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A hypercontractive family of the Ornstein–Uhlenbeck semigroup and its connection with Φ-entropy inequalities*

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Abstract

The purpose of this manuscript is twofold: (i) to provide a family of inequalities that unifies the hypercontractivity and its exponential variant of the Ornstein–Uhlenbeck semigroup; and (ii) to reveal a connection between the above-mentioned family and a family of Φ-entropy inequalities.

1 Introduction and main result

Given a positive integer d, let γ_d be the d-dimensional standard Gaussian measure. For every $p \geq 1$, define $L^p(\gamma_d)$ to be the set of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\|f\|_p^p := \int_{\mathbb{R}^d} |f(x)|^p \, \gamma_d(dx) < \infty$. We denote by $Q = \{Q_t\}_{t\geq 0}$ the Ornstein–Uhlenbeck semigroup acting on $L^1(\gamma_d)$: for $f \in L^1(\gamma_d)$ and $t \geq 0$,

$$(Q_t f)(x) := \int_{\mathbb{R}^d} f\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) \gamma_d(dy), \quad x \in \mathbb{R}^d.$$

It is well known that Q enjoys the hypercontractivity: if $f \in L^p(\gamma_d)$ for some $p > 1$, then

$$\|Q_t f\|_{q(t)} \leq \|f\|_p \quad \text{for all } t \geq 0, \tag{HC}$$

where $q(t) = e^{2t}(p-1)+1$. The hypercontractivity (HC) was firstly observed by Nelson [7] and found later by Gross [4] to be equivalent to the (Gaussian) logarithmic Sobolev inequality[1]:

$$\int_{\mathbb{R}^d} |f|^2 \log|f| \, d\gamma_d \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma_d + \|f\|_2^2 \log \|f\|_2 \,, \tag{LSI}$$

which holds true for any weakly differentiable function f in $L^2(\gamma_d)$ with $|\nabla f| \in L^2(\gamma_d)$. It is also known (see [1, Proposition 4]) that (HC) is equivalent to the exponential hypercontractivity: for any $f \in L^1(\gamma_d)$ with $e^f \in L^1(\gamma_d)$, it holds that

$$\|\exp(Q_t f)\|_{e^{2t}} \leq \|e^f\|_1 \quad \text{for all } t \geq 0. \tag{eHC}$$

One of the objectives of this manuscript is to show, by employing stochastic analysis, that two hypercontractivities (HC) and (eHC) are unified into

*This manuscript surveys the paper [6] by the author and is based on his talk given at Probability Symposium (確率論シンポジウム) held at RIMS, Kyoto University, from December 17 to December 20, 2018.

[1]The Gaussian logarithmic Sobolev inequality goes back to Stam [8].

Theorem 1 ([6], Theorem 1.1)**.** *Let a positive function c in $C^1((0,\infty))$ satisfy*

$$c' > 0 \text{ and } c/c' \text{ is concave on } (0,\infty), \tag{C}$$

and set

$$u(t,x) := \int_0^x c(y)^{e^{2t}} dy, \quad t \geq 0, \ x > 0. \tag{1}$$

Then for any nonnegative, measurable function f on $\mathbb{R}^d$ such that $u(0,f) \in L^1(\gamma_d)$, we have

$$v\left(t, \|u(t,Q_t f)\|_1\right) \leq v\left(0, \|u(0,f)\|_1\right) \quad \textit{for all } t \geq 0. \tag{uHC}$$

Here for every $t \geq 0$, the function $v(t,\cdot)$ is the inverse function of $u(t,x)$, $x > 0$.

The theorem asserts that if a nonnegative, measurable function f on $\mathbb{R}^d$ is such that $u(0,f) \in L^1(\gamma_d)$, then so is $u(t,Q_t f)$ for any $t \geq 0$ thanks to monotonicity of the function $u(t,x)$ in spatial variable x. We give examples of c fulfilling the condition (C).

