An extension of the Hartshorne theorem on the characterization of cofinite complexes

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Abstract

This report is the announcement of the result on the characterization of cofinite complexes (the first one is in the talk at the RIMS conference). Let $A$ be a homomorphic image of a Gorenstein ring of finite Krull dimension, $J$ an ideal of $A$, and $N^\bullet$ a bounded-below complex of $A$-modules. Suppose that $A$ is complete with respect to a $J$-adic topology. In this report, we introduce that $N^\bullet$ is a $J$-cofinite complex if and only if the support of $H^i(N^\bullet)$ is in $V(J)$ for all $i$, and $\text{Ext}^j(A/J, N^\bullet)$ is of finite type for all $j$.

1 Introduction

In this report, we shall introduce the following theorem.

**Theorem 1** Let $A$ be a ring, which is a homomorphic image of a Gorenstein ring of finite Krull dimension, $J$ an ideal of $A$. Suppose that $A$ is complete with respect to a $J$-adic topology. Let $N^\bullet$ be a complex of $A$-modules in $\mathcal{D}^+(A)$, where $\mathcal{D}^+(A)$ is the derived category consisting of complexes bounded below. Then the following conditions are equivalent:

1. The complex $N^\bullet$ is $J$-cofinite.
2. The complex $N^\bullet$ satisfies the following conditions:
   a) $\text{Supp} \ H^i(N^\bullet) \subseteq V(J)$ for each $i$,
   b) $\text{Ext}^j(A/J, N^\bullet)$ is of finite type for each $j$.

We assume that all rings are commutative and noetherian with identity throughout this report.

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2 Preliminaries

We shall recall the definitions on cofiniteness. For the basic definitions, the reader is referred to [15] and [16] (see [3], [4], [2] and [14] for the notation on derived categories and derived functors).

Before defining the $J$-cofiniteness on complexes, we introduce the definition of the dualizing functor (cf. [14, § 4.3, p. 70]):

**Definition 1** Let $A$ be a ring, equipped with a dualizing complex $D^*$ for $A$ and $J$ an ideal of $A$ (see [19, (1.1) Definition, p. 215] for the definition of the (fundamental) dualizing complex). We denote by $D_J(-)$ the functor $\mathbb{R}\text{Hom}^*(-, \mathbb{R}\Gamma_J(D^*))$ on the derived category $\mathcal{D}(A)$. In this paper, we call this functor $D_J(-)$ the $J$-dualizing functor (cf. [14, p. 70]). Note that the $J$-dualizing functor is defined over unbounded complexes by [20, Theorem C, p. 125]. In the case $J = (0)$, then we simply denote $D_J(-)$ by $D(-)$.

The $J$-cofiniteness on complexes is defined as follows (see [3, § 2, p. 149] for the definition over regular rings):

**Definition 2** Let $A$ be a ring, equipped with a dualizing complex $D^*$ for $A$ and $J$ an ideal of $A$. Let $N^\cdot$ be an object of the derived category $\mathcal{D}(A)$. We say $N^\cdot$ is $J$-cofinite, if there exists $M^\cdot \in \mathcal{D}_{ft}(A)$, such that $N^\cdot \cong D_J(M^\cdot)$ in $\mathcal{D}(A)$. Here $D_J(-)$ is the $J$-dualizing functor on $\mathcal{D}(A)$ defined as above and $\mathcal{D}_{ft}(A)$ is the derived category consisting of complexes with cohomology modules of finite type over $A$.

Over regular rings $R$ of finite Krull dimension, it is proved by Hartshorne (cf. [3, Theorem 5.1, p. 154]). Over the rings $A$ which is a homomorphic image of a Gorenstein ring of finite Krull dimension, the theorem as a variant of [3, Theorem 5.1, p. 154] is proved under the additional conditions (ii) and (ii) (cf. [11, Theorem 5, p. 318]). Our aim in the report, we introduce an extension of the Hartshorne theorem over the ring $A$, without the conditions (i) and (ii).

