An extension of the Hartshorne theorem on the characterization of cofinite complexes

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Abstract

This report is the announcement of the result on the characterization of cofinite complexes (the first one is in the talk at the RIMS conference♮). Let A be a homomorphic image of a Gorenstein ring of finite Krull dimension, J an ideal of A, and $N^\bullet$ a bounded-below complex of A-modules. Suppose that A is complete with respect to a J-adic topology. In this report, we introduce that $N^\bullet$ is a J-cofinite complex if and only if the support of $H^i(N^\bullet)$ is in $V(J)$ for all i, and $\mathrm{Ext}^j(A/J, N^\bullet)$ is of finite type for all j.

1 Introduction

In this report, we shall introduce the following theorem.

Theorem 1 *Let A be a ring, which is a homomorphic image of a Gorenstein ring of finite Krull dimension, J an ideal of A. Suppose that A is complete with respect to a J-adic topology. Let $N^\bullet$ be a complex of A-modules in $\mathcal{D}^+(A)$, where $\mathcal{D}^+(A)$ is the derived category consisting of complexes bounded below. Then the following conditions are equivalent:*

(1) *The complex $N^\bullet$ is J-cofinite.*

(2) *The complex $N^\bullet$ satisfies the following conditions:*

  a) $\mathrm{Supp}\, H^i(N^\bullet) \subseteq V(J)$ *for each i,*

  b) $\mathrm{Ext}^j(A/J, N^\bullet)$ *is of finite type for each j.*

We assume that all rings are commutative and noetherian with identity throughout this report.

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2 Preliminaries

We shall recall the definitions on cofiniteness. For the basic definitions, the reader is referred to [15] and [16] (see [3], [4], [2] and [14] for the notation on derived categories and derived functors).

Before defining the J-cofiniteness on complexes, we introduce the definition of the dualizing functor (cf. [14, § 4.3, p. 70]):

Definition 1 *Let A be a ring, equipped with a dualizing complex $\mathbf{D}^\bullet$ for A and J an ideal of A (see [19, (1.1) Definition, p. 215] for the definition of the (fundamental) dualizing complex). We denote by $D_J(\ -\ )$ the functor $\mathbb{R}\operatorname{Hom}^\bullet(\ -\ ,\mathbb{R}\Gamma_J(\mathbf{D}^\bullet))$ on the derived category $\mathcal{D}(A)$. In this paper, we call this functor $D_J(\ -\ )$ the* J**-dualizing functor** *(cf. [14, p. 70]). Note that the J-dualizing functor is defined over unbounded complexes by [20, Theorem C, p. 125]. In the case $J = (0)$, then we simply denote $D_J(\ -\ )$ by $D(\ -\ )$.*

The J-cofiniteness on *complexes* is defined as follows (see [3, §2, p. 149] for the definition over regular rings):

Definition 2 *Let A be a ring, equipped with a dualizing complex $\mathbf{D}^\bullet$ for A and J an ideal of A. Let $N^\bullet$ be an object of the derived category $\mathcal{D}(A)$. We say $N^\bullet$ is* J**-cofinite**, *if there exists $M^\bullet \in \mathcal{D}_{ft}(A)$, such that $N^\bullet \simeq D_J(M^\bullet)$ in $\mathcal{D}(A)$. Here $D_J(\ -\ )$ is the J-dualizing functor on $\mathcal{D}(A)$ defined as above and $\mathcal{D}_{ft}(A)$ is the derived category consisting of complexes with cohomology modules of finite type over A.*

Over regular rings R of finite Krull dimension, it is proved by Hartshorne (cf. [3, Theorem 5.1, p. 154]). Over the rings A which is a homomorphic image of a Gorenstein ring of finite Krull dimension, the theorem as a variant of [3, Theorem 5.1, p. 154] is proved under the additional conditions (ii) and (ii) (cf. [11, Theorem 5, p. 318]). Our aim in the report, we introduce an extension of the Hartshorne theorem over the ring A, without the conditions (i) and (ii).

