Approximate Minimality in Set Optimization
(集合最適化における近似最適性)

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ABSTRACT. This is a reserch note of appriximate minimality for set optimization. We consider a new concept of approximate efficiency for set optimization in terms of set order intervals by using Tanaka's approximate minimality for vector optimization.

1. Introduction

On optimization theory, necessity of weak solutions is associated with solving problems in which we cannot reach exact solutions. Seeing vector cases, we usually assume the pointwise partial ordering given by a convex cone. Loridan [2] introduced ε-efficiency in 1984 and it is used as a standard of weak optimality in vector optimization. However, Loridan's weakness charactorized by an error toward a specific direction plays strange roles in particular cases. In other words, the essenciality of this weakness strongly depends on the shape of given sets. The same can be applied to set cases.

In order to improve it, we apply Tanaka's approximate minimality [1] to set optimality. This minimality is dependent on an ε-neighborhood of each point. Under this notion, an optimal solution has no better points "far away" as opposed to Loridan's where the set of better solutions than an given point may not be bounded. If any better solutions exist nearby, this coincides with Loridan's efficiency.

We introduce this idea to set optimization by defining several neighborhoods of a set and contrast it with known idea.

2. Preliminaries

Throughout this paper, we let

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X : a topological vector space,
C : a convex solid (i.e., $\text{int}C \neq \emptyset$) cone in X,
$\leq_C$: the pointwise ordering between two vectors in X
($x \leq_C y \;:\Leftrightarrow\; y - x \in C$ for $x, y \in X$),
$\preceq_C$: a binary relation between two subsets of X.

Note that $\leq_C$ and $\preceq_C$ are usually denoted by $\leq$ and $\preceq$, respectively.

3. Motivation

Let S be a nonempty subset of $\mathbb{R}^n$, $\mathbb{R}^n_+$ the positive orthant of $\mathbb{R}^n$.

Definition 3.1 (ε-efficient point (Loridan [2], 1984))**.** $\bar{x} \in S$ is an ε-efficient point toward $d \in \mathbb{R}^n$ iff $(\bar{x} - \mathbb{R}^n_+) \cap (\varepsilon d + S \setminus \{\bar{x}\}) = \emptyset$ or equivalently, $\nexists x \in S$ such that $x + \varepsilon d \leq \bar{x}$ and $x \neq \bar{x}$.

Definition 3.2 (ε-approximately efficient point (Tanaka [1], 1996))**.** $\bar{x} \in S$ is an ε-approximately efficient point of S w.r.t.C iff $(\bar{x} - C) \cap (S \setminus B_\varepsilon(\bar{x})) = \emptyset$.

Definition 3.1 is a basic concept of weak optimalities in vector optimization. However, the given error ε toward a specific direction plays strange roles in particular cases. In other words, the essenciality of this weakness strongly depends on the shape of given sets. In this research, we focused on difference between Definition 3.1 and 3.2. Particularly, the following cases distinguish the definitions.

Example. Let $S_1 := \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$ and $S_2 := -\mathbb{R}^2_{++} = \text{int}\mathbb{R}^2_+$.
$(-2/3, -2/3)$ is a $(1/10)$-efficient point toward $(1,1)$ of S_1 and so is a $(1/10)$-approximately efficient point with respect to $\mathbb{R}^2_+$. On the other hand, $(1/10, 1/10)$ is a $(1/10)$-efficient point toward $(1,1)$ of S_2 while it cannot be an ε-approximately efficient point for any $\varepsilon > 0$.

4. Main Research

In this section, our generalized approximate efficiency of set optimization is considered. We let X be a topological vector space, $\mathcal{A}$ a family of bounded subsets of X, $\preceq_C$ a set relation defined as $A \preceq_C B :\Longleftrightarrow (A \subset B - C) \wedge (B \subset A + C)$. This definition called "set-less relation" is commonly introduced in recent papers, which is originally used by Young [3] and Nishnianidze [4], independently.

To begin with, we introduce known weak optimality which is a generalization of Definition 3.1.

Definition 4.1. Let $d \in X$, $\varepsilon > 0$. $\bar{A} \in \mathcal{A}$ is an ε-minimal set with respect to $\preceq$ toward d iff $\varepsilon d + A \preceq \bar{A}$ for some $A \in \mathcal{A} \Longrightarrow \bar{A} \preceq \varepsilon d + A$.

Usually, minimality of sets is given in the form of "non-dominated" optimality. In set-to-set comparison, it is for the most part, too strong that we settled with dominated solutions even if a convex ordering cone is pointed. If specification is needed, we denote $(A \preceq B) \wedge (B \not\preceq A)$ by $A \prec_C B$.

What we have to consider the most to extend approximate efficiency in Definition 3.2 is idea of neighborhoods for sets. in this paper, we prepare different three types of neighborhoods and propose new approximate minimalities for set optimization. The first one describes neighborhoods by the existence of intersection.

Definition 4.2. Let $N(A) := \{S \subset X \mid S \cap A \neq \emptyset\}$. $\bar{A} \in \mathcal{A}$ is an approximately minimal set with respect to $\preceq$ iff $A \preceq \bar{A}$ for some $A \in \mathcal{A} \setminus N(A) \Longrightarrow \bar{A} \preceq A$.

