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レヴィ市場におけるデジタルオプションに対する局所的リスク最小化問題について

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Abstract

The purpose of this paper is to announce the results of my research on local risk minimization problem for digital options in Lévy markets ongoing now (2019 now).

In this paper, we consider a local risk minimization problem for digital option in a Lévy market. To solve the problem, we next consider Malliavin differentiability of indicator functions on canonical Lévy spaces. By using it, we obtain explicit representation of a locally risk-minimizing hedging strategy for digital option in market driven Lévy process.

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1 Introduction

Locally risk-minimizing (LRM) is a well-known hedging method for contingent claims in a quadratic way. By using Malliavin calculus, we can obtain explicit representations of LRM for incomplete market models whose asset price process is described by a solution to a stochastic differential equation (SDE) driven by a Lévy process ([2]).

On the other hand, there is one important derivative security describe by indicator function called digital option. A digital option pays a fixed cash amount if some condition is realized. Mathematical representation of digital (or binary) options are given by

$$\mathbf{1}_{[K,\infty)}(S_T) = \begin{cases} 1 & \text{for } S_T \geq K, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{S_t\}_{t\in[0,T]}$ is a stock price process and $K > 0$ is a constant number that is fixed by the contract. It is popular and important derivative security. Therefore, to study digital options, we consider Malliavin differentiability of indicator functions ([13]).

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In this paper, we first review local risk minimization problem. We next consider Malliavin differentiability of indicator functions on canonical Lévy spaces. By using it, we obtain explicit representation of a locally risk-minimizing hedging strategy for digital option in market driven Lévy process.

2 Local risk minimization

In this section, we review basic notions of local risk minimization problem.

We now consider a incomplete financial market being composed of one risk-free asset and one risky asset with finite time horizon T. For simplicity, we assume that the interest rate of the market is given by 0, that is, the price of the risk-free asset is 1 at all times. The fluctuation of the risky asset is assumed to be given by a semi-martingale S on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$, where the filtration is supposed to be right-continuous, complete and $\mathcal{F}_0$ is trivial. The semi-martingale S has the following decomposition

$$S = S_0 + M + A,$$

where M a square-integrable martingale for which $M_0 = 0$, and with A a predictable process of finite variation $|A|$. We also assume the following assumption.

Assumption 2.1 *S satisfying the so-called structure condition (SC, for short). That is S satisfies*

$$\left\| [M]_T^{1/2} + \int_0^T |dA_s| \right\|_{L^2(\mathbb{P})} < \infty, \tag{2.1}$$

A is absolutely continuous with respect to $\langle M \rangle$ with a density λ satisfies $\mathbb{E}[\langle \int \lambda dM \rangle] < \infty$, we can rewrite the canonical decomposition as $S = S_0 + M + \int \lambda d\langle M \rangle$. Thirdly, the mean-variance trade-off process $K_t := \int_0^t \lambda_s^2 d\langle M \rangle_s$ is finite, that is, K_T is finite $\mathbb{P}$-a.s.

We define locally risk-minimizing (LRM, for short) for a contingent claim $F \in L^2(\mathbb{P})$. We first define L^2-strategy and cost process.

Definition 2.2 1. *Θ_S denotes the space of all $\mathbb{R}$-valued predictable processes ξ satisfying*

$$\mathbb{E}\left[\int_0^T \xi_t^2 d\langle M \rangle_t + \left(\int_0^T |\xi_t dA_t|\right)^2\right] < \infty$$

.

2. *An L^2-strategy is given by a pair $\varphi = (\xi, \eta)$, where $\xi \in \Theta_S$ and η is an adapted process such that $V(\varphi) := \xi S + \eta$ is a right continuous process with $\mathbb{E}[V_t^2(\varphi)] < \infty$ for every $t \in [0, T]$. Note that ξ_t (resp. η_t) represents the amount of units of the risky asset (resp. the risk-free asset) an investor holds at time t.*

3. *For $F \in L^2(\mathbb{P})$, the process $C^F(\varphi)$ defined by $C_t^F(\varphi) := F1_{\{t=T\}} + V_t(\varphi) - \int_0^t \xi_s dS_s$ is called the cost process of $\varphi = (\xi, \eta)$ for F.*

We next introduce the definition of a small perturbation.

