inequality [14].

Ito, Yamazaki, Yanagida, Furuta [17, 15, 28], Yoshino [29] defined generalized Aluthge transform $T(s,t) = |T|^sU|T|^t$ for $0 < s,t$ and studied class $wA(s,t)$ operator definded by

$$|T(s,t)^*|^{\frac{2t}{s+t}} \leq |T|^{2s}, |T|^{2t} \leq |T(s,t)|^{\frac{2t}{s+t}},$$

and it is known that p-hyponormal, log-hyponormal operators are class $wA(s,t)$ for all $s,t > 0$.

Prasad and Tanahashi [22] defined class p-$wA(s,t)$ operator as

$$|T(s,t)^*|^{\frac{2pt}{s+t}} \leq |T|^{2ps}, |T|^{2pt} \leq |T(s,t)|^{\frac{2pt}{s+t}}$$

for $0 < p \leq 1, 0 < s,t$. This is a generalization of $wA(s,t)$ operator and class p-$wA(s,t)$ operators have many interesting properties.

2. Results

Next theorem [8] shows that class of p-$wA(s,t)$ operators are decreasing with $0 < p \leq 1$ and increasing with $0 < s, t \leq 1$. The proof is essentially due to C. Yang and J. Yuan ([30] Proposition 3.4). We showed this theorem at 2016 RIMS conference [9], so we omit the proof.

Theorem 2.1. *If $0 < p_1 < p_2 \leq 1, 0 < s_2 < s_1, 0 < t_2 < t_1$, then a class p_2-$wA(s_2, t_2)$ operator is class p_1-$wA(s_1, t_1)$.*

Next proposition is a direct result of Theorem 2.6 of [22].

Proposition 2.2. *Let $T \in B(\mathcal{H})$ be class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then*

$$|T(s,t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$$

and

$$|T|^{2sp} \geq |T(s,t)^*|^{\frac{2sp}{s+t}}.$$

Hence

$$|T(s,t)|^{\frac{2\rho p}{s+t}} \geq |T|^{2\rho p} \geq |T(s,t)^*|^{\frac{2\rho p}{s+t}} \tag{2.1}$$

for any $\rho \in (0, \min\{s,t\}]$.

Next theorem is Theorem 2.2 of [8].

Theorem 2.3. *Let $T \in B(\mathcal{H})$ be class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. If $Tx = \rho e^{i\theta}x$ for $x \in \mathcal{H}$ with $\rho e^{i\theta} \in \mathbb{C}$ and $0 < \rho$. Then $|T|x = \rho x, Ux = e^{i\theta}x, U^*x = e^{-i\theta}x$ and $T^*x = \rho e^{-i\theta}x$.*

Proof. We may assume $s + t = 1$ by Theorem 2.1. Since

$$T(s,t)|T|^s x = |T|^s Tx = \rho e^{i\theta}|T|^s x,$$

we have

$$T(s,t)^*|T|^s x = \rho e^{-i\theta}|T|^s x$$

by Theorem 4 of [5], because $T(s,t)$ is rp-hyponormal for all $r \in (0, \min\{s,t\}]$. Hence

$$|T(s,t)|^2|T|^s x = T(s,t)^*T(s,t)|T|^s x = \rho^2|T|^s x.$$

This implies

$$|T(s,t)||T|^s x = \rho|T|^s x.$$

Similarly,

$$|T(s,t)^*||T|^s x = \rho|T|^s x.$$

Then

$$\begin{aligned}
\rho^{2rp}\langle|T|^s x, |T|^s x\rangle &= \langle|T(s,t)|^{2rp}\,|T|^s x, |T|^s x\rangle \\
&\geq \langle|T|^{2rp}|T|^s x, |T|^s x\rangle \\
&\geq \langle|T(s,t)^*|^{2rp}\,|T|^s x, |T|^s x\rangle \\
&= \rho^{2rp}\langle|T|^s x, |T|^s x\rangle.
\end{aligned}$$

Since $|T(s,t)|^{2rp} - |T|^{2rp} \geq 0$ and

$$\langle \left(|T(s,t)|^{2rp} - |T|^{2rp}\right) |T|^s x, |T|^s x\rangle = 0,$$

we have

$$|T|^{2rp}|T|^s x = |T(s,t)|^{2rp}|T|^s x = \rho^{2rp}|T|^s x.$$

Hence $|T||T|^s x = \rho|T|^s x$ and $|T|^s \left(|T| - \rho\right) x = 0$. This implies

$$\left(|T| - \rho\right) x \in \ker |T|^s = \ker |T| = \ker U.$$

Hence

$$0 = U \left(|T| - \rho\right) x = \rho e^{i\theta} x - \rho U x,$$

and so

$$Ux = e^{i\theta} x.$$

Also,

$$\begin{aligned} \|(U - e^{i\theta})^* x\|^2 &= \|U^* x\|^2 - \langle U^* x, e^{-i\theta} x\rangle - \langle e^{-i\theta} x, U^* x\rangle + \|x\|^2 \\ &= \|U^* x\|^2 - e^{i\theta}\langle x, Ux\rangle - e^{-i\theta}\langle Ux, x\rangle + \|x\|^2 \\ &\leq \|x\|^2 - \|x\|^2 = 0. \end{aligned}$$

Thus $U^* x = e^{-i\theta} x$ and $T^* x = |T|U^* x = \rho e^{-i\theta} x$.

□

The following theorem is Theorem 2.5 of [8]. The proof is similar to Theorem 2.3, so we omit.