3 Sketch proof of the main theorem

The purpose of this section is to give a sketch proof of the main theorem as an extension of [3, Theorem 5.1, p. 154]) without regularity assumption. The argument on the way out functors is used in [3, Theorem 5.1, p. 154]) over the regular rings. On the other hands, we use the spectral sequences. First let us prove the lemma:

**Lemma 2** Let $A$ be a ring, which is a homomorphic image of a Gorenstein ring of finite Krull dimension, $J$ an ideal of $A$, and $M^\cdot$ in $\mathcal{D}_{ft}^{-}(A)$. Suppose that $A$ is complete with respect to a $J$-adic topology. Then $\text{Ext}^j(T^\cdot, D_J(M^\cdot))$ is of finite type for all $j$, for each $T^\cdot$ in $\mathcal{D}_{ft}^{-}(A)$ with $\text{Supp} H^l(T^\cdot) \subset V(J)$ for all $l$. Here $\mathcal{D}_{ft}^{-}(A)$ is the derived category consisting of complexes bounded above with cohomology modules of finite type over $A$. 


Proof. We have an isomorphism in the derived category \( D(A) \) using the Hom-tensor adjunction (cf. [14, the Hom-tensor adjunction, p. 53]):

\[
\mathbb{R} \text{Hom}^\bullet(T^\bullet, D_J(M^\bullet)) \cong \mathbb{R} \text{Hom}^\bullet(M^\bullet, D(T^\bullet)).
\]

Then we have the lemma, using the spectral sequences and by Yoneda’s lemma. See [12] for the detailed calculation.

Proof of Theorem 1: See [12] for the detailed proof.

4 An application of the theorem

We recall the following claim (cf. [Kaw3, Claim 1]):

Claim 3 Let \( R \) be a ring, and \( J \) an ideal of \( R \). Let \( N^\bullet \in D^+(R) \) be a bounded-below complex. Then the following conditions are equivalent:

(i) \( \text{Ext}^j(R/J, N^\bullet) \) is of finite type over \( R \) for all \( j \);

(ii) \( \text{Ext}^j(R/\sqrt{J}, N^\bullet) \) is of finite type over \( R \) for all \( j \);

(iii) \( \text{Ext}^j(R/P, N^\bullet) \) is of finite type over \( R \) for all \( j \) and for each \( P \in \text{Min}(R/J) \);

(iv) \( \text{Ext}^j(W, N^\bullet) \) is of finite type over \( R \) for all \( j \) and for each \( R \)-module \( W \) of finite type over \( R \) such that \( \text{Supp}(W) \subseteq V(J) \);

(v) \( \text{Ext}^j(W^\bullet, N^\bullet) \) is of finite type over \( R \) for all \( j \) and for each \( W^\bullet \in D^b_{ft}(R) \) such that \( \text{Supp}(H^l(W^\bullet)) \subseteq V(J) \) for all \( l \);

(vi) \( \text{Ext}^j(W^\bullet, N^\bullet) \) is of finite type over \( R \) for all \( j \) and for each \( W^\bullet \in D^{-}_{ft}(R) \) such that \( \text{Supp}(H^l(W^\bullet)) \subseteq V(J) \) for all \( l \).

We can give the following corollary as an application from Theorem 1.

Corollary 4 Let \( A \) be a ring, which is a homomorphic image of a Gorenstein ring of finite Krull dimension, \( J \) an ideal of \( A \). Suppose that \( A \) is complete with respect to a \( J \)-adic topology. Let \( N^\bullet \) be a complex of \( A \)-modules in \( D^+(A) \). Then the following conditions are equivalent:

1. The complex \( N^\bullet \) is \( J \)-cofinite.
2. The complex \( N^\bullet \) satisfies the following conditions:
   a) \( \text{Supp} H^i(N^\bullet) \subseteq V(J) \) for each \( i \),
   b) the equivalence conditions in Claim 3.
References


\* The data of the talk in the conference:
Date: February 15, in 2018, 11:00∼12:00, Location: the room No. 111 in RIMS, Kyoto, in Japan, RIMS Conference Researches on theory of isometries and preserver problems from a various point of view.