Definition 2.3 (Small Perturbation) *A trading strategy $\Delta = (\delta, \varepsilon)$ is called a small perturbation if it satisfies the following:*

1. *δ is bounded,*
2. *$\int_0^T |\delta_t dA_t|$ is bounded,*
3. *$\delta_T = \varepsilon_T = 0$.*

For any subinterval $(s, t]$ of $[0, T]$, we define the small perturbation

$$\Delta|_{(s,t]} := (\delta \mathbf{1}_{(s,t]}, \varepsilon \mathbf{1}_{[s,t)}).$$

We also define partitions $\tau = (t_i)_{0 \le i \le N}$ of the interval $[0, T]$. A partition of $[0, T]$ is a finite set $\tau = \{t_0, t_1, \cdots, t_k\}$ of times with $0 = t_0 < t_1 < \cdots < t_k = T$ and the mesh size of τ is $|\tau| := \max_{t_i, t_{i+1} \in \tau}(t_{i+1} - t_i)$. A sequence $(\tau_n)_{n \in \mathbb{N}}$ is called increasing if $\tau_n \subseteq \tau_{n+1}$ for all n and it tends to the identity if $\lim_{n \to \infty} |\tau_n| = 0$. We next define the locally risk-minimizing.

Definition 2.4 (Locally Risk-minimizing) *For a trading strategy φ, a small perturbation Δ and a partition τ of $[0, T]$ the risk quotient $r^\tau[\varphi, \Delta]$ is defined as follows:*

$$r^\tau(\varphi, \Delta) := \sum_{t_i, t_{i+1} \in \tau} \frac{R_{t_i}(\varphi + \Delta|_{(t_i, t_{i+1}]}) - R_{t_i}(\varphi)}{\mathbb{E}[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} | \mathcal{F}_{t_i}]} \mathbf{1}_{(t_i, t_{i+1}]},$$

where $R_{t_i} = \mathbb{E}[(C_T - C_{t_i})^2 | \mathcal{F}_{t_i}]$. A trading strategy φ is called locally risk-minimizing if

$$\liminf_{n \to \infty} r^{\tau_n}(\varphi, \Delta) \ge 0$$

$\mathbb{P} \otimes \langle M \rangle$-a.e. on $\Omega \times [0, T]$ for every small perturbation Δ and every increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of partitions of $[0, T]$ tending to the identity.

The definition of LRM is very complicated to use. However, under Assumption 2.1, Theorem 1.6 of Schweizer [8] implies that the following definition of LRM is equivalent to original one:

Definition 2.5 *An L^2-strategy φ is said locally risk-minimizing for F if $V_T(\varphi) = 0$ and $C^F(\varphi)$ is a martingale orthogonal to M, that is, $C^F(\varphi)M$ is a martingale.*

Remark 2.6 *Note that φ is not self-financing. In fact, if φ is self-financing, then $C(\varphi)$ is a constant. If there exists a self-financing φ s.t. $V_T(\varphi) = 0$, we have $F = V_0(\varphi) + \int_0^T \xi_s dS_s$. This is a contradiction.*

We next define Föllmer-Schweizer decomposition (FS decomposition, for short).

Definition 2.7 *An $F \in L^2(\mathbb{P})$ admits a Föllmer-Schweizer decomposition if it can be described by*

$$F = F_0 + \int_0^T \xi_t^F dS_t + L_T^F, \tag{2.2}$$

where $F_0 \in \mathbb{R}$, $\xi^F \in \Theta_S$ and L^F is a square-integrable martingale orthogonal to M with $L_0^F = 0$.