Example 1. (i) For each $p > 1$, the power function $c(x) = x^{p-1}$ fulfills (C); indeed,

$$\frac{c(x)}{c'(x)} = \frac{x}{p-1},$$

and hence $(c/c')'' \equiv 0$. Therefore (uHC) applies and yields (HC). Observe that the addition of 1 that appears in the definition of $q(t)$ may be seen as a consequence of the integration in (1).

(ii) The exponential function $c(x) = e^x$ fulfills (C); indeed, we have $c/c' \equiv 1$, hence $(c/c')'' \equiv 0$. This choice of c in (uHC) yields (eHC) in the form

$$e^{-2t} \log \left\|\exp\left(e^{2t} Q_t f\right)\right\|_1 \leq \log \|e^f\|_1 \quad \text{for all } t \geq 0.$$

Note that if c satisfies $(c/c')'' \equiv 0$, then it is identical with either x^α for some $\alpha \neq 0$ or e^x up to affine transformation for variable x.

(iii) The third example deals with a mixture of (HC) and (eHC). For two exponents p, α such that $p + \alpha \geq 1$ and $0 < \alpha \leq 1$, take

$$c(x) = x^{p+\alpha-1} \exp\left(x^\alpha\right), \quad x > 0,$$

which fulfills (C). By L'Hôpital's rule, the corresponding u admits the asymptotics

$$u(t,x) \sim \frac{e^{-2t}}{\alpha} x^{q(t)+(e^{2t}-1)\alpha} \exp\left(e^{2t} x^\alpha\right) \quad \text{as } x \to \infty$$

for every $t \geq 0$ (here we abuse the notation $q(t)$ when $p \leq 1$). Therefore Theorem 1 entails that the following implication is true: for any nonnegative, measurable function f on $\mathbb{R}^d$,

$$f^p \exp\left(f^\alpha\right) \in L^1(\gamma_d) \ \Rightarrow \ (Q_t f)^{q(t)+(e^{2t}-1)\alpha} \exp\left\{e^{2t}(Q_t f)^\alpha\right\} \in L^1(\gamma_d), \ \forall t \geq 0.$$

2 Outline of proof of Theorem 1

To prove Theorem 1, we employ stochastic analysis. For this purpose, we prepare a d-dimensional standard Brownian motion $W = \{W_t\}_{0\le t\le 1}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $\{\mathcal{F}_t\}_{0\le t\le 1}$ the augmentation of the natural filtration of W: $\mathcal{F}_t = \sigma(W_s, s \le t) \vee \mathcal{N}$. For each $f \in L^1(\gamma_d)$, define

$$\begin{aligned} M_t \equiv M_t(f) :=& \mathbb{E}\left[f(W_1)|\mathcal{F}_t\right] \\ \equiv& \mathbb{E}\left[f(W_{1-t}+x)\right]\big|_{x=W_t}, \quad 0 \le t \le 1, \end{aligned}$$

where the second line is due to the Markov property of W. The last expression reveals the identity in law:

$$(Q_t f, \gamma_d) \overset{(d)}{=} (M_{e^{-2t}}(f), \mathbb{P})$$

for every fixed $t \ge 0$ and what in fact we are going to prove is

Proposition 1 ([6], Proposition 3.1)**.** *For a positive c in $C^1((0,\infty))$ satisfying* (C)*, set*

$$\mathsf{u}(t,x) := \int_0^x c(y)^{1/t}\, dy, \quad t \in (0,1],\ x > 0. \tag{1'}$$

Then for any nonnegative, measurable function f such that $\mathsf{u}(1,f) \in L^1(\gamma_d)$, we have

$$\mathsf{v}\left(t, \mathbb{E}[\mathsf{u}(t, M_t(f))]\right) \le \mathsf{v}\left(1, \mathbb{E}[\mathsf{u}(1, M_1(f))]\right) \quad \text{for all } t \in (0,1]. \tag{uHC'}$$

Here for every $0 < t \le 1$, we denote by $\mathsf{v}(t,\cdot)$ the inverse function of $\mathsf{u}(t,\cdot)$.