Theorem 2.4. *Let $T \in B(\mathcal{H})$ be class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Let $(T - \rho e^{i\theta})x_n \to 0$ for $x_n \in \mathcal{H}$ with $\|x_n\| = 1$ and $\rho e^{i\theta} \in \mathbb{C}, 0 < \rho$. Then $(|T| - \rho)x_n, (U - e^{i\theta})x_n, (U - e^{i\theta})^* x_n, (T - \rho e^{i\theta})^* x_n \to 0$.*

We say that $\lambda \in \sigma(T)$ belongs to the (Xia's) residual spectrum $\sigma_r^X(T)$ of T if $(T - \lambda)\mathcal{H} \neq \mathcal{H}$ and there exists a positive number $c > 0$ such that

$$\|(T - \lambda)x\| \geq c\|x\| \quad \text{for} \quad x \in \mathcal{H}.$$

By the definition, $\sigma(T)$ is a disjoint union of $\sigma_a(T)$ and $\sigma_r^X(T)$.

Lemma 2.5. *Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T with* $\ker U = \ker |T|$ *and let $T_\alpha = U|T|^\alpha$ with $0 < \alpha$. Then*

$$\begin{gathered} 0 \in \sigma_a(T) \Longleftrightarrow 0 \in \sigma_a(T_\alpha), \\ 0 \in \sigma_r^X(T) \Longleftrightarrow 0 \in \sigma_r^X(T_\alpha), \\ 0 \in \sigma(T) \Longleftrightarrow 0 \in \sigma(T_\alpha). \end{gathered}$$

Proof. Let $0 \in \sigma_a(T)$. Then there exist unit vectors x_n such that $Tx_n \to 0$. Then $|T|x_n = U^*U|T|x_n = U^*Tx_n \to 0$. Hence $T_\alpha x_n = U|T|^\alpha x_n \to 0$ and $0 \in \sigma_a(T_\alpha)$. The converse is similar. Let $0 \notin \sigma(T)$. Then $|T|$ is invertible and U is unitary. Hence $T_\alpha = U|T|^\alpha$ is invertible and $0 \notin \sigma(T_\alpha)$. The converse is similar. Since $\sigma(T)$ is a disjoint union of $\sigma_a(T)$ and $\sigma_r^X(T)$, the proof is completed. □

The following theorem is Theorem 2.5 of [21].

Theorem 2.6. *If $T = U|T| \in B(\mathcal{H})$ is class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$ and if $T_\alpha = U|T|^\alpha$ with $s + t \leq \alpha$, then*

$$\sigma_a(T_\alpha) = \{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma_a(T)\}, \tag{2.2}$$

$$\sigma_r^X(T_\alpha) = \{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma_r^X(T)\}, \tag{2.3}$$

$$\sigma(T_\alpha) = \{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma(T)\}. \tag{2.4}$$

Proof. Let $T = U|T|$ be class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Let $\lambda = re^{i\theta} \in \sigma_a(T) \setminus \{0\}$ with $0 < r$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $(T - re^{i\theta})x_n \to 0$. Hence $(T - re^{i\theta})^* x_n \to 0, (|T| - r)x_n \to 0, (U - e^{i\theta})x_n \to 0$ and $(U - e^{i\theta})^* x_n \to 0$ by Theorem 2.4. Hence $\lambda_\alpha := r^\alpha e^{i\theta} \in \sigma_{ja}(T_\alpha) \subset \sigma_a(T_\alpha)$. Conversely, let $\mu = r' e^\phi \in \sigma_a(T_\alpha) \setminus \{0\}$ with $0 < r'$. Then there exists a sequence unit vectors $\{x_n\}$ such that $(T_\alpha - r' e^\phi)x_n \to 0$. Since T_α is p-$wA(s/\alpha, t/\alpha)$ and $0 < s/\alpha + t/\alpha \leq 1$, we have that $\mu = r' e^\phi \in \sigma_{ja}(T_\alpha)$ by Theorem 2.4. Hence $\mu_{1/\alpha} = (r')^{1/\alpha} e^{i\phi} \in \sigma_{ja}(T) \subset \sigma_a(T)$. Therefore

$$\sigma_a(T_\alpha) \setminus \{0\} = \{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma_a(T)\} \setminus \{0\}. \tag{2.5}$$

Hence we have (2.2) by Lemma 2.5.

Next we show (2.3). Let $\lambda = re^{i\theta} \in \sigma_r^X(T) \setminus \{0\}$ with $0 < r$. We claim $\lambda_\alpha = r^\alpha e^{i\theta} \in \sigma(T_\alpha)$.

Assume that $\lambda_\alpha = r^\alpha e^{i\theta} \notin \sigma(T_\alpha)$. Let J be a closed interval $[1, \alpha]$ (or $[\alpha, 1]$) and let f be an operator valued continuous function $f(x) := T_x - r^x e^{i\theta}$ $(x \in J)$. Then $f(1)$ is semi-Fredholm operator with the Fredholm index

$$\mathrm{ind}(f(1)) = \dim(\ker(T - re^{i\theta})) - \dim(\ker(T - re^{i\theta})^*) \leq -1,$$

and $f(\alpha)$ is invertible (so, it is Fredholm with index 0).

We claim that there exists a real number $x_0 \in J$ such that $f(x_0)$ is not semi-Fredholm. Assume that there exists no such $x \in J$. Since $F(J) = \{f(x) | x \in J\}$ is connected in the set of all semi-Fredholm operators of $\mathcal{H}$ and every operator in $F(J)$ has the same Fredholm index, we have that $f(1)$ and $f(\alpha)$ have same Fredholm index. But this is a contradiction.