Proposition 5.2 of Schweizer [8] shows the following:

Proposition 2.8 (Proposition 5.2 of Schweizer [8]) *Under Assumption 2.1, an LRM $\varphi = (\xi, \eta)$ for F exists if and only if F admits an FS decomposition, and its relationship is given by*

$$\xi_t = \xi_t^F, \quad \eta_t = F_0 + \int_0^t \xi_s^F dS_s + L_t^F - F1_{\{t=T\}} - \xi_t^F S_t.$$

3 Malliavin calculus for canonical Lévy processes

Throughout this report, we consider Malliavin calculus for canonical Lévy processes, based on [4] and [10]. We now begin with preparation of the probabilistic framework and the underlying Lévy process X under which we discuss Malliavin calculus in the sequel. Let $T > 0$ be a finite time horizon, $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ the Wiener space, that is, the usual canonical space for a one-dimensional standard Brownian motion, with the space of continuous functions on $[0, T]$, the σ-algebra generated by the topology of uniform convergence and Wiener measure; and W its coordinate mapping process, that is, a one-dimensional standard Brownian motion with $W_0 = 0$. Let $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J)$ be the canonical Lévy space (see Solé et al. [10], Delong-Imkeller [4] and Suzuki [11, 12]) for a pure jump Lévy process on $[0, T]$ with Lévy measure ν. Now, we assume that $\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$, where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and denote $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_W \times \Omega_J, \mathcal{F}_W \times \mathcal{F}_J, \mathbb{P}_W \times \mathbb{P}_J)$ and we call it canonical Lévy space. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ be the canonical filtration completed for $\mathbb{P}$. Let X be a square integrable centered Lévy process on $(\Omega, \mathcal{F}, \mathbb{P})$. We now denote N the Poisson random measure defined as

$$N(t, A) := \sum_{s \leq t} \mathbf{1}_A(\Delta X_s),$$

$A \in \mathcal{B}(\mathbb{R}_0)$ and $t \in [0, T]$, where $\Delta X_s := X_s - X_{s-}$. In addition, we define its compensated measure as $\widetilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$. Then, Lévy-Itô decomposition implies that

$$X_t = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z \widetilde{N}(ds, dz), \tag{3.3}$$

where $\sigma \geq 0$.

We consider the finite measure q defined on $[0, T] \times \mathbb{R}$ by

$$q(E) = \sigma^2 \int_{E(0)} dt \delta_0(dz) + \int_{E'} z^2 dt \nu(dz), \quad E \in \mathcal{B}([0, T] \times \mathbb{R}),$$

where $E(0) = \{(t,0) \in [0,T] \times \mathbb{R}; (t,0) \in E\}$ and $E' = E - E(0)$, and the random measure Q on $[0,T] \times \mathbb{R}$ by

$$Q(E) = \sigma \int_{E(0)} dW_t \delta_0(dz) + \int_{E'} z\tilde{N}(dt,dz), \quad E \in \mathcal{B}([0,T] \times \mathbb{R}).$$

Let $L^2_{T,q,n}(\mathbb{R})$ denote the set of product measurable, deterministic functions $h : ([0,T] \times \mathbb{R})^n \to \mathbb{R}$ satisfying

$$\|h\|^2_{L^2_{T,q,n}} = \int_{([0,T]\times\mathbb{R})^n} |h((t_1,z_1),\cdots,(t_n,z_n))|^2 q(dt_1,dz_1)\cdots q(dt_n,dz_n) < \infty.$$

For $n \in \mathbb{N}$ and $h_n \in L^2_{T,q,n}(\mathbb{R})$, we denote

$$I_n(h_n) = \int_{([0,T]\times\mathbb{R})^n} h((t_1,z_1),\cdots,(t_n,z_n)) Q(dt_1,dz_1)\cdots Q(dt_n,dz_n).$$

It is easy to see that $\mathbb{E}[I_0(h_0)] = h_0$ and $\mathbb{E}[I_n(h_n)] = 0$, for $n \geq 1$.
In this setting, we introduce the following chaos expansion (see Section 2 of [10] and Section 3 of [4]).

