On multimap classes in the KKM theory

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Abstract

In the last two decades, we introduced the admissible multimap class $\mathfrak{A}_c^\kappa$, the better admissible class $\mathfrak{B}$, and the KKM admissible classes $\mathfrak{KC}$, $\mathfrak{KO}$ in the frame of the KKM theory. Our aim in this talk is to introduce the basic properties of our multimap classes and some mutual relations among them in general topological spaces or our abstract convex spaces. We add some new remarks and further comments to improve many of those results, and introduce some recent applications of our multimap classes.

1. Introduction

Since Kakutani obtained his celebrated fixed point theorem for convex-valued u.s.c. multimaps in 1941 and Eilenberg and Montgomerry extended it for acyclic maps in 1948, there have appeared many types of multimaps with applications in various fields in mathematics, economics, game theory, natural sciences, engineering, and others. In 1992, the author [13] obtained some coincidence theorems on acyclic maps and their applications to the newly named KKM theory originated from the celebrated intersection theorem of Knaster, Kuratowski and Mazurkiewitz in 1929. Since then a large number of applications or generalizations of some results in [13] have appeared; see [22, 25, 28] and the references therein.

Moreover, in the last two decades, we introduced several multimap classes in the frame of the KKM theory; namely, the admissible multimap class $\mathfrak{A}_c^\kappa$, the better admissible class $\mathfrak{B}$, and the KKM admissible classes $\mathfrak{KC}$, $\mathfrak{KO}$. Each of these classes contains a large number of particular multimaps.

In our previous work [22], we reviewed applications of our fixed point theorems for the multimap class of compact compositions of acyclic maps and, in [25], we collected most of fixed point theorems related to the KKM theory due to the author. Moreover, applications of our versions of the Fan-Browder fixed point theorem were introduced in [26]. Furthermore, in a recent work [28], we reviewed applications of our fixed point

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theorems and our multimap classes, appeared mainly in other authors' works. Most of them are not treated in [22, 25, 26].

Our aim in this talk is to introduce the basic properties of our multimap classes and some mutual relations among them in general topological spaces or our abstract convex spaces. We add some new remarks and further comments to improve some of those results, and introduce some recent applications of our multimap classes. This would be informative to peoples working in certain related fields.

For a preliminary on abstract convex spaces, the reader can consult with our previous works; see [24] for the references. Since 2007 such spaces became the main theme of the KKM theory and many new results on them have appeared mainly by the present author.

This review article is organized as follows. Section 2 deals with the admissible multimap class $\mathfrak{A}_c^\kappa$, which has been studied by a large number of authors. In Section 3, basic facts on the better admissible multimap classes $\mathfrak{B}$ and various fixed point theorems on them are introduced. Finally, Section 4 deals with the KKM admissible multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$ and some of their applications by other authors.

2. Admissible map class $\mathfrak{A}_c^\kappa$

Let X and Y be topological spaces. A polytope is a homeomorphic image of a simplex. The following due to the author is well-known:

Definition. An *admissible class* $\mathfrak{A}_c^\kappa(X,Y)$ of maps $T : X \multimap Y$ is the one such that, for each compact subset K of X, there exists a map $S \in \mathfrak{A}_c(K,Y)$ satisfying $S(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite compositions of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(1) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(2) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and

(3) for each polytope P, each $T \in \mathfrak{A}_c(P,P)$ has a fixed point, where the intermediate spaces of compositions are suitably chosen for each $\mathfrak{A}$.

Example. Examples of the multimap class $\mathfrak{A}$ are the classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$, the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$, the Powers maps $\mathbb{V}_c$, the O'Neil maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of uniform spaces), admissible maps of Górniewicz, the Simons maps $\mathbb{K}_c$, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, and others. Further, the Fan-Browder maps (codomains are convex sets), locally selectionable maps having convex values, $\mathbb{K}_c^+$ due to Lassonde, $\mathbb{V}_c^+$ due to Park et al., and approximable maps $\mathbb{A}_c^\kappa$ due to Ben-El-Mechaiekh and Idzik are examples of the multimap class $\mathfrak{A}_c^\kappa$.

For the literature, see Park [14-16], Park and H. Kim [30, 31] and the references therein.

Applications. The admissible class due to Park was first applied to the KKM theory and fixed point problems. Later many authors applied the class to various problems; see [28]. We introduce *some* of recent works applying our admissible class.