By density arguments, it suffices to show (uHC′) for $f \in C_b^1(\mathbb{R}^d)$ with $\inf_{x\in\mathbb{R}^d} f(x) > 0$. Here $C_b^1(\mathbb{R}^d)$ is the set of bounded C^1-functions on $\mathbb{R}^d$ with bounded derivatives. Set a d-dimensional process $\theta = \{\theta_t\}_{0\le t\le 1}$ by

$$\theta_t = \mathbb{E}\left[\nabla f(W_{1-t}+x)\right]\big|_{x=W_t}.$$

By the Clark–Ocone formula,

$$M_t = \mathbb{E}\left[f(W_1)\right] + \int_0^t \theta_s \cdot dW_s \quad \text{for all } 0 \le t \le 1,\ \mathbb{P}\text{-a.s.}$$

In fact, denoting $F(W) = f(W_1)$, we see that θ_t is nothing but

$$\mathbb{E}\left[D_t F(W)|\mathcal{F}_t\right]$$

with $DF(W)$ the *Malliavin derivative* of $F(W)$. In what follows we write

$$N_t \equiv N_t(f) := \mathsf{u}\left(t, M_t(f)\right).$$

What to do is to show that

$$\frac{d}{dt}\mathsf{v}(t, \mathbb{E}[N_t]) \ge 0, \quad 0 < t \le 1,$$

via the following two lemmas: set for $(t,x) \in (0,1] \times (0,\infty)$,

$$U(t,x) := \left\{\left(\frac{\mathsf{u}_{tx}}{\mathsf{u}_x}\right)_x \frac{1}{\mathsf{u}_x}\right\}(t,x) \qquad \text{and} \qquad \varphi(t,x) := -\frac{1}{U(t, \mathsf{v}(t,x))},$$

where in the definition of U, subscripts stand for partial differentiations with respect to corresponding variables.

Lemma 1. *We have for* $0 < t \le 1$,

$$\begin{aligned}
&2\mathsf{u}_x(t, \mathsf{v}(t, \mathbb{E}[N_t]))\, \frac{d}{dt}\mathsf{v}(t, \mathbb{E}[N_t]) \\
&= \int_0^1 \mathbb{E}\left[U(t, \mathsf{v}(t, \mathbb{E}[N_t|\mathcal{F}_s]))\, |\mathbb{E}[D_s N_t|\mathcal{F}_s]|^2\right] ds + \mathbb{E}\left[\mathsf{u}_{xx}(t, M_t)|\theta_t|^2\right].
\end{aligned}$$

Lemma 2. *We have for* $0 < t \le 1$ *and* $0 \le s \le 1$,

$$\mathbb{E}\left[U(t, \mathsf{v}(t, \mathbb{E}[N_t|\mathcal{F}_s]))\, |\mathbb{E}[D_s N_t|\mathcal{F}_s]|^2\right] \ge -\mathbb{E}\left[\frac{|D_s N_t|^2}{\varphi(t, N_t)}\right].$$

We postpone proofs of these two lemmas to the next section.

Proof of Proposition 1. By Lemmas 1 and 2, we have

$$\begin{aligned}
&2\mathsf{u}_x(t, \mathsf{v}(t, \mathbb{E}[N_t]))\, \frac{d}{dt}\mathsf{v}(t, \mathbb{E}[N_t]) \\
&\ge -\int_0^1 \mathbb{E}\left[\frac{|D_s N_t|^2}{\varphi(t, N_t)}\right] ds + \mathbb{E}\left[\mathsf{u}_{xx}(t, M_t)|\theta_t|^2\right]. \qquad (2)
\end{aligned}$$

By chain rule for D,

$$\begin{aligned}
D_s N_t &= \mathsf{u}_x(t, M_t) D_s M_t \\
&= \mathbf{1}_{[0,t]}(s)\mathsf{u}_x(t, M_t)\theta_t
\end{aligned}$$

as $M_t = \mathbb{E}\left[f(W_{1-t} + x)\right]\big|_{x=W_t}$. Hence the right-hand side of (2) is rewritten as

$$\mathbb{E}\left[\left\{-t\frac{(\mathsf{u}_x(t,x))^2}{\varphi(t, \mathsf{u}(t,x))} + \mathsf{u}_{xx}(t,x)\right\}\Bigg|_{x=M_t} \times |\theta_t|^2\right].$$