Theorem 3.1 *Any $\mathcal{F}$-measurable square integrable random variable F on the canonical space has a unique representation*

$$F = \sum_{n=0}^{\infty} I_n(h_n), \mathbb{P}\text{−a.s.}$$

with functions $h_n \in L^2_{T,q,n}(\mathbb{R})$ that are symmetric in the n pairs $(t_i,z_i), 1 \leq i \leq n$ and we have the isometry

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n!\|h_n\|^2_{L^2_{T,q,n}}.$$

Definition 3.2 *(1) Let $\mathbb{D}^{1,2}$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(h_n)$ satisfying*

$$\sum_{n=1}^{\infty} nn!\|h_n\|^2_{L^2_{T,q,n}} < \infty.$$

(2) Let $F \in \mathbb{D}^{1,2}$. Then the Malliavin derivative $DF : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ of a random variable $F \in \mathbb{D}^{1,2}$ is a stochastic process defined by

$$D_{t,z}F := \sum_{n=1}^{\infty} nI_{n-1}(h_n((t,z),\cdot)), \quad \text{valid for } q\text{−a.e. } (t,z) \in [0,T] \times \mathbb{R}, \mathbb{P} - a.s.$$

(3) For $\sigma \neq 0$, let $\mathbb{D}^{1,2}_0$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying

$$\sum_{n=1}^{\infty} nn! \int_0^T \|f_n(\cdot,(t,0))\|^2_{L^2_{T,q,n-1}} \sigma^2 dt < \infty.$$

Then, for $F \in \mathbb{D}_0^{1,2}$, we can define

$$D_{t,0}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n((t,0),\cdot)), \quad \text{valid for } q\text{−a.e. } (t,0) \in [0,T] \times \{0\}, \mathbb{P} - a.s.$$

(4) For $\nu \neq 0$, let $\mathbb{D}_1^{1,2}$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying

$$\sum_{n=1}^{\infty} nn! \int_0^T \int_{\mathbb{R}_0} \|f_n(\cdot,(t,z))\|^2_{L^2_{T,q,n-1}} z^2 \nu(dz)dt < \infty.$$

Then, for $F \in \mathbb{D}_1^{1,2}$, we can define

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n((t,z),\cdot)), \quad \text{valid for } q\text{−a.e. } (t,z) \in [0,T] \times \mathbb{R}_0, \mathbb{P} - a.s.$$

(5) Let D^W be the classical Malliavin derivative with respect to the Brownian motion W and Dom D^W *be the domain of D^W (for more details see [6]). We define*

$$\mathbb{D}^W := \left\{ F \in L^2(\mathbb{P}); F(\cdot,\omega_N) \in \text{Dom } D^W \text{ for } \mathbb{P}^N\text{−a.e. } \omega_N \in \Omega_N \right\}.$$

(6) Let F be a random variable on $\Omega_W \times \Omega_N$. Then we define the increment quotient operator

$$\Psi_{t,z}F := \frac{F(\omega_W, \omega_N^{t,z}) - F(\omega_W, \omega_N)}{z}, z \neq 0,$$

where $\omega_N^{t,z}$ transforms a element $\omega_N = ((t_1,z_1),(t_2,z_2),\cdots) \in \Omega_N$ into a new element

$$\omega_N^{t,z} = ((t,z),(t_1,z_1),(t_2,z_2),\cdots) \in \Omega_N,$$

by adding a jump of size z at time t into the trajectory. Moreover, we denote

$$\mathbb{D}^J := \left\{ F \in L^2(\mathbb{P}); \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} |\Psi_{t,z}F|^2 z^2 \nu(dz)dt \right] < \infty \right\}.$$

By Propositions 2.6.1, 2.6.2 in [3] and result of [1] (see section 3.3), we can derive the following:

Proposition 3.3 1. *If $F \in \mathbb{D}^W$, then $F \in \mathbb{D}_0^{1,2}$ and*

$$D_{t,0}F = \mathbf{1}_{\{\sigma>0\}} \sigma^{-1} D_t^W F(\cdot,\omega_N)(\omega_W)$$

for q -a.e. $(t,z) \in [0,T] \times \{0\}, \mathbb{P}$ -a.s.