Agarwal and O'Regan, *Top. Meth. Nonlinear Anal.* 21 (2003) [1]
Agarwal and O'Regan, *Comment. Math.* XLIV(1) (2004) [2]
Agarwal and O'Regan, *Fixed Point Theory Appl.* (2009) [3]
Agarwal, O'Regan and Taoudi, *Asia-European J. Math.* 4 (2011) [4]
O'Regan, *Appl. Math. Comp.* 219 (2012) [9]

3. Better admissible map class $\mathfrak{B}$

The following is the concept of a slightly new multimap class related to the KKM theory:

Definition. Let X and Y be topological spaces. We define *the better admissible class* $\mathfrak{B}$ of maps from X into Y as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ is a map such that, for any natural $n \in \mathbb{N}$, any continuous function $\phi : \Delta_n \to X$, and any continuous function $p : F\phi(\Delta_n) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi} \phi(\Delta_n) \subset X \overset{F}{\multimap} F\phi(\Delta_n) \xrightarrow{p} \Delta_n$$

has a fixed point.

Proposition 3.1. *For any topological spaces X, Y, we have $\mathfrak{A}_c^\kappa(X,Y) \subset \mathfrak{B}(X,Y)$.*

When X is a subset of an abstract convex space, the preceding definition reduces to the following previous one in [23]:

Definition. Let $(E, D; \Gamma)$ be an abstract convex space, X a nonempty subset of E, and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of maps from X into Y as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle D \rangle$ with the cardinality $|N| = n+1$, and for any continuous function $p : F(\Gamma_N) \to \Delta_n$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n) \subset X$.

This concept extends the corresponding one for G-convex spaces appeared in [23], where lots of examples were given.

The above definition also works for ϕ_A-spaces $(X, D; \Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D \rangle$ with the cardinality $|A| = n+1$.

Let X be a convex space and Y a Hausdorff space. More early in 1997 [15], we introduced a better admissible class $\mathfrak{B}$ of multimaps as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ such that, for any polytope P in X and any continuous map $f : F(P) \to P$, $f(F|_P)$ has a fixed point.

The following KKM theorem is due to the author [15, Theorem 3]:

Theorem 3.2. *Let X be a convex space, Y a Hausdorff space, $F \in \mathfrak{B}(X,Y)$ a compact map, and $S : X \multimap Y$ a map. Suppose that*

(1) *for each $x \in X$, $S(x)$ is closed; and*

(2) *for each $N \in \langle X \rangle$, $F(\operatorname{co} N) \subset S(N)$.*

Then $\overline{F(X)} \cap \bigcap\{S(x) \mid x \in X\} \neq \emptyset$.

Later this KKM theorem was applied to a minimax inequality related to admissible multimaps, from which we deduced generalized versions of lopsided saddle point theorems, fixed point theorems, existence of maximizable linear functionals, the Walras excess demand theorem, and the Gale-Nikaido-Debreu theorem.

In 2010, Balaj and Lin [6] showed that the preceding theorem is equivalent to some existence theorems of variational inclusion problems. These are applied to existence theorems of common fixed point, generalized maximal element theorems, a general coincidence theorems and a section theorem.

Example. For a G-convex space $(X, D; \Gamma)$ and any space Y, an admissible class $\mathfrak{A}_c^\kappa(X,Y)$ is a subclass of $\mathfrak{B}(X,Y)$ with some possible exceptions such as Kakutani maps. There are maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$, for example, the connectivity map due to Nash and Girolo; see [16]. Some other examples; see [25].

Recall that a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every nonempty compact subset K of X and every 0-neighborhood $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

In 1998, we obtained the following [16, Theorem 10.1]:

Theorem 3.3. *Let E be a Hausdorff t.v.s. and X an admissible* (*in the sense of Klee*) *convex subset of E. Then any compact closed map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

In [16], it was shown that Theorem 3.3 subsumes more than sixty known or possible particular cases and generalizes them in terms of the involving spaces and multimaps as well. Later, further examples of maps in the class $\mathfrak{B}$ were known.

It is not known whether the admissibility of X can be eliminated in Theorem 3.3. However, Theorem 3.3 can be generalized by switching the admissibility of domain of the map to the Klee approximability of its ranges as follows:

Let X be a subset of a t.v.s. E. A compact subset K of X is said to be *Klee approximable into* X if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope in X.

Example. We give some examples of Klee approximable sets:

(1) If a subset X of E is admissible (in the sense of Klee), then every compact subset K of X is Klee approximable into E.

(2) Any polytope in a subset X of a t.v.s. is Klee approximable into X.

(3) Any compact subset K of a convex subset X in a locally convex t.v.s. is Klee approximable into X.

(4) Any compact subset K of a convex and locally convex subset X of a t.v.s. is Klee approximable into X.

(5) Any compact subset K of an admissible convex subset X of a t.v.s. is Klee approximable into X.

(6) Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then every compact subset K of Y is Klee approximable into X.

Note that (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3).