Since there exists $x_0 \in J$ such that $f(x_0)$ is not semi-Fredholm, we have

$$r^{x_0} e^{i\theta} \in \sigma(T_{x_0}) \setminus \sigma_r^X(T_{x_0}) = \sigma_a(T_{x_0}).$$

Since $s + t \leq x_0$ and $0 < r$, we have $\lambda = re^{i\theta} \in \sigma_a(T)$ by (2.2). But it is a contradiction. Hence $\lambda_\alpha = r^\alpha e^{i\theta} \in \sigma(T_\alpha)$.

We claim $\lambda_\alpha = r^\alpha e^{i\theta} \notin \sigma_a(T_\alpha)$. Assume $\lambda_\alpha = r^\alpha e^{i\theta} \in \sigma_a(T_\alpha)$. Then $\lambda = re^{i\theta} \in \sigma_a(T)$ by (2.2). But it is a contradiction. Hence

$$\{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma_r^X(T) \setminus \{0\}\} \subset \sigma_r^X(T_\alpha) \setminus \{0\}.$$

Similarly we have

$$\{(r')^{1/\alpha} e^{i\theta} \mid r' e^{i\theta} \in \sigma_r^X(T_\alpha) \setminus \{0\}\} \subset \sigma_r^X(T) \setminus \{0\}.$$

Hence (2.3) holds by Lemma 2.5. Since $\sigma(T)$ is a disjoint union of $\sigma_a(T)$ and $\sigma_r^X(T)$, the proof of (2.4) is completed. $\square$

The following theorem is proved in [21].

Theorem 2.7. *Let $T \in B(\mathcal{H})$ be class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then T is normaloid*

$$\|T\| = r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$$

and isoloid (isolated point of spectrum is a point spectrum).

Proof. Since $T(s,t)$ is $\frac{\rho p}{s+t}$-hyponormal and satisfies

$$|T(s,t)|^{\frac{2\rho p}{s+t}} \geq |T|^{2\rho p} \geq |T(s,t)^*|^{\frac{2\rho p}{s+t}} \tag{2.6}$$

for all $\rho \in (0, \min\{s,t\}]$ by Proposition 2.2, we have

$$\sigma(T(s,t)) = \sigma(|T|^s U |T|^t) = \sigma(U|T|^{s+t}) = \{r^{s+t}e^{i\theta} \mid re^{i\theta} \in \sigma(T)\}$$

by Lemma 6 of [26] and Theorem 2.6. Since $T(s,t)$ is normaloid, we have

$$\begin{aligned} \||T(s,t)|^{\frac{2\rho p}{s+t}}\| &= \||T(s,t)|\|^{\frac{2\rho p}{s+t}} = \|T(s,t)\|^{\frac{2\rho p}{s+t}} \\ &= r\left(T(s,t)\right)^{\frac{2\rho p}{s+t}} = \left(r(T)^{s+t}\right)^{\frac{2\rho p}{s+t}} = r(T)^{2\rho p}, \end{aligned}$$

and

$$\|T\|^{2\rho p} = \||T|\|^{2\rho p} = \||T|^{2\rho p}\| \leq \||T(s,t)|^{\frac{2\rho p}{s+t}}\| = r(T)^{2\rho p}$$

by (2.6). Hence $\|T\| \leq r(T)$ and therefore $\|T\| = r(T)$. Thus T is normaloid.

Next we prove T is isoloid. Let $re^{i\theta}$ be an isolated point of $\sigma(T)$ with $0 \leq r$. Since

$$\sigma(T(s,t)) = \sigma(|T|^s U |T|^t) = \sigma(U|T|^{s+t})$$

by Lemma 6 of [26] and

$$\sigma(U|T|^{s+t}) = \{r^{s+t}e^{i\theta} \mid re^{i\theta} \in \sigma(T)\}$$

by Theorem 2.6, we have $r^{s+t}e^{i\theta}$ is an isolated point of $\sigma(T(s,t))$. We remark $T(s,t)$ is $\frac{\rho p}{s+t}$-hyponormal for any $\rho \in (0, \min\{s,t\}]$ by Proposition 2.2.

Assume $re^{i\theta} = 0$. Since $T(s,t)$ is $\frac{\rho p}{s+t}$-hyponormal, we have $E_0(s,t) = \ker T(s,t)$ where $E_0(s,t)$ is the Riesz idempotent of $T(s,t)$ for $0 \in \operatorname{iso}\sigma(T(s,t))$ by Theorem 5 of [10]. Hence there exists non-zero vector $x \in \mathcal{H}$ such that $T(s,t)x = 0$. Hence $Tx = 0$ by (2.6).

Assume $re^{i\theta} \neq 0$. Then

$$E_{r^{s+t}e^{i\theta}}(s,t) = \ker(T(s,t) - r^{s+t}e^{i\theta}) = \ker((T(s,t) - r^{s+t}e^{i\theta})^*)$$

where $E_{r^{s+t}e^{i\theta}}(s,t)$ is the Riesz idempotent of $T(s,t)$ for $r^{s+t}e^{i\theta} \in \operatorname{iso}\sigma(T(s,t))$ by Theorem 5 of [10]. Hence there exists non-zero vector $x \in \ker(T(s,t) - r^{s+t}e^{i\theta})$ such that $T(s,t)^*x = r^{s+t}e^{-i\theta}x$ and $|T(s,t)|x = |T(s,t)^*|x = r^{s+t}x$ by Theorem 5 of [10]. Then we have

$$\begin{aligned} 0 &= \left\langle \left(|T(s,t)|^{\frac{2\rho p}{s+t}} - r^{2\rho p} \right) x, x \right\rangle \geq \left\langle \left(|T|^{2\rho p} - r^{2\rho p} \right) x, x \right\rangle \\ &\geq \left\langle \left(|T(s,t)^*|^{\frac{2\rho p}{s+t}} - r^{2\rho p} \right) x, x \right\rangle = 0 \end{aligned}$$