Example. Let $\mathcal{A}_1 := \{A(x) \mid x \in \mathbb{R}^2_{++}\}$ where $A(x) := \{y \in \mathbb{R}^2 \mid (y_1 - x_1)^2 + (y_2 - x_2)^2 \leq 1\}$. In this case, $x' \leq_{\mathbb{R}^2_+} x \Leftrightarrow A(x') \preceq_{\mathbb{R}^2_+} A(x)$. Then, $A(1/10, 1/10)$ is a approximately minimal set with respect to $\mathbb{R}^2_+$ since $A(1/10, 1/10) \cap A(x) \neq \emptyset$ for all $x \leq (1/10, 1/10)$. In this case, $A(1/10, 1/10)$ also satisfies the condition of $(1/10)$-minimality in Definition 4.1.

Example. Let $\mathcal{A}_2 := \{A(x) \mid x \in \mathbb{R}^2_{++}, x_1 > 0\}$. Then, for any $\varepsilon > 0$ and $x \in \mathbb{R}^2$ satisfying $x_1 = \varepsilon$, $A(x)$ is an ε-minimal set toward $(1, 1)$. However, $A(x)$ is not an approximately minimal set since $A(y) \preceq A(x)$ for $y \in \{x \in \mathbb{R}^2 \mid (x_1 = y_1) \wedge (y_2 < x_2 - 2)\}$,which means $A(y) \not\subset N(A(x))$.

However, we have an example which seems to be strange. Consider $\mathcal{A}' := \{A'(\lambda) \mid \lambda \geq 0\}$ where $A'(\lambda) := \{y \in \mathbb{R}^2 \mid y_1 + y_2 \geq -(\lambda + 1), y_2 + y_1 \leq 1, y_1 - y_2 \geq -1, y_1 - y_2 \leq 1\}$. In this case, $\lambda' \leq \lambda \Leftrightarrow A'(\lambda') \preceq_{\mathbb{R}^2_+} A'(\lambda)$. Then, $A'(0)$ is approximately minimal set in $\mathcal{A}'$ as opposed to the fact that we can improve solutions by taking far larger $\lambda \geq 0$. This implies Definition 4.2 may lose sense when dealing with a given family consisting on sets having infinitely many kinds of shapes.

Next, let us impose the order interval of a convex ordering cone. From a vectorial point of view, order intervals $[x, y] := (x + C) \cap (y - C)$ form a Hausdorff topology under some algebraical assumptions ([5]). Note that intervals $[x, y]$ may not be bounded (but algebraically bounded) even if C is pointed. In this paper, an extended form between two sets $[A, B]$ is used to define neighborhoods.

Definition 4.3. Let C be a convex solid pointed ordering cone, $k \in \mathrm{int}C$, $\varepsilon > 0$, and $[A, B] := (A + C) \cap (B - C)$ for $A, B \subset X$. $\bar{A} \in \mathcal{A}$ is an ε-approximately minimal set toward k with respect to $\preceq$ iff $A \setminus [-\varepsilon k + \bar{A}, \varepsilon k + \bar{A}] \preceq \bar{A}$ for some $A \in \mathcal{A} \Longrightarrow \bar{A} \preceq A \setminus [-\varepsilon k + \bar{A}, \varepsilon k + \bar{A}]$.

Definition 4.3 states getting rid of "near points" before considering minimality. Under this definition, $A'(0)$ is clearly not ε-approximately minimal for any $\varepsilon > 0$ in the previous example.

Example. Let $\mathcal{B} := \{B_1, B_2\}$ where $B_1 = \mathbb{R}^2_+ \cap ((1,1) - \mathbb{R}^2_+)$, $B_2 = ((-1,-1) + \mathbb{R}^2_+) \cap (-\mathbb{R}^2_+)$. These two sets contain only $(0, 0)$ in common and $B_2 \prec B_1$, moreover for all $x_1 \in B_1$ and $x_2 \in B_2$, $x_2 \leq_{\mathbb{R}^2_+} x_1$ holds. Then, B_1 cannot be ε-approximately minimal toward $(1, 1)$ for all $\varepsilon \in (0, 1)$ while it is approximately minimal by the Definition 4.2.

Finally, we show an abstract idea. Generally in optimization, we cannot fully recognize how errors come and what it causes. Definition 4.2 and 4.3 provide certain ways of approximation in ideal cases. However each error bound should be treated more flexibly in general case.

Definition 4.4. Let $E \subset X$ satisfying $\theta_X \in \mathrm{cl}E$ and $E_S := E + S$. $\bar{A} \in \mathcal{A}$ is an E-approximately minimal with respect to $\preceq$ iff $A \setminus E_{\bar{A}} \preceq \bar{A}$ for some $A \in \mathcal{A} \Longrightarrow \bar{A} \preceq A \setminus E_{\bar{A}}$.

Example. Let $\mathcal{C} := \{C_1, C_2\}$ where $C_2 := \{x \in -\mathbb{R}^2_+ \mid x_1^2 + x_2^2 \leq 1\}$, $C_1 := \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$. Then, Definition 4.3 cannot distinguish these sets due to the existence of $\varepsilon > 0$. On the other hand, if we assume $E \subset \mathbb{R}^2$, then C_1 is E-approximate minimal in $\mathcal{C}$ but C_2.

By Definition 4.4, E stands for an error bound including the origin at least in the closure of E since any error occurs with a little deviation from its ideal point. Note that this definition does not coincide with the natural non-dominated minimality induced by $\preceq_C$ when $E = \{\theta\}$.

Example. Let $\mathcal{D} := \{D_1, D_2\}$ where $D_1 = \{x \in \mathbb{R}^2 \mid \|x\| \geq 1\}$, $D_2 = D_1 \cap (-\mathbb{R}^2_+)$. Then, $D_2 \prec D_1$ holds while D_1 is $\{\theta\}$-approximately minimal.

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