In 2004 [18], Theorem 3.3 is generalized as follows:

Theorem 3.4. *Let X be a subset of a Hausdorff t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed multimap. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

The following were obtained in 2007 [19], where it should be $\mathfrak{B}^p = \mathfrak{B}$:

Corollary 3.5. *Let X be an almost convex admissible subset of a Hausdorff t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed map. Then F has a fixed point.*

Corollary 3.6. *Let X be an almost convex subset of a locally convex Hausdorff t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed map. Then F has a fixed point.*

One of the most simple known example is that every compact continuous selfmap on an almost convex subset in a Euclidean space has a fixed point. This generalizes the Brouwer fixed point theorem.

Moreover, since the class $\mathfrak{B}(X, X)$ contains a large number of special types of multimap classes, we can apply Theorem 3.4 to them. For example, since any Kakutani map belongs to $\mathfrak{B}$, Theorem 3.4 and Corollaries 3.5 and 3.6 can be applied to them.

Applications. We list some works which treat relatively new applications of our better admissible multimap class:

Park, *PanAm. Math. J.* 18 (2008) [20]

O'Regan and Perán, *J. Math. Anal. Appl.* 380 (2011) [11]

Lu and Zhang, *Comp. Math. Appl.* 64 (2012) [8]

O'Regan and Shahzad, *Advan. Fixed Point Theory* 2 (2012) [12]

O'Regan, *Appl. Anal.* 92 (2013) [10]

4. KKM admissible classes $\mathfrak{KC}$, $\mathfrak{KO}$

Recall that, early in 1994 [14], for a convex space (X, D) and a Hausdorff space Y, it was indicated that an acyclic map $F : X \multimap Y$ or, more generally, a map $F \in \mathfrak{A}_c^\kappa(X, Y)$ belongs to the class $\mathfrak{KC}$. This was the origin of the study of the so-called KKM class of multimaps. Later, in 1997 [31], the fact was extended to G-convex spaces $(X, D; \Gamma)$ instead of convex spaces.

Since then, in the KKM theory on abstract convex spaces, there have appeared multimap classes $\mathfrak{A}_c^\kappa$, KKM, S-KKM, s-KKM, $\mathfrak{B}$, $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$, and various modifications of them. Park [23] reviewed certain mutual relations among such classes. In fact, we showed that the multimap class S-KKM is included in the class $\mathfrak{KC}$, and that most of known fixed point theorems on s-KKM maps follow from the corresponding ones on $\mathfrak{B}$-maps. Consequently, we could unify all the classes KKM, S-KKM and s-KKM to $\mathfrak{KC}$-maps. Note that compact closed maps in the classes KKM and s-KKM belong to the class $\mathfrak{B}$; see [18].

The following is known [21, Lemma 6]:

Proposition 4.1. *Let $(E, D; \Gamma)$ be a G-convex space and Z a topological space. Then*
(1) $\mathbb{C}(E, Z) \subset \mathfrak{A}_c^\kappa(E, Z) \subset \mathfrak{B}(E, Z)$;
(2) $\mathbb{C}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z)$; *and*
(3) $\mathfrak{A}_c^\kappa(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z)$ *if Z is Hausdorff.*

Consider the following condition for a G-convex space $(E \supset D; \Gamma)$:

$(*)$ $\Gamma_{\{x\}} = \{x\}$ for each $x \in D$; and, for each $N \in \langle D \rangle$ with the cardinality $|N| = n+1$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that $\phi_N(\Delta_n) = \Gamma_N$ and that $J \in \langle N \rangle$ implies $\phi_N(\Delta_J) = \Gamma_J$.

Note that every convex space satisfies the condition $(*)$. We had the following [21, Theorem 16]:

Theorem 4.2. *Let $(E, D; \Gamma)$ be a G-convex space and Z a topological space.*

(1) *If Z is a Hausdorff space, then every compact map $F \in \mathfrak{B}(E, Z)$ belongs to $\mathfrak{KC}(E, Z)$.*

(2) *If $F : E \multimap Z$ is a closed map such that $F\phi_N \in \mathfrak{KC}(\Delta_n, Z)$ for any $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, then $F \in \mathfrak{B}(E, Z)$.*

(3) *In the class of closed maps defined on a G-convex space $(E \supset D; \Gamma)$ satisfying condition $(*)$ into a space Z, a map $F \in \mathfrak{KC}(E, Z)$ belongs to $\mathfrak{B}(E, Z)$.*

Remark. In (2), note that for any map $F \in \mathfrak{A}_c^\kappa(E, Z)$, we have $F\phi_N \in \mathfrak{A}_c^\kappa(\Delta_n, Z) \subset \mathfrak{KC}(\Delta_n, Z) \cap \mathfrak{KO}(\Delta_n, Z)$ when Z is Hausdorff.