2. *If $F \in \mathbb{D}^J$, then $F \in \mathbb{D}_1^{1,2}$ and $D_{t,z}F = \Psi_{t,z}F$ for q -a.e. $(t,z) \in [0,T] \times \mathbb{R}_0, \mathbb{P}$ -a.s.*
3. *$\mathbb{D}^{1,2} = \mathbb{D}^W \cap \mathbb{D}^J$ holds.*

4 Malliavin differentiability of indicator functions on canonical Lévy spaces

In this section, we consider Malliavin differentiability of indicator functions on canonical Lévy spaces, based on the papers [13] and [5].

In the case of the Wiener functionals we have already known the following:

Proposition 4.1 ([9]) *If $A \in \mathcal{F}_W$, then, the indicator function of A belongs to* $\mathrm{Dom}D^W$ *if and only if $\mathbb{P}_W(A)$ is equal to zero or one.*

However, in the case of the functionals of canonical Lévy processes, this result is not generally satisfied. We first consider the following examples.

Example 4.2 *Consider now the case $0 \neq \nu(\mathbb{R}) < \infty$. In this case, we can also show that if $\mathbb{P}(A) = 0$ or 1, then $\mathbf{1}_A \in \mathbb{D}^{1,2}$. However, the reverse is not always true. We give a counterexample of it. We take $\sigma = 0$, $\nu = \lambda\delta_{\{1\}}, \lambda > 0$ and $X_t = N_t$, where $\{N_t\}_{t\in[0,T]}$ be the Poisson process with $\mathbb{P}(N_t = k) = \frac{(\lambda t)^k}{k!}e^{-\lambda t}, k \in \mathbb{N}$. Let $F = \mathbf{1}_A$, $A = \{N_T = k_0\}$, $k_0 \in \mathbb{N}$. Then, $N_T \in \mathbb{D}^{1,2}$ and $D_{t,z}N_T = 1$. Moreover, Proposition 3.3 implies that*

$$\begin{aligned}\Psi_{t,z}F &= \frac{\mathbf{1}_{\{k_0\}}(N_T(\omega_N^{t,z})) - \mathbf{1}_{\{k_0\}}(N_T)}{z} \\ &= \frac{\mathbf{1}_{\{k_0\}}(N_T + z\frac{N_T(\omega_N^{t,z}) - N_T}{z}) - \mathbf{1}_{\{k_0\}}(N_T)}{z} \\ &= \frac{\mathbf{1}_{\{k_0\}}(N_T + zD_{t,z}N_T) - \mathbf{1}_{\{k_0\}}(N_T)}{z} \\ &= \frac{\mathbf{1}_{\{k_0\}}(N_T + z) - \mathbf{1}_{\{k_0\}}(N_T)}{z}, z \neq 0\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} |\Psi_{t,z}F|^2 z^2 \nu(dz)dt\right] &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} |\mathbf{1}_{\{k_0\}}(N_T + z) - \mathbf{1}_{\{k_0\}}(N_T)|^2 \nu(dz)dt\right] \\ &\leq 4T\nu(\mathbb{R}_0) < \infty.\end{aligned}$$

Hence, $F \in \mathbb{D}^J$. Therefore, by Proposition 3.3, we have $F \in \mathbb{D}_1^{1,2}$. In this case, since $\mathbb{D}_1^{1,2} = \mathbb{D}^{1,2}$, we obtain $F \in \mathbb{D}^{1,2}$. However, $\mathbb{P}(X_T = k_0) = \frac{(\lambda T)^{k_0}}{k_0!}e^{-\lambda T} \neq 0$ or 1.