3 Sketch proof of the main theorem

The purpose of this section is to give a sketch proof of the main theorem as an extension of [3, Theorem 5.1, p. 154]) without regularity assumption. The argument on the way out functors is used in [3, Theorem 5.1, p. 154]) over the regular rings. On the other hands, we use the spectral sequences. First let us prove the lemma:

Lemma 2 *Let A be a ring, which is a homomorphic image of a Gorenstein ring of finite Krull dimension, J an ideal of A, and $M^\bullet$ in $\mathcal{D}^-_{ft}(A)$. Suppose that A is complete with respect to a J-adic topology. Then $\operatorname{Ext}^j(T^\bullet, D_J(M^\bullet))$ is of finite type for all j, for each $T^\bullet$ in $\mathcal{D}^-_{ft}(A)$ with $\operatorname{Supp} H^l(T^\bullet) \subset V(J)$ for all l. Here $\mathcal{D}^-_{ft}(A)$ is the derived category consisting of complexes bounded above with cohomology modules of finite type over A.*

Proof. We have an isomorphism in the derived category $\mathcal{D}(A)$ using the Hom-tensor adjunction (cf. [14, the Hom-tensor adjunction, p. 53]):

$$\mathbb{R}\operatorname{Hom}^\bullet(T^\bullet, D_J(M^\bullet))) \simeq \mathbb{R}\operatorname{Hom}^\bullet(M^\bullet, D(T^\bullet)).$$

Then we have the lemma, using the spectral sequences and by Yoneda's lemma. See [12] for the detailed calculation. □

Proof of Theorem 1: See [12] for the detailed proof. □

4 An application of the theorem

We recall the following claim (cf. [Kaw3, Claim 1]):

Claim 3 *Let R be a ring, and J an ideal of R. Let $N^\bullet \in \mathcal{D}^+(R)$ be a bounded-below complex. Then the following conditions are equivalent:*

(i) $\operatorname{Ext}^j(R/J, N^\bullet)$ *is of finite type over R for all j;*

(ii) $\operatorname{Ext}^j(R/\sqrt{J}, N^\bullet)$ *is of finite type over R for all j;*

(iii) $\operatorname{Ext}^j(R/P, N^\bullet)$ *is of finite type over R for all j and for each $P \in \operatorname{Min}(R/J)$;*

(iv) $\operatorname{Ext}^j(W, N^\bullet)$ *is of finite type over R for all j and for each R-module W of finite type over R such that* $\operatorname{Supp}(W) \subseteq V(J)$*;*

(v) $\operatorname{Ext}^j(W^\bullet, N^\bullet)$ *is of finite type over R for all j and for each $W^\bullet \in \mathcal{D}^b_{ft}(R)$ such that* $\operatorname{Supp}(H^l(W^\bullet)) \subseteq V(J)$ *for all l;*

(vi) $\operatorname{Ext}^j(W^\bullet, N^\bullet)$ *is of finite type over R for all j and for each $W^\bullet \in \mathcal{D}^-_{ft}(R)$ such that* $\operatorname{Supp}(H^l(W^\bullet)) \subseteq V(J)$ *for all l.*

We can give the following corollary as an application form Theorem 1.

Corollary 4 *Let A be a ring, which is a homomorphic image of a Gorenstein ring of finite Krull dimension, J an ideal of A. Suppose that A is complete with respect to a J-adic topology. Let $N^\bullet$ be a complex of A-modules in $\mathcal{D}^+(A)$. Then the following conditions are equivalent:*

(1) *The complex $N^\bullet$ is J-cofinite.*

(2) *The complex $N^\bullet$ satisfies the following conditions:*

   a) $\operatorname{Supp} H^i(N^\bullet) \subseteq V(J)$ *for each i,*

   b) *the equivalence conditions in Claim 3.*

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♮ The data of the talk in the conference:
Date: February 15, in 2018, 11:00∼12:00, Location: the room No. 111 in RIMS, Kyoto, in Japan, RIMS Conference Researches on theory of isometries and preserver problems from a various point of view.