The following are [21, Corollaries 16.1 and 16.2], respectively.

Corollary 4.3. *In the class of compact closed maps defined on a G-convex space* $(E \supset D;\Gamma)$ *satisfying condition* $(*)$ *into a Hausdorff space* Z*, two subclasses* $\mathfrak{KC}(E,Z)$ *and* $\mathfrak{B}(E,Z)$ *are identical.*

Corollary 4.4. *In the class of compact closed maps defined on a convex space* (X,D) *into a Hausdorff space* Z*, two subclasses* $\mathfrak{KC}(X,Z)$ *and* $\mathfrak{B}(X,Z)$ *are identical.*

Remark. 1. This is noted in [15] by a different method. In view of Corollary 4.4, the class $\mathfrak{B}$ is favorable to use for convex spaces since it has already plenty of examples and is much easier to find examples.

2. Proposition 4.1, Theorem 4.2, Corollaries 4.3 and 4.4 hold also for ϕ_A-spaces $(X,D;\Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D \rangle$ with $|A| = n+1$.

Corollary 4.5. *Let* X *be a subset of a Hausdorff t.v.s.,* I *a nonempty set,* $s : I \to X$ *a map such that* $\operatorname{co} s(I) \subset X$*, and* $T \in s\text{-}KKM(I,X,X)$*. If* T *is closed and compact, then* $T \in \mathfrak{B}(X,X)$*.*

In 2004, the author [18] showed that a compact closed s-KKM map from a convex subset of a t.v.s. into itself belongs to $\mathfrak{B}$ whenever $s : I \to X$ is a surjection.

Corollary 4.6. *Let* X *be a subset of a Hausdorff t.v.s.,* I *a nonempty set,* $s : I \to X$ *a map such that* $\operatorname{co} s(I) \subset X$*, and* Y *a Hausdorff space. Then, in the class of closed compact maps, four classes* $\mathfrak{KC}(X,Y)$*,* $\mathrm{KKM}(X,Y)$*,* $s\text{-}\mathrm{KKM}(I,X,Y)$*, and* $\mathfrak{B}(X,Y)$ *coincide.*

In view of Corollary 4.6, all fixed point theorems on s-KKM maps on a Hausdorff t.v.s. are consequences of corresponding ones on $\mathfrak{B}$-maps.

Moreover, if $F : X \to Y$ is a continuous single-valued map or if $F : X \multimap Y$ has a continuous selection, then it is easy to check that $F \in \mathfrak{KC}(X,Y) \cap \mathfrak{KO}(X,Y)$. Note that there are many known selection theorems due to Michael and others; see [27].

For convex subsets of a t.v.s., from the KKM principle, we had the following almost fixed point theorems for the class $\mathfrak{KC}$ and $\mathfrak{KO}$ [17]:

Theorem 4.7. *Let* X *be a convex subset of a t.v.s.* E *and* $F \in \mathfrak{KC}(X,X)$ *such that* $F(X)$ *is totally bounded. Then for any convex neighborhood* V *of* 0 *in* E*, there exists an* $x_* \in X$ *such that* $F(x_*) \cap (x_* + V) \neq \emptyset$*.*

Theorem 4.8. *Let* X *be a totally bounded convex subset of a t.v.s.* E *and* $F \in \mathfrak{KO}(X,X)$*. Then for each closed convex neighborhood* V *of* 0 *in* E*, there exists an* $x_* \in X$ *such that* $F(x_*) \cap (x_* + V) \neq \emptyset$*.*

Note that E is not necessarily Hausdorff in Theorems 4.7 and 4.8. From Theorem 4.7, we immediately have the following with a routine proof:

Corollary 4.9 *Let* X *be a convex subset of a locally convex Hausdorff t.v.s.* E*. Then any compact closed map* $F \in \mathfrak{KC}(X,X)$ *has a fixed point.*

This is a far-reaching generalization of the Himmelberg fixed point theorem.

Applications. Some relatively recent works related the classes $\mathfrak{KC}$ and $\mathfrak{KO}$ are as follows:

Shahzad, *Nonlinear Anal.* 56 (2004) [32]
Amini, Fakhar and Zafarani, *Nonlinear Anal.* 66 (2007) [5]
Yang, Xu and Huang, *Fixed Point Theory Appl.* (2011) [34]
Yang and Huang, *Bull. Korean Math. Soc.* 49 (2012) [33]
Lu and Hu, *J. Function Spaces Appl.* (2013) [7]

Final Remark. This talk was intended to introduce the contents of our recent work [29], where the reader can find the details.

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