Because of expressions

$$\frac{1}{\varphi(t, \mathsf{u}(t,x))} = \frac{1}{t^2}\frac{c'(x)}{c(x)}c(x)^{-1/t}, \quad \mathsf{u}_x(t,x) = c(x)^{1/t} \quad \text{and} \quad \mathsf{u}_{xx}(t,x) = \frac{1}{t}\frac{c'(x)}{c(x)}c(x)^{1/t},$$

we have for any $x > 0$,

$$\begin{aligned}
-t\frac{(\mathsf{u}_x(t,x))^2}{\varphi(t, \mathsf{u}(t,x))} + \mathsf{u}_{xx}(t,x) &= \left(-t \times \frac{1}{t^2} + \frac{1}{t}\right)\frac{c'(x)}{c(x)}c(x)^{1/t} \\
&= 0,
\end{aligned}$$

which shows that the right-hand side of (2) is identical with 0. Since $\mathsf{u}_x(t,x)$ is positive for all $0 < t \le 1$ and $x > 0$, we obtain from (2),

$$\frac{d}{dt}\mathsf{v}(t, \mathbb{E}[N_t]) \ge 0$$

as desired. $\square$

3 Proof of Lemmas 1 and 2

In this section we prove Lemmas 1 and 2.

Proof of Lemma 1. Since $dM_t = \theta_t \cdot dW_t$ by the Clark–Ocone formula, Itô's formula entails that

$$d\mathsf{u}(t, M_t) = \mathsf{u}_t(t, M_t)\, dt + \mathsf{u}_x(t, M_t)\theta_t \cdot dW_t + \frac{1}{2}\mathsf{u}_{xx}(t, M_t)|\theta_t|^2\, dt,$$

hence

$$\frac{d}{dt}\mathbb{E}\left[\mathsf{u}(t, M_t)\right] = \mathbb{E}\left[\mathsf{u}_t(t, M_t)\right] + \frac{1}{2}\mathbb{E}\left[\mathsf{u}_{xx}(t, M_t)|\theta_t|^2\right].$$

Recall $N_t = \mathsf{u}\,(t, M_t)$. As v is the inverse function of u in spatial variable, there holds the relation

$$\begin{aligned} &\mathsf{u}_x(t, \mathsf{v}(t, \mathbb{E}[N_t]))\, \frac{d}{dt}\mathsf{v}(t, \mathbb{E}[N_t]) \\ &= \mathbb{E}\left[\mathsf{u}_t(t, M_t)\right] - \mathsf{u}_t(t, \mathsf{v}(t, \mathbb{E}[N_t])) + \frac{1}{2}\mathbb{E}\left[\mathsf{u}_{xx}(t, M_t)|\theta_t|^2\right]. \end{aligned} \tag{3}$$

Noting $\mathsf{u}_t(t, M_t) = \mathsf{u}_t\left(t, \mathsf{v}(t, \mathbb{E}[N_t|\mathcal{F}_1])\right)$, we develop the process

$$\mathsf{u}_t\left(t, \mathsf{v}(t, \mathbb{E}[N_t|\mathcal{F}_\tau])\right), \quad 0 \le \tau \le 1,$$

via the Clark–Ocone formula for $\mathbb{E}[N_t|\mathcal{F}_\tau]$:

$$\mathbb{E}[N_t|\mathcal{F}_\tau] = \mathbb{E}[N_t] + \int_0^\tau \mathbb{E}[D_s N_t|\mathcal{F}_s] \cdot dW_s, \quad 0 \le \tau \le 1,\ \mathbb{P}\text{-a.s.},$$

together with Itô's formula, to see that

$$\begin{aligned} d_\tau \mathsf{u}_t\left(t, \mathsf{v}(t, \mathbb{E}[N_t|\mathcal{F}_\tau])\right) = &\frac{\mathsf{u}_{tx}}{\mathsf{u}_x}(t, \mathsf{v}(t, \mathbb{E}[N_t|\mathcal{F}_\tau]))\mathbb{E}[D_\tau N_t|\mathcal{F}_\tau] \cdot dW_\tau \\ &+ \frac{1}{2}U(t, \mathsf{v}(t, \mathbb{E}[N_t|\mathcal{F}_\tau]))\left|\mathbb{E}[D_\tau N_t|\mathcal{F}_\tau]\right|^2\, d\tau. \end{aligned}$$