Example 4.3 *Consider now the case $\nu(\mathbb{R}) = \infty$. In this case, we can also show that if $\mathbb{P}(A) = 0$ or 1, then $\mathbf{1}_A \in \mathbb{D}^{1,2}$. However, the reverse is not always true. The counterexample is provided by ([5]). We take $\sigma = 0$, $\nu(dx) = |x|^{-\alpha-1}\mathbf{1}_{[-1,1]\setminus\{0\}}(x)dx$ for $\alpha \in (0,1)$ and $A = \{X_T \geq K\}$ for some $K \in \mathbb{R}$. Then, $\nu(\mathbb{R}) = \infty$ and Proposition 28.3 and Theorem 24.10 (ii) in [7] imply that X_T has a density f with support $\mathbb{R}$ such that $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)| < \infty$. Hence, $\mathbb{P}(A) \in (0,1)$.*

Since $X_T(\omega_N^{t,z}) = X_T(\omega) + z$, *then, we have*

$$\begin{aligned}\Psi_{t,z}\mathbf{1}_A = \Psi_{t,z}\mathbf{1}_{[K,\infty)}(X_T) &= \frac{\mathbf{1}_{[K,\infty)}(X_T(\omega_N^{t,z})) - \mathbf{1}_{[K,\infty)}(X_T)}{z}\mathbf{1}_{\mathbb{R}_0}(z) \\ &= \frac{\mathbf{1}_{[K,\infty)}(X_T+z) - \mathbf{1}_{[K,\infty)}(X_T)}{z}\mathbf{1}_{\mathbb{R}_0}(z) \\ &= -\frac{1}{z}\mathbf{1}_{\{K\le X_T<K-z\}}\mathbf{1}_{\{z<0\}} + \frac{1}{z}\mathbf{1}_{\{K-z\le X_T<K\}}\mathbf{1}_{\{z>0\}}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}|\Psi_{t,z}\mathbf{1}_{[K,\infty)}(X_T)|^2 q(dt,dz)\right] \\ &= \int_0^T\int_{\mathbb{R}_0}\frac{1}{z^2}\{\mathbb{P}(K\le X_T<K-z)\mathbf{1}_{\{z<0\}} + \mathbb{P}(K-z\le X_T<K)\mathbf{1}_{\{z>0\}}\}z^2\nu(dz)dt \\ &= T\int_{-1}^0\left(\int_K^{K-z}f(x)dx\right)\nu(dz) + T\int_0^1\left(\int_{K-z}^K f(x)dx\right)\nu(dz) \\ &\le T\int_{-1}^0\|f\|_\infty|z|\nu(dz) + T\int_0^1\|f\|_\infty|z|\nu(dz) < \infty.\end{aligned}$$

Hence, $\mathbf{1}_A \in \mathbb{D}^{1,2}$.

From Examples 4.2 and 4.3, we have the following:

Theorem 4.4 *If* $\nu(\mathbb{R}) \neq 0$, *we have then the following: If* $\mathbb{P}(A) = 0$ or 1, *then* $\mathbf{1}_A \in \mathbb{D}^{1,2}$, *but the reverse is not always true.*

Moreover, we can get the following:

Theorem 4.5 *Assume that* $\nu(\mathbb{R}) = 0$, $\sigma \neq 0$ *and* $A \in \mathcal{F}$. *Then,* $\mathbf{1}_A \in \mathbb{D}^{1,2}$ *if and only if* $\mathbb{P}(A) = 0$ or 1.

5 LRM for digital option under geometric Lévy model

In this section, we consider local risk minimization problem for digital option under geometric Lévy model. We consider the following:

$$S_t := S_0 \exp\left\{\mu t + \int_0^t\int_{\mathbb{R}_0} x\tilde{N}(dt,dx)\right\}, t\in[0,T],\ S_0>0.$$

Moreover, S is also a solution to the stochastic differential equation

$$dS_t = S_{t-}\left[\mu^S\,dt + \int_{\mathbb{R}_0}(e^x-1)\tilde{N}(dt,dx)\right],$$

where $\mu^S := \mu + \int_{\mathbb{R}_0}(e^x - 1 - x)\nu(dx)$. Now, defining $L_t := \log S_t$ for all $t \in [0,T]$, we obtain a Lévy process L. Moreover, $dM_t := S_{t-}\int_{\mathbb{R}_0}(e^x-1)\tilde{N}(dt,dx)$ is the martingale part of S and $dA_t = S_{t-}\mu^S dt$.

