THE FIXED POINT PROPERTY OF
A-DIRECT SUMS OF
N UNIFORMLY NON-SQUARE
BANACH SPACES

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Abstract
We shall show the fixed point property of A-direct sums of N uniformly non-square Banach spaces by characterizing the nontrivialness of Dominguez-Benavides coefficient $R(1, (X_1 \oplus \cdots \oplus X_N)_A)$, that is, $R(1, (X_1 \oplus \cdots \oplus X_N)_A) < 2$.

A norm $\| \cdot \|$ on $\mathbb{R}^N$ is called monotone if $\|a\| \leq \|b\|$ for all $a = (a_j), b = (b_j) \in \mathbb{R}^N$ with $|a_j| \leq |b_j|$ $(j = 1, \ldots, N)$. For $a = (a_j)$ $|a|$ is defined by $|a| = (|a_j|) \in \mathbb{R}^N$. A norm $\| \cdot \|$ on $\mathbb{R}^N$ is called absolute if $\|a\| = \|(|a|)\|$ for all $a \in \mathbb{R}^N$ and normalized if $\|e_j\| = 1$ for all $1 \leq j \leq N$, where $e_j$ is the $j$-th unit vector in $\mathbb{R}^N$.

In [4] and [10] A-direct sums and AN-direct sums of $N$ Banach spaces were introduced respectively by the following: Let $\| \cdot \|_A$ be an arbitrary norm on $\mathbb{R}^N$. The A-direct sum $(X_1 \oplus \cdots \oplus X_N)_A$ is the direct sum of $X_1, \ldots, X_N$ equipped with the norm

$$\|(x_1, \ldots, x_N)_A\| = \|(\|x_1\|, \ldots, \|x_N\|)_A\|, \quad (x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N$$

and an AN-direct sum is an A-direct sum whose norm is defined from some absolute normalized norm $\| \cdot \|_{AN}$ on $\mathbb{R}^N$. It is known that a norm $\| \cdot \|_A$ on $\mathbb{R}^N$ is absolute if and only if it is monotone ([2],[4],[14]).

In [13] Z-direct sums were introduced by the following: Let $\| \cdot \|_Z$ be an monotone norm on $\mathbb{R}^N$. The Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ is the direct sum of $X_1, \ldots, X_N$ equipped with the norm

$$\|(x_1, \ldots, x_N)_Z\| = \|(\|x_1\|, \ldots, \|x_N\|)_Z\|, \quad (x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N$$

Then we see that Z-direct sum and AN-direct sum are A-direct sum. Since an A-direct sum is isometric isomorphic to some AN-direct sum ([4]), then we have the following theorem.

Theorem 1 (cf. [4]). Let $X_1, \ldots, X_N$ be Banach spaces. Let $\| \cdot \|_A$ be an arbitrary norm on $\mathbb{R}^N$. Then the norm of $(X_1 \oplus \cdots \oplus X_N)_A$ is monotone, that is,

$$\|(x_1, \ldots, x_N)_A\| \leq \|(y_1, \ldots, y_N)_A\|$$
for \((x_1, \ldots, x_N), (y_1, \ldots, y_N) \in (X_1 \oplus \cdots \oplus X_N)_A\) with \(\|x_j\| \leq \|y_j\| (j = 1, \ldots, N)\).

As usual \(S_X\) and \(B_X\) stand for the unit sphere and the closed unit ball of \(X\), respectively. A Banach space \(X\) is said to have the fixed point property (resp. weak fixed point property) for nonexpansive mappings if every nonexpansive self-mapping \(T\) of any nonempty bounded closed (resp. weakly compact) convex subset \(C\) of \(X\) has a fixed point \((T\) is called nonexpansive if \(\|Tx - Ty\| \leq \|x - y\|\) for all \(x, y \in C\). In [6] the coefficient \(R(a, X)\) called as Domínguez-Benavides coefficient (cf. [3]) was introduced by the following: For \(0 < a \leq 1\) let

\[
R(a, X) = \sup \left\{ \liminf_{n \to \infty} \|x_n + x\| \right\},
\]

where the supremum is taken over all \(x \in X\) with \(\|x\| \leq a\) and all weakly null sequences \(\{x_n\}_n\) in the unit ball of \(X\) such that

\[
\lim_{n,m \to \infty; n \neq m} \|x_n - x_m\| \leq 1.
\]

In this paper we shall show the fixed point property for nonexpansive mappings of \(A\)-direct sums of \(N\) uniformly non-square Banach spaces by characterizing the nontrivialness of Domínguez-Benavides coefficient \(R(1, (X_1 \oplus \cdots \oplus X_N)_A)\), that is, \(R(1, (X_1 \oplus \cdots \oplus X_N)_A) < 2\).

The following theorem was proved in [6].

**Theorem 2 ([6]).** Let \(X\) be a Banach space. If \(R(a, X) < 1 + a\) for some \(a > 0\), then \(X\) has the weak fixed point property for nonexpansive mappings.

A Banach space \(X\) is called uniformly non-square ([9]) if there exists a constant \(\varepsilon > 0\) such that

\[
\min\{\|x + y\|, \|x - y\|\} \leq 2(1 - \varepsilon) \quad \text{for all } x, y \in S_X.
\]

By Theorem 2 García-Falset et. al. [8] obtained the following remarkable result.

**Theorem 3 ([8]).** Let \(X\) be a uniformly non-square Banach space. Then \(R(1, X) < 2\) and hence \(X\) has the fixed point property for nonexpansive mappings.

In [7] the following notions were introduced.