by (2.6). Hence $\langle (|T|^{2\rho p} - r^{2rp})\, x, x\rangle = 0$. Since $0 < \rho \le \min\{s,t\}$ is arbitrary, we have $\langle (|T|^{\rho p} - r^{\rho p})\, x, x\rangle = 0$ by the same arguement. Then

$$\begin{aligned}\| (|T|^{\rho p} - r^{\rho p})\, x\|^2 &= \langle (|T|^{\rho p} - r^{\rho p})^2\, x, x\rangle \\ &= \left\langle \left(|T|^{2\rho p} - r^{2\rho p}\right) x, x\right\rangle - 2r^{\rho p}\langle (|T|^{\rho p} - r^{\rho p})\, x, x\rangle = 0.\end{aligned}$$

Hence $(|T|^{\rho p} - r^{\rho p})\, x = 0$ and this implies $|T|x = rx$. Then $U^*Ux = U^*U|T|r^{-1}x = |T|r^{-1}x = x$. Since $r^{s+t}e^{-i\theta}x = T(s,t)^*x = |T|^tU^*|T|^sx = |T|^tU^*r^sx$, we have $|T|^tU^*x = r^te^{-i\theta}x = |T|^te^{-i\theta}x$. Hence $\left(U^* - e^{-i\theta}\right) x \in \ker |T|^t = \ker |T| = \ker U$. Hence $U\left(U^* - e^{-i\theta}\right) x = 0$ and $UU^*x = e^{-i\theta}Ux$. Then

$$U^*x = U^*UU^*x = e^{-i\theta}U^*Ux = e^{-i\theta}x$$

because $U^*Ux = x$. Then

$$\begin{aligned}\| \left(U - e^{i\theta}\right) x\|^2 &= \left\langle \left(U - e^{i\theta}\right) x, \left(U - e^{i\theta}\right) x\right\rangle \\ &= \left\langle \left(U - e^{i\theta}\right)^* \left(U - e^{i\theta}\right) x, x\right\rangle \\ &= \left\langle U^*Ux - e^{-i\theta}\left(U - e^{i\theta}\right) x - e^{i\theta}\left(U^* - e^{-i\theta}\right) x - x, x\right\rangle \\ &= \left\langle -e^{-i\theta}x, \left(U - e^{i\theta}\right)^* x\right\rangle = 0.\end{aligned}$$

Hence $Ux = e^{i\theta}x$. Thus $Tx = U|T|x = re^{i\theta}x$ and T is isoloid.

□

Theorem 2.8. *Let $T \in B(\mathcal{H})$ be class p-$wA(s,t)$ with $0 < p \le 1$ and $0 < s, t, s+t \le 1$ and $\sigma(T) = \{\lambda\}$. Then $T = \lambda$.*

Proof. Let $\lambda = 0$. Since T is normaloid by Theorem 2.7, we have $\|T\| = r(T) = 0$. Hence $T = 0$. Let $\lambda \ne 0$. Then $S := T/\lambda$ is class p-$wA(s,t)$ and $\sigma(S) = \{1\}$. Hence $\|S\| = r(S) = 1$ by Theorem 2.7. Since S^{-1} is class p-$wA(t,s)$ by [22], we have $\|S^{-1}\| = r(S^{-1}) = 1$ by Theorem 2.7. This implies $S = 1$. Hence $T = \lambda$.

□

The following theorem is proved in [25].

Theorem 2.9. *Let $T \in B(\mathcal{H})$ be class p-$wA(s,t)$ with $0 < p \le 1$ and $0 < s, t, s+t \le 1$. If $T(s,t)$ is quasinormal, then T is quasinormal. Also, if $T(s,t)$ is normal, then T is normal.*

Proof. Since T is a class p-$wA(s,t)$ operator,

$$|T(s,t)|^{\frac{2rp}{s+t}} \ge |T|^{2rp} \ge |T(s,t)^*|^{\frac{2rp}{s+t}} \tag{2.7}$$

for all $r \in (0, \min\{s,t\}]$. Then Douglas's theorem [11] implies that

$$\operatorname{ran} |T(s,t)|^{\frac{rp}{s+t}} \supset \operatorname{ran} |T|^{rp} \supset \operatorname{ran} |T(s,t)^*|^{\frac{rp}{s+t}}.$$

Hence

$$[\operatorname{ran} |T(s,t)|] \supset [\operatorname{ran} |T|] \supset [\operatorname{ran} |T(s,t)^*|] = [\operatorname{ran} T(s,t)]$$

where $[\mathcal{M}]$ denotes the norm closure of $\mathcal{M} \subset \mathcal{H}$. Since $\ker |T| \subset \ker(|T|^s U |T|^t) = \ker T(s,t)$, we have

$$\begin{aligned}[\operatorname{ran} |T|] = (\ker |T|)^{\perp} &\supset (\ker T(s,t))^{\perp} \\ &= (\ker |T(s,t)|)^{\perp} = [\operatorname{ran} |T(s,t)|].\end{aligned}$$

Hence

$$[\operatorname{ran} |T(s,t)|] = [\operatorname{ran} |T|].$$

Let $T(s,t) = W|T(s,t)|$ be the polar decomposition of $T(s,t)$. Then

$$\begin{aligned} E := W^*W &= U^*U \\ &= \text{ the orthogonal projection onto } [\operatorname{ran} |T|] \\ &\geq \text{ the orthogonal projection onto } [\operatorname{ran} T(s,t)] = WW^* =: F. \end{aligned}$$