To consider the LRM, we first calculate $\Psi_{t,z}S_T$: By the definition of $\Psi_{t,z}$, we obtain

$$\begin{aligned}\Psi_{t,z}S_T &= \frac{S_T(\omega^{t,z}) - S_T}{z}\mathbf{1}_{\mathbb{R}_0}(z)\\ &= \frac{S_0 e^{L_T(\omega^{t,z})} - S_T}{z}\mathbf{1}_{\mathbb{R}_0}(z)\\ &= \frac{S_0 e^{z\frac{L_T(\omega^{t,z})-L_T}{z}+L_T} - S_T}{z}\mathbf{1}_{\mathbb{R}_0}(z)\\ &= \frac{S_0 e^{z\Psi_{t,z}L_T+L_T} - S_T}{z}\mathbf{1}_{\mathbb{R}_0}(z)\\ &= \frac{S_0 e^{z+L_T} - S_T}{z}\mathbf{1}_{\mathbb{R}_0}(z) = z^{-1}S_T(e^z-1)\mathbf{1}_{\mathbb{R}_0}(z),\end{aligned}$$

and we can see that

$$S_T(\omega^{t,z}) = S_0 e^{z+L_T}, z \neq 0.$$

Hence, we have

$$\begin{aligned}&\Psi_{t,z}\mathbf{1}_{\{S_T\geq K\}}\\ &= \frac{\mathbf{1}_{[K,\infty)}(S_T(\omega^{t,z})) - \mathbf{1}_{[K,\infty)}(S_T)}{z}\mathbf{1}_{\mathbb{R}_0}(z)\\ &= \frac{\mathbf{1}_{[K/S_0,\infty)}(e^{z+L_T}) - \mathbf{1}_{[K/S_0,\infty)}(e^{L_T})}{z}\mathbf{1}_{\mathbb{R}_0}(z)\\ &= \frac{\mathbf{1}_{[\log K/S_0,\infty)}(z+L_T) - \mathbf{1}_{[\log K/S_0,\infty)}(L_T)}{z}\mathbf{1}_{\mathbb{R}_0}(z)\\ &= \left(-\frac{1}{z}\mathbf{1}_{\{\log K/S_0\leq L_T<\log K/S_0-z\}}\mathbf{1}_{\{z<0\}} + \frac{1}{z}\mathbf{1}_{\{\log K/S_0-z\leq L_T<\log K/S_0\}}\mathbf{1}_{\{z>0\}}\right)\mathbf{1}_{\mathbb{R}_0}(z).\end{aligned}$$

Hence, by using the main results of [2], we can get the following:

Theorem 5.1 *Under the geometric Lévy model, LRM for digital option $\mathbf{1}_{\{S_T\geq K\}} \in \mathbb{D}^{1,2}, K>0$ is given by*

$$\begin{aligned}&\xi_t^{\mathbf{1}_{\{S_T\geq K\}}}\\ &= \frac{\int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{P}^*}[z\Psi_{t,z}F|\mathcal{F}_{t-}](e^z-1)\nu(dz)}{S_{t-}\left(\int_{\mathbb{R}_0}(e^x-1)^2\nu(dx)\right)}\\ &= \frac{\int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{P}^*}[-\mathbf{1}_{\{\log K/S_0\leq L_T<\log K/S_0-z\}}\mathbf{1}_{\{z<0\}} + \mathbf{1}_{\{\log K/S_0-z\leq L_T<\log K/S_0\}}\mathbf{1}_{\{z>0\}}|\mathcal{F}_{t-}](e^z-1)\nu(dz)}{S_{t-}\left(\int_{\mathbb{R}_0}(e^x-1)^2\nu(dx)\right)},\end{aligned}$$

where $\lambda_t = \frac{\mu t}{S_{t-}(\int_{\mathbb{R}_0}(e^z-1)\nu(dz))}$, $dM_t = S_{t-}[\int_{\mathbb{R}_0}(e^z-1)\tilde{N}(dt,dz)]$ *and* $dZ_t = -\lambda_t Z_{t-} dM_t$, $d\mathbb{P}^* = Z_T d\mathbb{P}$.

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