Integrating both sides from 0 to 1 relative to τ and taking expectations lead to

$$\begin{aligned} &\mathbb{E}\left[\mathsf{u}_t(t, M_t)\right] - \mathsf{u}_t(t, \mathsf{v}(t, \mathbb{E}[N_t])) \\ &= \frac{1}{2}\int_0^1 \mathbb{E}\left[U(t, \mathsf{v}(t, \mathbb{E}[N_t|\mathcal{F}_\tau]))\left|\mathbb{E}[D_\tau N_t|\mathcal{F}_\tau]\right|^2\right] d\tau. \end{aligned}$$

Plug the last expression into (3) to obtain

$$\begin{aligned} &\mathsf{u}_x(t, \mathsf{v}(t, \mathbb{E}[N_t]))\, \frac{d}{dt}\mathsf{v}(t, \mathbb{E}[N_t]) \\ &= \frac{1}{2}\int_0^1 \mathbb{E}\left[U(t, \mathsf{v}(t, \mathbb{E}[N_t|\mathcal{F}_\tau]))\left|\mathbb{E}[D_\tau N_t|\mathcal{F}_\tau]\right|^2\right] d\tau + \frac{1}{2}\mathbb{E}\left[\mathsf{u}_{xx}(t, M_t)|\theta_t|^2\right] \end{aligned}$$

as claimed. □

Proof of Lemma 2. As $\varphi(t,x) = -1/U(t,\mathsf{v}(t,x))$ by definition, what to show is

$$\mathbb{E}\left[\frac{|\mathbb{E}[D_sN_t|\mathcal{F}_s]|^2}{\varphi(t,\mathbb{E}[N_t|\mathcal{F}_s])}\right] \le \mathbb{E}\left[\frac{|D_sN_t|^2}{\varphi(t,N_t)}\right]. \tag{4}$$

Recall from [6, Lemma 3.1] that $\varphi > 0$ and $\varphi(t,\cdot)$ is concave for every $t \in (0,1]$ under the condition (C). Observe a.s.,

$$\begin{aligned} 0 &\le \mathbb{E}\left[\varphi(t,N_t)\left|\frac{D_sN_t}{\varphi(t,N_t)} - \frac{\mathbb{E}[D_sN_t|\mathcal{F}_s]}{\varphi(t,\mathbb{E}[N_t|\mathcal{F}_s])}\right|^2\middle|\mathcal{F}_s\right] \\ &= \mathbb{E}\left[\frac{|D_sN_t|^2}{\varphi(t,N_t)}\middle|\mathcal{F}_s\right] - 2\frac{|\mathbb{E}[D_sN_t|\mathcal{F}_s]|^2}{\varphi(t,\mathbb{E}[N_t|\mathcal{F}_s])} + \mathbb{E}\left[\varphi(t,N_t)|\mathcal{F}_s\right]\frac{|\mathbb{E}[D_sN_t|\mathcal{F}_s]|^2}{\{\varphi(t,\mathbb{E}[N_t|\mathcal{F}_s])\}^2} \\ &\le \mathbb{E}\left[\frac{|D_sN_t|^2}{\varphi(t,N_t)}\middle|\mathcal{F}_s\right] - \frac{|\mathbb{E}[D_sN_t|\mathcal{F}_s]|^2}{\varphi(t,\mathbb{E}[N_t|\mathcal{F}_s])} \end{aligned}$$

because of

$$\mathbb{E}\left[\varphi(t,N_t)|\mathcal{F}_s\right] \le \varphi(t,\mathbb{E}[N_t|\mathcal{F}_s]) \quad \text{a.s.}$$