Put

$$|T(s,t)^*|^{\frac{1}{s+t}} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$$

on $\mathcal{H} = [\operatorname{ran} T(s,t)] \oplus \ker T(s,t)^*$. Then X is injective and has a dense range. Since $W \subset [\operatorname{ran} T(s,t)]$, we have

$$W = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}.$$

Since $T(s,t)$ is quasinormal, W commutes with $|T(s,t)|$ and

$$\begin{aligned}|T(s,t)|^{\frac{2rp}{s+t}} &= W^*W|T(s,t)|^{\frac{2rp}{s+t}} = W^*|T(s,t)|^{\frac{2rp}{s+t}}W \\ &\geq W^*|T|^{2rp}W \geq W^*|T(s,t)^*|^{\frac{2rp}{s+t}}W = |T(s,t)|^{\frac{2rp}{s+t}}.\end{aligned}$$

Hence

$$\begin{aligned}|T(s,t)|^{\frac{2rp}{s+t}} &= W^*|T(s,t)|^{\frac{2rp}{s+t}}W \\ &= W^*|T(s,t)^*|^{\frac{2rp}{s+t}}W = W^*|T|^{2rp}W\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} X^{2rp} & 0 \\ 0 & 0 \end{pmatrix} &= |T(s,t)^*|^{\frac{2rp}{s+t}} = W|T(s,t)|^{\frac{2rp}{s+t}}W^* \\ &= WW^*|T(s,t)|^{\frac{2rp}{s+t}}WW^* = WW^*|T|^{2rp}WW^*.\end{aligned} \tag{2.8}$$

Since $WW^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, (2.8) implies that $|T(s,t)|^{\frac{2rp}{s+t}}$ and $|T|^{2rp}$ are of the forms

$$|T(s,t)|^{\frac{2rp}{s+t}} = \begin{pmatrix} X^{2rp} & 0 \\ 0 & Y^{2rp} \end{pmatrix} \geq |T|^{2rp} = \begin{pmatrix} X^{2rp} & 0 \\ 0 & Z^{2rp} \end{pmatrix} \tag{2.9}$$

where $Y, Z \geq 0$. Since X is injective and has a dense range and $[\operatorname{ran} |T(s,t)|] = [\operatorname{ran} |T|]$, we have

$$[\operatorname{ran} Y] = [\operatorname{ran} Z] = [\operatorname{ran} |T|] \ominus [\operatorname{ran} T(s,t)] = \ker T(s,t)^* \ominus \ker T.$$

Since W commutes with $|T(s,t)|$ and $|T(s,t)|^{\frac{1}{s+t}}$, we have

$$\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} W_1X & W_2Y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} XW_1 & XW_2 \\ 0 & 0 \end{pmatrix}.$$

So $W_1X = XW_1$ and $W_2Y = XW_2$, and hence $[\operatorname{ran} W_1]$ and $[\operatorname{ran} W_2]$ are reducing subspaces of X. Since $W^*W|T(s,t)| = |T(s,t)|$, we have $W^*W|T(s,t)|^{\frac{1}{s+t}} = |T(s,t)|^{\frac{1}{s+t}}$. Then

$$\begin{pmatrix} W_1^*W_1X & W_1^*W_2Y \\ W_2^*W_1X & W_2^*W_2Y \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

Hence $W_1^*W_1 = 1, W_2^*W_2Y = Y$ and

$$X^k = W_1^*W_1X^k = W_1^*X^kW_1,$$
$$Y^k = W_2^*W_2Y^k = W_2^*X^kW_2$$

for all $k = 1, 2, \cdots$. Put $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$. Then $T(s,t) = |T|^sU|T|^t = W|T(s,t)|$ implies

$$\begin{pmatrix} X^s & 0 \\ 0 & Z^s \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} X^t & 0 \\ 0 & Z^t \end{pmatrix} = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X^{s+t} & 0 \\ 0 & Y^{s+t} \end{pmatrix}$$

and

$$\begin{pmatrix} X^sU_{11}X^t & X_sU_{12}Z^t \\ Z^sU_{21}X^t & Z_sU_{22}Z^t \end{pmatrix} = \begin{pmatrix} W_1X^{s+t} & W_2Y^{s+t} \\ 0 & 0 \end{pmatrix}.$$

Then

$$X^sU_{11}X^t = W_1X^{s+t} = X^sW_1X^t,$$
$$X^sU_{12}Z^t = W_2Y^{s+t} = X^{s+t}W_2$$

and

$$X^s(U_{11} - W_1)X^t = 0,$$
$$X^s(U_{12}Z^t - X^tW_2) = 0.$$

Since X is injective and has a dense range, we have $U_{11} = W_1$ and $U_{12}Z^t = X^tW_2$. Hence $U_{11}^*U_{11} = W_1^*W_1 = 1$. Since U^*U is the orthogonal projection onto $[\operatorname{ran}|T|] \supset [\operatorname{ran} T(s,t)]$ and

$$U^*U = \begin{pmatrix} 1 + U_{21}^*U_{21} & U_{11}^*U_{12} + U_{21}^*U_{22} \\ U_{12}^*U_{11} + U_{22}^*U_{21} & U_{12}^*U_{12} + U_{22}^*U_{22} \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

on $\mathcal{H} = [\operatorname{ran} T(s,t)] \oplus \ker T(s,t)^*$, we have $U_{21} = 0$, $U_{12}^*U_{11} = 0$ and

$$U^*U = \begin{pmatrix} 1 & 0 \\ 0 & U_{12}^*U_{12} + U_{22}^*U_{22} \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $U_{12}Z^t = X^tW_2$, we have

