Zero set theorem of a definable closed set

Tomohiro Kawakami
Department of Mathematics, Wakayama University
Sakaedani, Wakayama 640-8510, Japan
kawa@center.wakayama-u.ac.jp

1 Introduction

Let $\mathcal{M}=(\mathbb{R}, +, \cdot, <, \ldots)$ be an o-minimal expansion of the standard structure $\mathcal{R}=(\mathbb{R}, +, \cdot, <)$ of $\mathbb{R}$. Note that if $\mathcal{M}=\mathcal{R}$, then a definable $C^r$ manifold is a $C^r$ Nash manifold. Definable $C^r$ categories based on $\mathcal{M}$ are generalizations of the $C^r$ Nash category.

For any definable closed subset $A$ of $\mathbb{R}^n$ and $1 \leq r < \infty$, there exists a definable $C^r$ function $f : \mathbb{R}^n \to R$ such that $A=f^{-1}(0)$ ([2]). We consider the case where $r=\infty$ and its applications.

General references on o-minimal structures are [1], [2], see also [11]. The term "definable" means "definable with parameters in $\mathcal{M}$".

Theorem 1.1. Let $X$ be an affine definable $C^\infty$ manifold and $V$ a definable subset closed in $X$. Then there exists a non-negative definable $C^\infty$ function $f : X \to \mathbb{R}$ such that $f^{-1}(0) = V$.

As applications of Theorem 1.1, we have the following results.

Theorem 1.2. Let $\mathcal{M}=(\mathbb{R}, +, \cdot, e^x, \ldots)$ be an exponential o-minimal expansion of the standard structure $\mathcal{R}=(\mathbb{R}, +, \cdot, <)$ of the field of real numbers with $C^\infty$ cell decomposition. Then every $n$-dimensional definable $C^\infty$ manifold $X$ is definably $C^\infty$ imbeddable into $\mathbb{R}^{2n+1}$.

2010 Mathematics Subject Classification. 14P10, 14P20, 57R55, 58A05, 03C64.
Key Words and Phrases. Zero sets, definable $C^\infty$ manifolds, o-minimal, affine.
Theorem 1.2 is proved in [3] and its definable $C^r$ case ($1 \leq r < \infty$) is proved in [8]. We give another proof of it.

Theorem 1.2 is the definable version of Whitney’s imbedding theorem (e.g. 2.14 [4]). Even in the Nash category (i.e. $\mathcal{M} = \mathcal{R}$), we cannot drop the assumption that $\mathcal{M}$ is exponential by Theorem 1.2 [10].

**Theorem 1.3 ([6]).** If $0 \leq s < \infty$ and $\mathcal{M}$ is an exponential o-minimal expansion of $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ with $C^\infty$ cell decomposition, then every definable $C^s$ map between definable $C^\infty$ manifolds is approximated in the definable $C^s$ topology by definable $C^\infty$ maps.

Its equivariant version is proved in [6].

Using Theorem 1.3 and by a way similar to the proof of Theorem 1.2 and 1.3 [5], we have another proof of the following theorem ([3]).

**Theorem 1.4 ([3]).** Let $1 \leq s < r \leq \infty$, then every definable $C^s$ manifold admits a unique definable $C^r$ manifold structure up to definable $C^r$ diffeomorphism.

2 Proof of our results.

**Proof of Theorem 1.1.** By definition of affineness and 3.2 [9], $X$ is definably $C^\infty$ diffeomorphic to a definable $C^\infty$ submanifold of some $\mathbb{R}^l$ which is closed in $\mathbb{R}^l$. We identify $X$ with its image. Thus $V$ is closed in $\mathbb{R}^l$. Since $\mathcal{M}$ admits $C^\infty$ cell decomposition, there exists a $C^\infty$ cell decomposition $D$ partitioning $V$. For every cell $C \in D$, the closure $\overline{C}$ of $C$ in $X$ lies in $V$. Thus if $V = C_1 \cup \cdots \cup C_m$, then $V = \overline{C_1} \cup \cdots \cup \overline{C_m}$. If $C_i$ is bounded and $k$-dimensional, then $\overline{C_i}$ is definably $C^\infty$ diffeomorphic to $[-1,1]^k$. Hence $\overline{C_i}$ is the zeros of a definable $C^\infty$ function. Thus the case where $V$ is compact is proved.

Let $C_i$ be unbounded. Replacing $\mathbb{R}^l$ by $\mathbb{R}^{l+1}$, we may assume that $0 \notin \overline{C_i}$. Let $i : \mathbb{R}^{l+1} - \{0\} \to \mathbb{R}^{l+1} - \{0\}, i(x) = \frac{x}{||x||^2}$, where $||x||$ denotes the norm of $x$. Then $C_i' = i(\overline{C_i}) \cup \{0\}$ is the one point compactification of $\overline{C_i}$. Thus there exists a definable $C^\infty$ function $\psi : \mathbb{R}^{l+1} \to \mathbb{R}$ with $C_i' = \psi^{-1}(0)$. Hence $\overline{C_i}$ is definably $C^\infty$ diffeomorphic to the set $C_i'' = \{(x, y) \in \mathbb{R}^{l+1} \times \mathbb{R} | \psi(x) = 0, ||x||^2 y = 1\}$. Therefore $\overline{C_i}$ is the zeros of a definable $C^\infty$ function. Since $V = \overline{C_1} \cup \cdots \cup \overline{C_m}$, $V$ is the zeros of a definable $C^\infty$ function $\phi$. Thus $f := \phi^2 : X \to \mathbb{R}$ is the required function.
The following is a definable $C^\infty$ partition of unity.