by the conditional Jensen inequality. This observation entails (4). □

Remark 1. (i) In each of two cases that $c(x) = x^{p-1}$ for some $p > 1$ and that $c(x) = e^x$, the corresponding φ is a linear function in spatial variable (see [6, Remark 3.1 (2)]), which entails that (4) holds as equality. This fact enables us to obtain the following "hypercontractive identities": for any $f \in C_b^1(\mathbb{R}^d)$ with $\inf_{x\in\mathbb{R}^d} f(x) > 0$,

$$\|Q_tf\|_{q(t)} = \|f\|_p \exp\left\{-\int_0^t \frac{e^{-2\tau}}{\|Q_\tau f\|_{q(\tau)}^{q(\tau)}}\Xi(e^{-2\tau})\,d\tau\right\},$$

$$\|\exp(Q_tf)\|_{e^{2t}} = \|e^f\|_1 \exp\left\{-\int_0^t \frac{e^{-2\tau}}{\|\exp(Q_\tau f)\|_{e^{2\tau}}^{e^{2\tau}}}\Xi(e^{-2\tau})\,d\tau\right\}$$

for all $t \ge 0$; see [6, Remark 3.2 (1)]. Here the nonnegative function $\Xi(t) \equiv \Xi_{c,f}(t)$, $t \in (0,1]$, is defined by

$$\Xi(t) = \int_0^1 \mathbb{E}\left[\varphi(t,N_t)\left|\frac{D_sN_t}{\varphi(t,N_t)} - \frac{\mathbb{E}[D_sN_t|\mathcal{F}_s]}{\varphi(t,\mathbb{E}[N_t|\mathcal{F}_s])}\right|^2\right] ds.$$

(ii) If we replace the definition $(1')$ of $\mathsf{u}(t,x)$ by

$$\mathsf{u}(t,x) = \int_0^x c(y)^{-1/t}\,dy,$$

then the inequality (4) is reversed, yielding a generalization of the *reverse hypercontractivity*: if we let a positive c in $C^1((0,\infty))$ satisfy (C) and $\lim_{x\to 0+} c(x) > 0$, and set the function u by

$$u(t,x) = \int_0^x c(y)^{-e^{2t}}\,dy, \quad t \ge 0,\ x > 0,$$

in place of (1), then for any $f \in C_b^1(\mathbb{R}^d)$ with $\inf_{x\in\mathbb{R}^d} f(x) > 0$, we have

$$v\left(t, \|u(t,Q_tf)\|_1\right) \ge v\left(0, \|u(0,f)\|_1\right) \quad \text{for all } t \ge 0.$$

Here $v(t,\cdot)$ is the inverse function of $u(t,\cdot)$ for every $t \ge 0$ as before. We refer to [6, Section 4] for more details.

4 Generalization of Gaussian logarithmic Sobolev inequality

Recall the fact ([4]) that differentiating the left-hand side of (HC) at $t = 0$ yields (LSI); the same argument enables us to obtain from (uHC) the following generalization of (LSI):

Corollary 1 ([6], Corollary 3.1)**.** *For a function c satisfying the assumptions in Theorem 1, set*

$$G(x) = \int_0^x c(y)\,dy \qquad \text{and} \qquad H(x) = \int_0^x c(y)\log c(y)\,dy$$

for $x > 0$. Then for any $f \in C_b^1(\mathbb{R}^d)$ with $\inf_{x\in\mathbb{R}^d} f(x) > 0$, we have

$$\int_{\mathbb{R}^d} H(f)\,d\gamma_d \leq \frac{1}{2}\int_{\mathbb{R}^d} c'(f)\,|\nabla f|^2\,d\gamma_d + H\circ G^{-1}\left(\|G(f)\|_1\right). \tag{gLSI}$$

Here G^{-1} is the inverse function of G.

Proof. Since the left-hand side of (3) is nonnegative as seen in the proof of Proposition 1, evaluation of its right-hand side at $t = 1$ yields (gLSI). □

Be aware that the initial value of $v\left(t, \|u(t, Q_t f)\|_1\right)$, $t \geq 0$, corresponds to the terminal value of $\mathsf{v}(t, \mathbb{E}[N_t])$, $0 < t \leq 1$.