$$Z^{2t} \geq Z^tU_{12}^*U_{12}Z^t = W_2^*X^{2t}W_2 = Y^{2t}.$$

Since $0 < \dfrac{rp}{t} \leq 1$, we have

$$\begin{aligned} Z^{2rp} &\geq (Z^t U_{12}^* U_{12} Z^t)^{\frac{rp}{t}} \\ &= \left(W_2^* X^{2t} W_2\right)^{\frac{rp}{t}} = Y^{2rp} \geq Z^{2rp} \end{aligned}$$

by Lowner-Heinz's inequality and (2.9). Hence

$$(Z^t U_{12}^* U_{12} Z^t)^{\frac{rp}{t}} = Z^{2rp} = Y^{2rp},$$

so $Z = Y$ and

$$|T(s,t)| = |T|^{s+t}.$$

Since

$$\begin{aligned} Z^{2t} &= Z^t U_{12}^* U_{12} Z^t \leq Z^t U_{12}^* U_{12} Z^t + Z^t U_{22}^* U_{22} Z^t \\ &= Z^t \left(U_{12}^* U_{12} + U_{22}^* U_{22}\right) Z^t \leq Z^{2t}, \end{aligned}$$

we have $Z^t U_{22}^* U_{22} Z^t = 0$ and $Z^t U_{22}^* = 0$. This implies that $[\operatorname{ran} U_{22}^*] \subset \ker Z$. On the other hand $U^* = U^* U U^*$ implies

$$\begin{aligned} \begin{pmatrix} U_{11}^* & 0 \\ U_{12}^* & U_{22}^* \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & U_{12}^* U_{12} + U_{22}^* U_{22} \end{pmatrix} \begin{pmatrix} U_{11}^* & 0 \\ U_{12}^* & U_{22}^* \end{pmatrix} \\ &= \begin{pmatrix} U_{11}^* & 0 \\ (U_{12}^* U_{12} + U_{22}^* U_{22}) U_{12}^* & (U_{12}^* U_{12} + U_{22}^* U_{22}) U_{22}^* \end{pmatrix}. \end{aligned}$$

Hence $U_{22}^* = (U_{12}^* U_{12} + U_{22}^* U_{22}) U_{22}^*$ and

$$\begin{aligned} \operatorname{ran} U_{22}^* &\subset [\operatorname{ran}\,(U_{12}^* U_{12} + U_{22}^* U_{22})] \\ &= [\operatorname{ran} U^* U] \ominus [\operatorname{ran} T(s,t)] \\ &= [\operatorname{ran} |T|] \ominus [\operatorname{ran} T(s,t)] = [\operatorname{ran} Z]. \end{aligned}$$

Hence

$$\operatorname{ran} U_{22}^* \subset \ker Z \cap [\operatorname{ran} Z] = \{0\}.$$

Hence $U_{22} = 0$. Then $U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} W_1 & U_{12} \\ 0 & 0 \end{pmatrix}$ and

$$\operatorname{ran} U \subset [\operatorname{ran} T(s,t)] \subset [\operatorname{ran} |T|] = \operatorname{ran} E.$$

Hence $EU = U$. Since W commutes with $|T(s,t)| = |T|^{s+t}$ and $|T|$, we have

$$|T|^s (W - U) |T|^t = W|T|^{s+t} - |T|^s U |T|^t = W|T(s,t)| - T(s,t) = 0.$$

Hence $E(W - U)E = EWE - EUE = 0$. Since $E = U^* U = W^* W$ and

$$[\operatorname{ran} W] \subset [\operatorname{ran} T(s,t)] \subset [\operatorname{ran} |T|] = \operatorname{ran} E,$$

we have $EW = W$. Then

$$\begin{aligned} U &= UU^*U = UE = EUE \\ &= EWE = WE = WW^*W = W. \end{aligned}$$

Thus $U = W$. Since W commutes with $|T(s,t)|$, we have U commutes with $|T|$. Therfore T is quasinormal.

If $T(s,t)$ is normal, then T is quasinormal by the preceeding arguments. Hence $T(s,t) = U|T|^{s+t}$ and $T(s,t)^* = |T|^{s+t}U^*$. Thus

$$|T|^{2(s+t)} = |T(s,t)|^2 = |T(s,t)^*|^2 = |T^*|^{2(s+t)}.$$

This implies that $|T| = |T^*|$ and therefore T is normal. □

The following theorem is Theorem 7.1 of [23].

Theorem 2.10. *Let $T \in B(\mathcal{H})$ be class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then*

$$\left\| |T(s,t)|^{\frac{2\min\{sp,tp\}}{s+t}} - |T|^{2\min\{sp,tp\}} \right\| \leq \left\| |T(s,t)|^{\frac{2\min\{sp,tp\}}{s+t}} - |(T(s,t))^*|^{\frac{2\min\{sp,tp\}}{s+t}} \right\|$$
$$\leq \frac{\min\{sp,tp\}}{\pi} \iint_{\sigma(T)} r^{2\min\{sp,tp\}-1}\, dr d\theta.$$

Moreover, if meas $(\sigma(T)) = 0$, then T is normal.