**Proposition 2.1.** Let $\{U_i\}_{i=1}^k$ be a definable open covering of a definable $C^\infty$ manifold $X$. Then there exist definable $C^\infty$ functions $\lambda_i : X \to \mathbb{R}$ ($1 \leq i \leq k$) such that $0 \leq \lambda_i \leq 1$, $\text{supp } \lambda_i \subset U_i$ and $\sum_{i=1}^k \lambda_i = 1$.

If $X$ is affine, then the definable $C^r$ version of Proposition 2.1 is known in 4.8 [7].

**Proof.** We now prove that there exists a definable open covering $\{V_i\}_{i=1}^k$ of $X$ such that $\overline{V_i} \subset U_i$, $(1 \leq i \leq k)$, where $\overline{V_i}$ denotes the closure of $V_i$ in $X$.

We proceed by induction on $k$. If $k = 1$, then there is nothing to prove. Assume that there exists a definable open covering $\{V_i\}_{i=1}^{k-1} \cup \{U_k\}$ of $X$ such that $\overline{V_i} \subset U_i$, $(1 \leq i \leq k-1)$.

Let $X_{k-1} := \bigcup_{i=1}^{k-1} V_i$. By the inductive hypothesis, there exists a definable open covering $\{W_i\}_{i=1}^{k-1}$ of $X_{k-1}$ such that $\text{cl } W_i \subset V_i$, where $\text{cl } W_i$ means the closure of $W_i$ in $X_{k-1}$.

We may assume that $U_k$ is affine. Let $X_k := U_k \cap \bigcup_{i=1}^{k-1} V_i$ and $\text{Cl } X_k$ denote the closure of $X_k$ in $U_k$. By Theorem 1.1, there exists a non-negative definable $C^\infty$ function $\phi_k : U_k \to \mathbb{R}$ such that $\phi_k^{-1}(0) = \text{Cl } X_k$. Since $\text{cl } W_i \subset V_i$, $\phi_k$ is extensible to a non-negative definable $C^\infty$ function $\phi_k : U_k \cup W_i \to \mathbb{R}$ such that $\phi_k^{-1}(0) = \text{Cl } Z_k \cup W_i$. Inductively, we have a non-negative definable $C^\infty$ function $\phi : X \to \mathbb{R}$ such that $\phi^{-1}(0) = \text{Cl } Z_k \cup W_i \cdots \cup W_{k-1}$. Let $V_k := \{x \in U_k | \phi(x) > 0\}$. Then $V_k = \{x \in X | \phi(x) > 0\}$, $\overline{V_k} \subset U_k$ and $\{V_i\}_{i=1}^k$ is the required definable open covering of $X$.

By Theorem 1.1, we have a non-negative definable $C^\infty$ function $\mu_i : U_i \to \mathbb{R}$ such that $\mu_i^{-1}(0) = U_i - V_i$. Thus $\mu_i$ is extensible to a non-negative definable $C^\infty$ function $\mu'_i : X \to \mathbb{R}$ such that $\mu'_i^{-1}(0) = X - V_i$. Therefore $\lambda_i := \mu'_i / \sum_{i=1}^k \mu'_i$ is the required definable $C^r$ partition of unity.

**Proof of Theorem 1.1.** Let $\{\phi_i : U_i \to \mathbb{R}^{n_i}\}_{i=1}^k$ be a definable $C^r$ atlas of $X$. By Proposition 2.1, we have definable $C^\infty$ functions $\lambda_i : X \to \mathbb{R}$, $(1 \leq i \leq k)$ such that $0 \leq \lambda_i \leq 1$, $\text{supp } \lambda_i \subset U$ and $\sum_{i=1}^k \lambda_i = 1$. Thus the map $F : X \to \mathbb{R}^{nk} \times \mathbb{R}^k$ defined by $F(x) = (\lambda_1(x)\phi_1(x), \ldots, \lambda_k(x)\phi_k(x), \lambda_1(x), \ldots, \lambda_k(x))$ is a definable $C^\infty$ imbedding. Hence $X$ is affine. Thus it is either compact or compactifiable by 1.2 [7]. Hence we may assume that $X$ is affine and compact at the beginning. A similar argument of the proof of 1.4 [12], every definable $C^\infty$ map $f : X \to \mathbb{R}^{2n+1}$ can be approximated in the $C^r$ topology by an injective definable $C^\infty$ immersion $h : X \to \mathbb{R}^{2n+1}$. Since $X$ is compact, $h$ is the required definable $C^\infty$ imbedding.
References