Remark 2. Taking $c(x) = x^{p-1}$ $(p > 1)$ and e^x, we recover (LSI) from (gLSI).

5 Connection with Φ-entropy inequalities

Let $\Phi \in C^2((0,\infty))$ be such that

$$\Phi'' > 0 \text{ and } 1/\Phi'' \text{ is concave on } (0,\infty). \tag{P}$$

Fix $f \in C_b^1(\mathbb{R}^d)$ with $\inf_{x\in\mathbb{R}^d} f(x) > 0$. Then

Proposition 2 ([6], Proposition 3.3)**.** (gLSI) *holds for any positive $c \in C^1((0,\infty))$ satisfying* (C) *if and only if for any $\Phi \in C^2((0,\infty))$ satisfying* (P)*, the* Φ-entropy inequality *holds:*

$$\int_{\mathbb{R}^d} \Phi(f)\,d\gamma_d - \Phi\left(\int_{\mathbb{R}^d} f\,d\gamma_d\right) \leq \frac{1}{2}\int_{\mathbb{R}^d} \Phi''(f)\,|\nabla f|^2\,d\gamma_d. \tag{ΦI}$$

The quantity on the left-hand side of (ΦI) is referred to as the Φ-*entropy* and gives a nonnegative value by Jensen's inequality when Φ is convex. Typical examples of Φ's fulfilling (P) are $\Phi(x) = x\log x$ and $\Phi(x) = x^2$ (if we consider it on $\mathbb{R}$), and these two choices in (ΦI) lead to (LSI) and Poincaré's inequality, respectively.

Proof of Proposition 2. We start with if part. Given a positive $c \in C^1((0,\infty))$ satisfying (C), take $\Phi = H\circ G^{-1}$ with H and G given in Corollary 1. Then it is readily seen that Φ fulfills (P). Writing f for $G^{-1}(f)$ leads to (gLSI).

We turn to only if part. For $\Phi \in C^2((0,\infty))$ satisfying (P), take $c = \exp(\Phi')$. Then c fulfills (C) and so does $c^\alpha = \exp(\alpha\Phi')$ for any $\alpha > 0$. We replace c by c^α in (gLSI), divide both sides by α and let $\alpha \to 0$. Then (ΦI) follows, which ends the proof. □

As already observed in Corollary 1, the hypercontractive family (uHC) implies (gLSI); the next proposition shows that the converse is also true.

Proposition 3 (cf. [6], Proposition 3.4)**.** (gLSI) *implies* (uHC).

An important observation is that if a positive $c \in C^1((0,\infty))$ fulfills (C), then so does c^α for any $\alpha > 0$ as has already been seen above in a restrictive setting. Then (gLSI) applied to c^α yields

$$\int_{\mathbb{R}^d} H_\alpha(f)\, d\gamma_d \le \frac{\alpha}{2} \int_{\mathbb{R}^d} (c^{\alpha-1}c')(f)|\nabla f|^2\, d\gamma_d + H_\alpha \circ G_\alpha^{-1}(\|G_\alpha(f)\|_1)\,, \tag{5}$$

where G_α and H_α are defined as in Corollary 1 with c therein replaced by c^α.

Proof of Proposition 3. Write $\alpha(t) = e^{2t}$, $t > 0$. Similarly to proof of Lemma 1, we compute

$$\begin{aligned}
&u_x\left(t, v\big(t, \|u(t,Q_tf)\|_1\big)\right) \frac{d}{dt} v\big(t, \|u(t,Q_tf)\|_1\big) \\
&= -u_t\left(t, v\big(t, \|u(t,Q_tf)\|_1\big)\right) + \frac{d}{dt}\, \|u(t,Q_tf)\|_1 \\
&= -2H_{\alpha(t)} \circ G_{\alpha(t)}^{-1}\big(\big\|G_{\alpha(t)}(Q_tf)\big\|_1\big) + \frac{d}{dt}\, \|u(t,Q_tf)\|_1\,.
\end{aligned} \tag{6}$$