Proof. Assume that $0 < t \leq s$. Since T is class p-$wA(s,t)$, we have

$$|T(s,t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp} \geq |T(s,t)^*|^{\frac{2tp}{s+t}}$$

by Proposition 2.2. Hence

$$\left\| |T(s,t)|^{\frac{2tp}{s+t}} - |T|^{2tp} \right\| \leq \left\| |T(s,t)|^{\frac{2tp}{s+t}} - |(T(s,t))^*|^{\frac{2tp}{s+t}} \right\|$$
$$\leq \frac{tp}{\pi(s+t)} \iint_{\sigma(T(s,t))} \rho^{\frac{2tp}{s+t}-1}\, d\rho d\theta.$$

where $\rho e^{i\theta} \in \sigma(T(s,t))$ by Theorem 5 of [5]. Since

$$\sigma(T(s,t)) = \sigma(|T|^s U |T|^t) = \sigma(U|T|^{s+t}) = \{r^{s+t}e^{i\theta} | re^{i\theta} \in \sigma(T)\}$$

by Lemma 6 of [26] and Theorem 2.6, we have

$$\frac{tp}{\pi(s+t)} \iint_{\sigma(T(s,t))} \rho^{\frac{2tp}{s+t}-1}\, d\rho d\theta = \frac{tp}{\pi} \iint_{\sigma(T)} r^{2tp-1}\, dr d\theta$$

by taking $re^{i\theta} = \rho^{\frac{1}{s+t}} e^{i\theta} \in \sigma(T)$. The proof of the case $0 < s \leq t$ is similar.

If $meas(\sigma(T)) = 0$, then $|T(s,t)| = |(T(s,t))^*|$ and T is normal by Theorem 2.9. □

Next, we investigate subscalarity of class p-$wA(s,t)$ operator. Let $\mathcal{X}$ be a complex Banach space and $\mathcal{U} \subset \mathbb{C}$ be an open subset. Let $\mathcal{O}(\mathcal{U}, \mathcal{X})$ denote the Fréchet space of all analytic $\mathcal{X}$-valued functions on $\mathcal{U}$ with the topology of uniform convergence on compact subsets of $\mathcal{U}$. Also, Let $\mathcal{E}(\mathcal{U}, \mathcal{X})$ denote the Fréchet space of all infinitely differentiable $\mathcal{X}$-valued functions on $\mathcal{U}$ with the topology of uniform convergence of all derivatives on compact subsets of $\mathcal{U}$. We say that T satisfy Bishop's property (β) if

$$(T - z)f_n(z) \to 0 \text{ in } \mathcal{O}(\mathcal{U}, \mathcal{X}) \Longrightarrow f_n(z) \to 0 \text{ in } \mathcal{O}(\mathcal{U}, \mathcal{X})$$

for every open set $\mathcal{U} \subset \mathbb{C}$ and $f_n(z) \in \mathcal{O}(\mathcal{U}, \mathcal{X})$. E. Albrecht and J. Eschmeier [1] proved that $T \in B(\mathcal{X})$ satisfies Beshop's property (β) if and only if T is subdecomposable, i.e., T is a restriction of a decomposable operator.

We say that T satisfy Eschmeier-Putinar-Bishop's property $(\beta)_\epsilon$ if

$$(T-z)f_n(z) \to 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X}) \Longrightarrow f_n(z) \to 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X})$$

for every open set $\mathcal{U} \subset \mathbb{C}$ and $f_n(z) \in \mathcal{E}(\mathcal{U},\mathcal{X})$. J. Eschmeier and M. Putinar [12] proved that $T \in B(\mathcal{X})$ satisfies Eschmeier-Putinar-Bishop's property $(\beta)_\epsilon$ if and only if T is subscalar, i.e., T is a restriction of a scalar operator.

The following theorem is Theorem 2.4 of [25].

Theorem 2.11. *If $T \in B(\mathcal{H})$ is class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, then T satisfies Bishop's property (β) and Eschmeier-Putinar-Bishop's property $(\beta)_\epsilon$. Hence T has single valued extension property and T is subscalar.*

Proof. We may assume $s+t=1$ by Theorem 2.1. Then $T(s,t)$ is $\frac{\min(sp,tp)}{2}$-hyponormal by Proposition 2.2. Hence $T(s,t)$ satisfies Bishop's property (β) and Eschmeier-Putinar-Bishop's property $(\beta)_\epsilon$ by [4, 18]. Then T satisfies Bishop's property (β), Eschmeier-Putinar-Bishop's property $(\beta)_\epsilon$ by Theorem 2.1 of [3] and T is subscalar. □

The following theorem is Theorem 5.1 of [23].

Theorem 2.12. *Let $T \in B(\mathcal{H})$ be class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then the following assertions hold.*

(i) *Weyl's theorem holds for T.*

(ii) *$\sigma_w(f(T)) = f(\sigma_w(f(T)))$ for every $f \in H(\sigma(T))$.*

(iii) *Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

To prove Theorem 2.12, we prepare the following result.

Lemma 2.13. *Let $T \in B(\mathcal{H})$ be class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. If T is Fredholm, then ind$(T) \leq 0$.*

Proof. Take a positive number $1 \leq \alpha$ such that $\alpha(s+t)=1$. Since T is Fredholm, $|T|^{\alpha s}$ is also Fredholm and $\mathrm{ind}(|T|^{\alpha s}) = 0$. Then

$$\mathrm{ind}(T) = \mathrm{ind}(|T|^{\alpha s}T) = \mathrm{ind}(T(\alpha s,\alpha t)|T|^{\alpha s}) = \mathrm{ind}(T(\alpha s,\alpha t)).$$

Since $T(\alpha s,\alpha t)$ is ρp-hyponormal for any $\rho \in (0,\min\{\alpha s,\alpha t\}]$ by Proposition 2.2, we have $\mathrm{ind}(T(\alpha s,\alpha t)) \leq 0$ by Theorem 4 of [5]. Thus $\mathrm{ind}(T) \leq 0$. □

Proof of Theorem 2.12. (i) Let $\lambda \in \sigma(T)\setminus\sigma_w(T)$, then $T-\lambda$ is Fredholm, $\mathrm{ind}(T-\lambda)=0$ and $0 < \dim\ker(T-\lambda) < \infty$. If λ is an interior point of $\sigma(T)$, there exists an open subset G such that $\lambda \in G \subset \sigma(T)\setminus\sigma_w(T)$. Then $\dim\ker(T-\mu) > 0$ for all $\mu \in G$ and T does not have the single valued extension property by Theorem 9 of [13]. But this is impossible by Theorem 2.11. Hence $\lambda \in \partial\sigma(T)$. Then $\lambda \in \mathrm{iso}\,\sigma(T)$ by Theorem XI 6.8 of [6]. Thus $\lambda \in \pi_{00}(T)$.