**Definition 4 ([7]).** For \(a = (a_j) \in \mathbb{R}^N\) let \(\text{supp } a = \{j : a_j \neq 0\}\).

(i) A norm \(\| \cdot \|\) on \(\mathbb{R}^N\) is said to have Property \(T_1^N\) if

\[
\|a\| = \|b\| = \frac{1}{2}\|a + b\| = 1, \ a, b \in \mathbb{R}^N \implies \text{supp } a \cap \text{supp } b \neq \emptyset.
\]

(ii) A norm \(\| \cdot \|\) on \(\mathbb{R}^N\) is said to have Property \(T_\infty^N\) if

\[
\|a\| = \|b\| = \|a + b\| = 1 \implies \text{supp } a \cap \text{supp } b \neq \emptyset.
\]
To show our key result we need the following propositions.

**Proposition 5** ([11]). Let \( \{x_n^{(k)}\}_{n,k}, \{y_n^{(k)}\}_{n,k} \) be double sequences with nonzero terms in a Banach space \( X \) such that

\[
\lim_{k \to \infty} \lim_{n \to \infty} \|x_n^{(k)}\| > 0, \quad \lim_{k \to \infty} \lim_{n \to \infty} \|y_n^{(k)}\| > 0.
\]

Then the following are equivalent.

(i) \( \lim_{k \to \infty} \liminf_{n \to \infty} \|x_n^{(k)} + y_n^{(k)}\| = \lim_{k \to \infty} \liminf_{n \to \infty} (\|x_n^{(k)}\| + \|y_n^{(k)}\|). \)

(ii) \( \lim_{k \to \infty} \liminf_{n \to \infty} \frac{x_n^{(k)}}{\|x_n^{(k)}\|} \|y_n^{(k)}\| = 2. \)

**Proposition 6** ([5]; see also [1, Chapter III, Theorem 1.5]). Let \( \{x_n\} \) be a bounded sequence in a Banach space \( X \). Then \( \{x_n\} \) contains a subsequence \( \{x_{n_k}\} \) such that \( \lim_{k \to \infty} \limsup_{n \to \infty} \|x_{n_k} - x_n\| \) exists.

**Proposition 7** ([16]). Let \( \{x_n\} \) be a weakly null sequence in a Banach space \( X \). Assume that \( \lim_{n,m \to \infty; n \neq m} \|x_n - x_m\| \) exists. Then \( \limsup_{n \to \infty} \|x_n\| \leq \lim_{n,m \to \infty; n \neq m} \|x_n - x_m\| \).

**Proposition 8** ([12]). Let \( a = (a_j), b = (b_j) \in \mathbb{R}^N \) and let a norm \( \| \cdot \|_A \) on \( \mathbb{R}^N \) be monotone. If \( \|a\| = \|b\|, |a_j| \leq |b_j| \) \( (j=1, \ldots, N) \) and \( |a_{j_0}| < |b_{j_0}| \) then \( \|(\chi_{N \setminus \{j_0\}}(j)a_j)\| = \|(b_j)\| \), where \( N = \{1, \ldots, N\} \) and \( \chi_{N \setminus \{j_0\}} \) is the characteristic function of \( N \setminus \{j_0\} \).

By Theorem 1, Propositions 5, 6, 7 and 8 we can prove the following key result.

**Theorem 9.** Let \( X_1, \ldots, X_N \) be Banach spaces and let a norm \( \| \cdot \|_A \) on \( \mathbb{R}^N \) have Property \( T_1^N \). Then \( R(1, (X_1 \oplus \cdots \oplus X_N)_A) < 2 \) if and only if \( R(1, X_j) < 2 \) for all \( 1 \leq j \leq N \).

**Corollary 10** (cf. [15]). Let \( X \) and \( Y \) be Banach spaces and let a norm on \( \mathbb{R}^2 \) be not \( \ell_1 \)-norm. Then \( R(1, (X \oplus A Y)) < 2 \) if and only if \( R(1, X) < 2 \) and \( R(1, Y) < 2 \).

Theorem 9 combined with Theorem 3 yields nontrivialness of Dominguez-Benavides coefficient of \( A \)-direct sums of \( N \) uniformly non-square Banach spaces.

**Theorem 11.** Let \( X_1, \ldots, X_N \) be uniformly non-square Banach spaces and let a norm \( \| \cdot \|_A \) on \( \mathbb{R}^N \) have Property \( T_1^N \). Then \( R(1, (X_1 \oplus \cdots \oplus X_N)_A) < 2 \).

By Theorem 11 and Theorem 2 we obtain our main results.

**Theorem 12.** Let \( X_1, \ldots, X_N \) be uniformly non-square Banach spaces and let a norm \( \| \cdot \|_A \) on \( \mathbb{R}^N \) have Property \( T_1^N \). Then \( (X_1 \oplus \cdots \oplus X_N)_A \) has the fixed point property for nonexpansive mappings.
Theorem 13. Let $X_1, \ldots, X_N$ be uniformly non-square Banach spaces and let a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ have Property $T^N_{\infty}$. Then $(X_1^* \oplus \cdots \oplus X_N^*)_A$ has the fixed point property for nonexpansive mappings, where $(X_1^* \oplus \cdots \oplus X_N^*)_A$ is an $A$-direct sum of $X_1^*, \ldots, X_N^*$ whose norm is defined by the dual norm $\|\cdot\|_A$.

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References
[12] M. Kato and T. Tamura, Weak nearly uniform smoothness of the $\psi$-direct sums $(X_1 \oplus \cdots \oplus X_N)_\psi$, Comment. Math. 52 (2012), 171-198.
[16] T. Tamura, On Domínguez-Benavides coefficient of $\psi$-direct sums $(X_1 \oplus \cdots \oplus X_N)_\psi$ of Banach spaces, Linear and Nonlinear Analysis 3 (2017), 87-99.