The last term is calculated and estimated as

$$\begin{aligned}
&\int_{\mathbb{R}^d} u_t(t,Q_tf)\, d\gamma_d + \int_{\mathbb{R}^d} u_x(t,Q_tf) LQ_tf\, d\gamma_d \\
&= 2\int_{\mathbb{R}^d} H_{\alpha(t)}(Q_tf)\, d\gamma_d + \int_{\mathbb{R}^d} \{c(Q_tf)\}^{\alpha(t)} LQ_tf\, d\gamma_d \\
&= 2\int_{\mathbb{R}^d} H_{\alpha(t)}(Q_tf)\, d\gamma_d - \alpha(t)\int_{\mathbb{R}^d} \big\{c^{\alpha(t)-1}c'\big\}(Q_tf)|\nabla Q_tf|^2\, d\gamma_d \\
&\le 2H_{\alpha(t)} \circ G_{\alpha(t)}^{-1}\big(\big\|G_{\alpha(t)}(Q_tf)\big\|_1\big),
\end{aligned}$$

where for the first and second lines, we used L to denote the Ornstein–Uhlenbeck operator $\Delta - x\cdot\nabla$, and for the third line, we used integration by parts (ibp for short) and chain rule for ∇, and for the last, we used (5). Combining the last estimate with (6), we have

$$\frac{d}{dt} v\big(t, \|u(t,Q_tf)\|_1\big) \le 0$$

for any $t > 0$, which proves (uHC). $\square$

6 Concluding remarks

In this manuscript, we have provided a framework that embraces (HC) and (eHC), as well as the family of Φ-entropy inequalities (ΦI) indexed by $\Phi \in C^2((0,\infty))$ fulfilling (P), on which we add specific comments as follows.

(i) The condition (C) is not artificial in view of Φ-entropy inequalities (ΦI). It should also be mentioned that (uHC) possesses a certain optimality (see [6, Subsection A.2]) observed by an anonymous referee of [6], who also pointed out to us that under (C) (with additional assumption that c is of class C^3), functionals as on the right-hand side of (uHC) are considered in [5, Theorem 106 (i)] to discuss their convexity in a discrete setting.

(ii) Equivalence between (uHC) and (ΦI) holds true in a general setting of *Markov triple* (E, μ, Γ) with associated Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, the notion elaborated in [2, Chapters 4–7]; in particular, if the triple (E, μ, Γ) is such that under the condition (P),

$$\int_E \Phi(f)\,d\mu - \Phi\left(\int_E f\,d\mu\right) \leq \frac{R}{2}\int_E \Phi''(f)\Gamma(f,f)\,d\mu \tag{ΦI$'$}$$

for any positive $f \in \mathcal{D}(\mathcal{E})$ for some $R > 0$, and that its *carré du champ* Γ satisfies

$$\int_E \Gamma(f,g)\,d\mu = -\int_E gLf\,d\mu, \tag{ibp}$$

$$\Gamma(\psi(f), g) = \psi'(f)\Gamma(f,g), \tag{chain rule}$$

then by rewriting (ΦI$'$) similarly to (5), the same reasoning as in the proof of Proposition 3 applies and leads to (uHC) with replacement:

$$Q_t \text{ by } e^{tL} \qquad \text{and} \qquad e^{2t} \text{ in (1) by } e^{2t/R}.$$

For instance, if a probability measure μ on $E = \mathbb{R}^d$ is given in the form $\mu(dx) = e^{-V(x)}dx$ with $V \in C^2(\mathbb{R}^d)$ whose Hessian matrix satisfies $y \cdot \mathrm{Hess}_V(x)y \geq \rho|y|^2$, $x, y \in \mathbb{R}^d$, for some $\rho > 0$, then the Φ-entropy inequality (ΦI$'$) for $\Gamma(f,f) = |\nabla f|^2$ is known (cf. [3, Corollary 2.1]) to hold with $R = 1/\rho$, and hence (uHC) holds true for the semigroup generated by $L = \Delta - \nabla V \cdot \nabla$, with exponent e^{2t} in (1) replaced by $e^{2\rho t}$. See [6, Subsection 3.2] for more detailed description.

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