Let $\lambda \in \pi_{00}(T)$. Take a positive number $1 \leq \alpha$ such that $\alpha(s+t)=1$. Since $\sigma(T) = \sigma(T(\alpha s,\alpha t))$, we have $\lambda \in \mathrm{iso}\,\sigma(T(\alpha s,\alpha t))$. Since $T(\alpha s,\alpha t)$ is ρp-hyponormal for any $\rho \in (0,\min\{\alpha s,\alpha t\}]$ by Proposition 2.2, we have $E_\lambda = E_\lambda(\alpha s,\alpha t)$ and $\dim(E_\lambda\mathcal{H}) = \dim(\ker(T-\lambda)) < \infty$ by Theorem 3.6 of [23]. Thus $\lambda \in \sigma(T)\setminus\sigma_w(T)$ by Proposition XI 6.9 of [6].

(ii) Since $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$ is always true for any operator by Theorem 2(b) of [16], we prove that $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$. We may assume that $f \in H(\sigma(T))$ is not constant. Let $\lambda \notin \sigma_w(f(T))$ and

$$f(z) - \lambda = (z - \lambda_1) \cdots (z - \lambda_k) g(z)$$

where $\{\lambda_i : i = 1, \cdots, k\}$ are the zeros of $f(z) - \lambda$ in G (listed according to multiplicity) and $g(z) \neq 0$ for each $z \in G$. Then

$$f(T) - \lambda = (T - \lambda_1) \cdots (T - \lambda_k) g(T).$$

Since $\lambda \notin \sigma_w(f(T))$ and $\sigma_e(f(T)) \subset \sigma_w(f(T))$, we have $\lambda \notin \sigma_e(f(T)) = f(\sigma_e(T))$. Hence $T - \lambda_j$ is Fredholm for all $j = 1, \cdots, k$. Then

$$\begin{aligned} 0 = \text{ind } (f(T) - \lambda) &= \text{ind } (g(T)) + \sum_{j=1}^{k} \text{ind } (T - \lambda_j) \\ &= \sum_{j=1}^{k} \text{ind } (T - \lambda_j) \leq 0 \end{aligned}$$

by Lemma 2.13. Hence ind $(T - \lambda_j) = 0$ for all $j = 1, \cdots, k$. This implies that $T - \lambda_j$ is Weyl and $\lambda_j \notin \sigma_w(T)$. Thus $\lambda \notin f(\sigma_w(T))$.

(iii) Since T is isoloid by Theorem 2.7, we have

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T))$$

from [20]. On the other hand, we have

$$f(\sigma(T) \setminus \pi_{00}(T)) = f(\sigma_w(T)) = \sigma_w(f(T))$$

by (ii). Thus Weyl's theorem holds for $f(T)$. □

Two operators $S \in B(\mathcal{H}), T \in B(\mathcal{K})$ is called quasisimilar if there exist injective operators $X \in B(\mathcal{H}, \mathcal{K}), Y \in B(\mathcal{K}, \mathcal{H})$ with dense rages such that $SX = XT$ and $YS = TY$. This equivalence relation of quasisimilarity was introduced by Sz.-Nagy and Foias and has received considerable attention. In general, quasisimilarity need not preserve the spectrum and essential spectrum. However, quasisimilarity preserves spectra in special classes of operators. For instance, if T and S are quasisimilar hyponormal operators then $\sigma(T) = \sigma(S)$ by Corollary 3 of [24] and $\sigma_e(T) = \sigma_e(S)$ by Theorem 2.4 of [27].

The following theorem is Corollary 1 of [7].

Theorem 2.14. *Let $S \in B(\mathcal{H})$ and $T \in B(\mathcal{K})$ be quasisimilar class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then $\sigma(S) = \sigma(T)$ and $\sigma_e(S) = \sigma_e(T)$.*

Proof. Since S and T satisfies Bishop's property (β) by Theorem 2.11, we have $\sigma(S) = \sigma(T)$ and $\sigma_e(S) = \sigma_e(T)$ by Theorem 3.7.15 of [19].

□

The following theorem is Theorem 6 of [7]. The proof is complicated, so we omit.

Theorem 2.15. *Let $S \in B(\mathcal{H})$ and $T^* \in B(\mathcal{K})$ be class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ and $\ker S \subset \ker S^*, \ker T^* \subset \ker T$. Let $SX = XT$ for some operator $X \in B(\mathcal{K},\mathcal{H})$. Then $S^*X = XT^*, [\text{ran } X]$ reduces $S, (\ker X)^\perp$ reduces T, and $S|_{[\text{ran } X]}, T|_{(\ker X)^\perp}$ are unitarily equivalent normal operators.*

Questions
(1) If T is class p-$wA(s,t)$ and $\mathcal{M}$ is T-invariant, then $T|_{\mathcal{M}}$ is p-$wA(s,t)$?
(2) If T is class p-$wA(s,t)$ and $T|_{\mathcal{M}}$ is normal, then $\mathcal{M}$ reduces T?

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